Dido's discrete problem and Bobylev's problem

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A) Preliminary tasks

The introductory series refers to the classical facts about Dido's minimal surface problem. Let Φ be a figure of the minimum perimeter with a given area S.

A1 Prove that any chord of the figure Φ bisecting its perimeter bisects its area and vice versa.

A2 Prove that the chord from the previous problem is perpendicular to the boundary of the figure Φ.

A3 (Dido's task) Prove that the figure Φ of the maximum area of a given perimeter is a circle.

A4 Specify a curve of minimum length that divides an equilateral triangle into two equal parts in area.

A5 Prove that among all n-gons of a given perimeter, the regular one has the maximum area.

A6 Prove that among all n-gons with given side lengths, the described one has the maximum area.

A7 Prove that the area of the n-gon from the previous problem does not depend on the order of its sides.

A8 (Dido's multidimensional problem) Prove that a body of a given surface area with a maximum volume is a ball.

A9 Prove that the tetrahedron of a given surface area with the maximum volume is regular.

A10 Prove that a parallelepiped of a given surface area with a maximum volume is a cube.

A11 (Research task) What can we say about multidimensional generalizations?

B) The discrete Dido problemThe discrete Dido problem

B1 A cellular polygon with cells of two colors will be called good if exactly a quarter of the cells in it are black. Is it true that any good square 12×12 can be cut into 9 good polygons?

Consider an infinite checkered plane (square, triangular, hexagonal). Let several cells be marked. A marked cell is called a boundary cell if it borders at least one unmarked cell. Let there be n marked boundary cells. The discrete problem of Dido will be called the question of maximizing the number of marked cells.

B2 Solve the discrete Dido problem for a square lattice (the boundary ones are on the side).

B3 Solve the discrete Dido problem for a square lattice (the boundary ones are on the side or corner).

B4 Solve the discrete Dido problem for a regular hexagonal lattice (the boundary ones are on the side).

B5 Solve the discrete Dido problem for a regular triangular lattice (the boundary ones are on the side).

B7 *Come up with a multidimensional generalization for a cubic lattice.

The problem can also be posed a little differently: for example, if the set of boundary cells is located inside a certain square, and we do not take into account the cells on the border of this square. The set of boundary cells that do not fall on the boundary of a square (or, similarly, a cube) will be called a free surface.The problem can also be posed a little differently: for example, if the set of boundary cells is located inside a certain square, and we do not take into account the cells on the border of this square. The set of boundary cells that do not fall on the boundary of a square (or, similarly, a cube) will be called a free surface.

B8 To solve the discrete Dido problem for the free surface of a square lattice section in the square $k \times k$ (that is, in the free surface of n cells, the boundary cells are on the side, we minimize the number of marked cells in the square kk).

B9 * Solve the discrete Dido problem for the free surface of a section of a cubic lattice in a $k \times k \times k$ cube (the boundary ones are along the face).

B10 * Solve the discrete Dido problem for the free surface of a section of a multidimensional cubic lattice in a cube $k \times k \times \cdots \times k$ (the boundary ones are along the face).

B11 (Open question) Derive the cube problem from the previous paragraph (see cycle C).

C) Properties of the multidimensionalProperties of the multidimensional

Let's start with the well-known problem of V.I. Arnold:

C1 What percentage of the volume is occupied by the pulp in a one-dimensional watermelon with a diameter of 1 meter, if the thickness of the crust is 1 cm?

Let's continue the topic.

C2 What does the volume of an n-dimensional ball of radius 2022 tend to at $n \to \infty$?

C3 Prove that the main building of the Moscow State University can be placed in an n-dimensional cube at $n \gg 1$. The distance between sets A and B will be called such a maximum number d that any distance between points x and y, where $x \in A$, $y \in B$, will be at least d.

C4 Prove that the section of a multidimensional cube plane can be a polygon arbitrarily close to the circle (that is, the distance between the polygon and the circle can be made less than any predetermined number $\delta > 0$).

C5 In a multidimensional unit cube, a set M of volume $0,99$ and a point A are given. Prove that the distance from M to A can be arbitrarily large.

Nevertheless, there is a conviction in a positive solution to the following problem.

The problem of a multidimensional cube. In the n-dimensional cube of a unit volume there are two sets M_1 and M_2 of volume ε each. Then the distance between them does not exceed some constant $F(\varepsilon)$.

Remark. The constant F depends only on ε, but not on the dimension.

We do not know the solution to the multidimensional cube problem, an attack on it is one of the goals of this project.

C6 ****. The multidimensional ball problem.** In an n-dimensional sphere of a unit volume there are two sets M_1 and M_2 of volume ε each. Then the distance between them does not exceed some constant $G(\varepsilon)$.

C7 Solve the multidimensional ball problem for convex bodies.

C8 ***** (Bobylev's problem.) Solve the multidimensional cube problem for convex bodies.

Remark. Similar questions can be posed for simplices (multidimensional tetrahedra) and multidimensional octahedra. However, such questions seem premature to us at the moment, at least until the cube problem is solved.

Functional analysis studies multidimensional and infinite-dimensional spaces. The problems of the cube and the sphere will undoubtedly shed additional light on the relevant problems, especially those related to the theory of measure and to the understanding of the structure of infinite-dimensional spaces.

What does **Dido's task** have to do with it?

An open question. The problem of the minimum free surface in a cube.

We describe the following class of subsets of points of a unit k-dimensional cube. We choose any natural number $n \leqslant k$, choose n coordinates and include in the subset all points whose selected coordinates are non—negative, and the sum of their squares does not exceed some C, and the remaining coordinates are numbers from 0 to 1. We include in the class all the sets obtained by all possible such choices. Then a subset of the cube having a fixed volume and a minimal free surface (that is, the area of the part of the surface that does not extend to the boundary of the cube is minimal) is achieved on one of the sets of this class.

C9 **The problem of the minimum free surface in the ball.** The set of volume V in a ball of unit volume B with the minimum area of the part of the surface that does not extend to the boundary of the ball is arranged as $B \cap B'$, where B' is a ball whose surface is perpendicular to the surface of B.

C10 Derive the cube problem from the problem of the minimum free surface in the cube.

Hint. If M is some set of points inside the cube, then for $\delta \to 0$ the volume The δ-neighborhood of M is asymptotically equal to $Vol(M) + \delta S(M)$.

Remark. The asymptotics of $Vol(M) + \delta S(M)$ is the basis for determining the surface area by Minkowski.

C11 Derive the ball problem from the problem of the minimum free surface in the ball.

On the discrete Dido problem. Although the problem of the minimal free surface in a cube seems to be very difficult, a discrete analogue of this problem (sufficient to solve the cube problem), it seems to us, can be solved.

D) Numbering of cube cells

D1 The numbers from 1 to 64 are placed in an 8×8 square. Prove that there are two adjacent cells on the side, the numbers in which differ by at least 5.

D2 In the cell, the boards $N \times N$ are numbered with numbers from 1 to N^2. Prove that there are two adjacent cells on the side whose number difference is not less than N.

D3 The same question for a cube with side N.

D4 The same question for a multidimensional cube.

E) The Peano curve

E1 Prove that there is a continuous mapping of a segment to a square.

E2 Prove that infinitely many pairs of points are bound to stick together.

E3 Is it true that infinitely many triples of points are bound to stick together?

E4 Is it true that infinitely many fours of points are bound to stick together?

E5 Explore multidimensional generalizations.

E6 Explore the systems of glued points and the distances between their prototypes.