Paint my hat in 3.5 colors!

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1 Let us introduce: the HATS game!

Let an undirected graph G be given, one sage and one chest with hats of different colors are located in each of its vertices. All the sages are acquainted with each other. Graph G, the location of the sages in the vertices of the graph and the colors of hats in all chests are fixed and known to everybody. In particular, each sage understands, in which vertex each of the others sages is located. The referee performs the following test with the sages. He puts a hat on the head of each sage, the hat is taken from the sage's chest. Each sage sees only the hats of the sages located in the neighbouring vertices of the graph, he does not see his own hat and does not know its color. The sages cannot communicate during the test. At the command of the referee each of the sages writes names of several colors on his paper simultaneously (how many colors the sages has to mention, is determined by the additional rule). We say that the sages have passed the test successfully = "have won", if at least one of them wrote the color of his hat in his paper.

The sages have been informed of the rules of the test before the testing and they have the possibility to hold a meeting, in which they must to define their public strategy. The publicity means that all the participants, including the referee, know this strategy. The strategy of the sages has to be deterministic, that is each sage has to write colors on his paper looking only the colors that he sees on his neighbours. We call the strategy *winning* if for any hats placement at least one sage will guess correctly the color of the hat on his head, i. e. mention this color in the his list of guesses. We say that the sages win, if they have a winning strategy, and that they lose, if they have not.

Therefore, the HATS game is not a game in a sense as it is ordinarily understood. This game lasts only one move.

1.1. The referee puts a hat of white, blue, red or green color on the head of each of two sages. Each of them sees the hat of the other, but does not see his own hat. Each of them writes on his own paper two colors simultaneously. They try to guess correctly the colors of their own hats. Prove that the sages can come to an agreement in the meeting before the test in such a way that at least one of them will guess correctly.

1.2. The referee puts a hat of five possible colors on the head of each of two sages. Each of them sees the hat of the other, but does not see his own hat. Each of them try to guess correctly the color of his own hat. The first sage writes on his own paper two colors and the second — three colors simultaneously. Prove that the sages can come to an agreement in the meeting before test in such a way that at least one of them will guess correctly.

1.3. Five sages stand around the non-transparent baobab. Shah has put red, blue, yellow or green hat the head of each of the sages. Sage does not know the color of his own hat and sees only the two neighbouring sages. As usual, without any communication each sages must makes one assumptions about the colors of his hat. But they fear be too lucky. How they should act to guarantee that for any placement of hats no more than two sages guess correctly the colors of their hats?

1.4. Sultan examines six court sages. By the rule of the examination the sultan locates 5 sages in 5 pits positioned around a circle, and locates the sixth sage in the tower in the center of the circle. The sultan writes one of the numbers 1, 2 or 3 on the forehead of each of the first five sages and writes a number from 1 to 243 on the forehead of the central sage. The sage in the tower sees the numbers of all the other sages, and these sages see his number (but do not see each other). All the sages must simultaneously try to guess correctly their numbers: the sages in the pits must say two numbers and the sage in the tower — one number. The sultan has explained to the sages the rules of the examination beforehand and has given time to communicate before the beginning of the examination. Can the sages act so that at least one of them certainly guess correctly his number?

We identify a vertex of graph G and the sage located in it. We assume that the colors are numbered by 0, 1, 2, 3, ... and that the chest of sage v contains hats of colors from 0 to some number $h(v) - 1$.

The HATS game is the triple $\langle G, h, g\rangle$, where $G = \langle V, E\rangle$ — a graph, $h\colon V \to \mathbb{N}$ — a function that for each vertex v equals the number of colors of hats keeping in the chest in this vertex, $g\colon V \to \mathbb{N}$ — a function equal to the number of guesses of each sage. We call function h a "hatness", and g — a function of guesses or the number of attempts. For each non negative integer h we denote by $\star h$ the function on V possessing the constant value h. Instead of the notation $\langle G, h, \star 1\rangle$ we will use shorter notation $\langle G, h\rangle$.

1.5. Prove that if the game $\langle G, h, g\rangle$ is winning, then for each non negative integer k the game $\langle G, k \cdot h, k \cdot g\rangle$ is winning, too.

1.6. Game $\langle G, h, g\rangle$ is given. Let $K \subset G$ is anticlique (a set of vertices such that there is no edge connected any pair of them) and for each $v \in K$ $h(v) > g(v)$. Prove that there exists a hats placement, for which none of the sages in K guesses correctly.

1.7. Let h and g be natural numbers, $G = \langle G, \star h, \star g\rangle$ be a winning game, $r' \leqslant \frac{h}{g}$ be a rational number. Prove that there exist natural numbers h' and g' such that $\frac{h'}{g'} = r'$ and game $\langle G, \star h', \star g'\rangle$ is winning.

1.8. Formulate and prove the analogue of the previous statement for non-constant functions of hatness and guessing.

1.9. Denote by K_n a complete graph on n vertices. Prove that the game $\langle K_n, h, g\rangle$ is winning if and only if

$$\sum_{v \in K_n} \frac{g(v)}{h(v)} \geqslant 1.$$

2 Paths and trees

The theory of HATS game on the complete graph K_3 is given by the problem statement 1.9. Now consider a path P_3 which is less complicated graph.

2.1. Prove that the sages lose in the game $\langle P_3, \star 3, \star 1\rangle$.

2.2. a) Prove that the game $\overset{\frac{3}{10}}{\bullet}\!\!-\!\!\!-\!\!\overset{\frac{3}{10}}{\bullet}\!\!-\!\!\!-\!\!\overset{\frac{3}{5}}{\bullet}$ is winning (the numerator is the number of guesses, and the denominator is the hatness).

b) Prove that the game $\overset{\frac{3}{11}}{\bullet}\!\!-\!\!\!-\!\!\overset{\frac{3}{10}}{\bullet}\!\!-\!\!\!-\!\!\overset{\frac{3}{5}}{\bullet}$ is losing.

c) Prove that the game $\overset{\frac{s}{t(s)}}{\bullet}\!\!-\!\!\!-\!\!\overset{\frac{s}{t(s)}}{\bullet}\!\!-\!\!\!-\!\!\overset{\frac{s}{t(s)}}{\bullet}$ is losing, where $t(s) = s^2 + s + 1$.

Let G be a graph and s be a non negative integer. Denote by $\mathrm{HG}_s(G)$ the s-hat number of G, i.e. the maximum number of hats h for which the game $\langle G, \star h, \star s\rangle$ is winning. For $s = 1$ this number is called hat number of G and is denoted by $\mathrm{HG}(G)$.

2.3. Prove that for any non negative integers n and s the game $\underset{A}{\overset{\frac{s}{2s}}{\bullet}}\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet\cdots$ on path P_n is losing. Here all vertices except the leftmost vertex A have hatness $4s - 1$ and s guesses.

2.4. Prove that one can find n such that $\mathrm{HG}_2(P_n) = 6$, $\mathrm{HG}_3(P_n) = 10$, $\mathrm{HG}_4(P_n) = 14$.

2.5. Prove that for any non negative integer s the game $\langle P_n, \star(4s - 2), \star s\rangle$ is winning for $n \geqslant 2s$.

2.6. a) Let $t(s) = s^2 + s + 1$. Prove that for each tree T the game $\langle T, \star t(s), \star s\rangle$ is losing.

b) Len $K_{1,n}$ be "a star" graph (i.e. a tree consisting of a root and n leaves). Prove that for sufficiently large n the game $\langle K_{1,n}, \star(s^2 + s), \star s\rangle$ is winning.

c) Prove that for any non negative integer h there exists integer n such that the game on graph $K_{1,n}$ is winning if all the sages in pendant vertices have one guess and hatness 3, and the central sage has 2 guesses and hatness h.

3 Constructors

3.1. Let $\langle G, h, g\rangle$ be a winning game, A_1 and A_2 be vertices of G, not connected with an edge and such that $h(A_1) = h(A_2)$. We glue vertices A_1 and A_2 of G into one new vertex A, denote by $\widetilde{G}$ the obtained graph. Let functions $\widetilde{h}$ and $\widetilde{g}$ defined on the set of vertices of $\widetilde{G}$ coincide with h and g in all vertices except A_1 and A_2, and $\widetilde{h}(A) = h(A_1)$, $\widetilde{g}(A) = g(A_1) + g(A_2)$. Then the game $\langle \widetilde{G}, \widetilde{h}, \widetilde{g}\rangle$ is also winning. (And then if the game $\langle \widetilde{G}, h, \widetilde{g}\rangle$ is losing, then the game $\langle G, h, g\rangle$ is losing too.)

Let $\mathcal{G}_1 = \langle G_1, h_1, g_1\rangle$, $\mathcal{G}_2 = \langle G_2, h_2, g_2\rangle$ be two games such that $V_1 \cap V_2 = \{v\}$. Let $G = G_1 +_v G_2$ be the union of graphs G_1 and G_2, in which both vertices v are glued into one new vertex. Define functions $h, g\colon V_1 \cup V_2 \to \mathbb{N}$:

$$h(u) = \begin{cases} h_i(u), & u \in V_i \setminus \{v\},\ (i = 1, 2), \\ h_1(v)h_2(v), & u = v, \end{cases} \qquad g(u) = \begin{cases} g_i(u), & u \in V_i \setminus \{v\},\ (i = 1, 2), \\ g_1(v)g_2(v), & u = v. \end{cases}$$

We say that the game $\mathcal{G} = \langle G, h, g\rangle$ is a product of games $\mathcal{G}_1$ and $\mathcal{G}_2$ and denote it by $\mathcal{G}_1 \times_v \mathcal{G}_2$ (fig. 1).

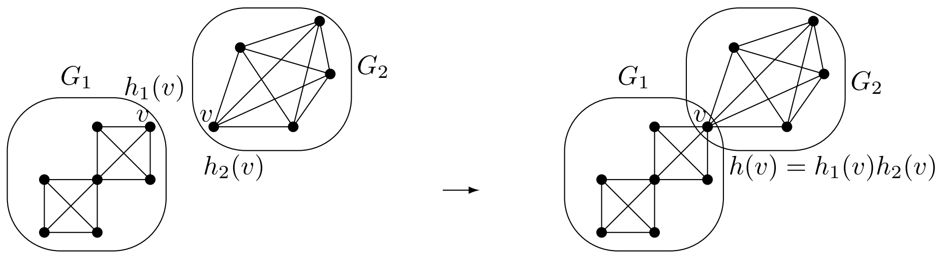

Figure 1. The product $G_1 \times_v G_2$

3.2. The theorem about the product. If the sages win in games $\mathcal{G}_1$ and $\mathcal{G}_2$, then they also win in game $\mathcal{G}_1 \times_v \mathcal{G}_2$.

3.3. Let $G = G_1 +_A G_2$, where G_1 and G_2 are graphs, for which $V(G_1) \cap V(G_2) = \{A\}$. Let games $\mathcal{G}_1 = \langle G_1, h_1, g_1\rangle$ and $\mathcal{G}_2 = \langle G_2, h_2, g_2\rangle$ be losing, and the following conditions hold:

$$g_1(A) = g_2(A) = s, \qquad h_1(A) \geqslant h_2(A) = s + 1.$$

Then game $\mathcal{G} = \langle G_1 +_A G_2, h, g\rangle$ is losing, where

$$h(x) = \begin{cases} h_i(x), & x \in V_i \setminus \{A\}\ (i = 1, 2), \\ h_1(A), & x = A, \end{cases} \qquad g(x) = \begin{cases} g_i(x), & x \in V_i \setminus \{A\}\ (i = 1, 2), \\ s, & x = A. \end{cases}$$

3.4. A half-edge removal. Let $\langle G, h, g\rangle$ be a winning game, AB be an edge of graph G, $\widetilde{G}$ be the graph obtained from G by replacing edge AB by directed edge $B \to A$ (i.e. sage A does not see sage B, but B sees A). Let function $\widetilde{g}$ on the vertices of graph G coincide with g in all vertices except A, and $\widetilde{g}(A) = h(B)g(A)$. Then game $\langle \widetilde{G}, h, \widetilde{g}\rangle$ is winning too. (And therefore, if game $\langle \widetilde{G}, h, \widetilde{g}\rangle$ is losing, then $\langle G, h, g\rangle$ is also losing.)

3.5. Substitution theorem. Let $\mathcal{G}_1 = \langle G_1, h_1, g_1\rangle$ be $\mathcal{G}_2 = \langle G_2, h_2, g_2\rangle$ be winning games. Let A be an arbitrary vertex of graph G_2. Consider the new graph G obtained from G_2 by substitution of graph G_1 on the place of vertex v (each vertex G_1 is connected with former neighbours of vertex A by new edges, see fig. 2). Then game $\langle G, h, g\rangle$ is winning, where

$$h(u) = \begin{cases} h_2(u), & u \in V(G_2) \setminus \{A\}, \\ h_1(u)h_2(A), & u \in V(G_1), \end{cases} \qquad g(u) = \begin{cases} g_2(u), & u \in V(G_2) \setminus \{A\}, \\ g_1(u)g_2(A), & u \in V(G_1). \end{cases}$$

3.6. Substitution with reducing. Let $\mathcal{G} = \langle G, h, \star s\rangle$, $\mathcal{G}' = \langle G', h', g'\rangle$ be winning games. Let A be a vertex of graph G', and $h'(A) = s$. Let $\langle \widetilde{G}, \widetilde{h}, \widetilde{g}\rangle$ be the winning game obtained by the substitution of

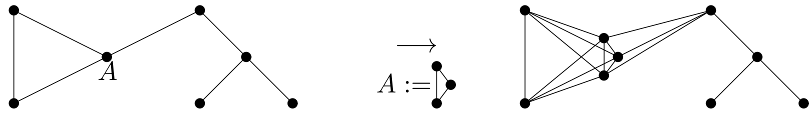

Figure 2. Substitution of a graph on the place of vertex A.

game $\mathcal{G}$ on the place of vertex A to game $\mathcal{G}'$ (as in problem 3.5). By the rule of the substitution for all substituting vertices v

$$\widetilde{h}(v) = h(v)h'(A) = s \cdot h(v), \qquad \widetilde{g}(v) = g(v)g'(A) = s \cdot g'(A).$$

Consider new functions h^*, g^* on graph $\widetilde{G}$, which differ from $\widetilde{h}$, $\widetilde{g}$ only by the values in substituting vertices v, and this difference is the cancellation by s:

$$h^*(v) = h(v), \qquad g^*(v) = g'(A).$$

Then game $\langle \widetilde{G}, h^*, g^* \rangle$ is also winning.

3.7. Blowing up of a vertex. Let $\mathcal{G} = \langle G, h, g \rangle$ be winning game, $A \in V(G)$, $\widetilde{G}$ be the graph obtained from G by the substitution of clique B consisting of $g(A)$ vertices on the place of vertex A. Then game $\langle \widetilde{G}, \widetilde{h}, \widetilde{g} \rangle$ is also winning, where

$$\widetilde{h}(v) = \begin{cases} h(v), & v \in V(G) \setminus \{A\}, \\ h(A), & v \in B, \end{cases} \qquad \widetilde{g}(v) = \begin{cases} g(v), & v \in V(G) \setminus \{A\}, \\ 1, & v \in B. \end{cases}$$

4 "Petals" and "petunias"

We define a *petal graph* to be a graph G obtained from a path by adding a vertex v adjacent to every vertex of this path, see fig. 3, we say that v is the *stem* of G.

Then, we define a *petunia* to be a graph constructed in the following way. Take two petals L_1 and L_2, denote one vertex in each of them by v_1, and construct a graph $M_2 = L_1 +_{v_1} L_2$. After that consider graph M_2 and a new petal L_3 denote one vertex in each of them by v_2, and construct a graph $M_3 = M_2 +_{v_2} L_3$ and so on.

A *royal petunia* is a petunia (рис. 4), for which the vertex v_i in each step of its construction were chosen as the stem of petal L_{i+1}.

4.1. Let G be a petal of n vertices, see fig. 3, let the stem has hatness 2 and the other vertices have hatness 7. Prove that the sages lose in the game $\langle G, h \rangle$.

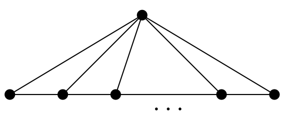

Figure 3. A petal of n vertices

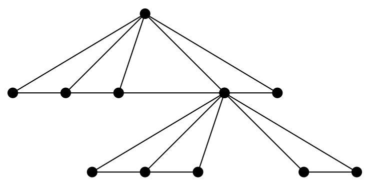

Figure 4. A royal petunia

4.2. Let G be a petal of n vertices, $f(s) = s^2 + s$. Prove that $\mathrm{HG}_s(G) \leqslant f(f(s))$.

4.3. Let M be a petunia, h_s be maximum integer such that the game $\langle M, \star h_s, \star s \rangle$ is winning. Prove that $h_s \leqslant f(f(f(s)))$.

4.4. a) Prove that $\mathrm{HG}_s(G) = 4s(s+1) - 2$, where G is a petal of n vertices for sufficiently large n (fig. 3).

b) Prove the same equality if G is a royal petunia that hat sufficiently big petals.

5 Problems after intermediate finish

To section 1

5.1. Let $K_{1,n}$ be a "star" graph, and $h = (h_0, \dots, h_n)$, $g = (g_0, \dots, g_n)$ be arbitrary functions of hatness and number of attempts, where $1 \leqslant g_i \leqslant h_i$ for all i, the zero index corresponds to the central vertex of the graph. Prove that the existence of k, for which $\langle K_{1,n}, k \cdot h, k \cdot g\rangle$ is winning, is equivalent to the inequality

$$\frac{g_0}{h_0} \geqslant \prod_{i=1}^{n}\left(1 - \frac{g_i}{h_i}\right).$$

To section 2

5.2. Let h and g be positive integers, and $g^2 - 3gh + h^2 < 0$. Prove that game $\langle P_3, \star h, \star g\rangle$ is winning.

Fractional hat guessing number of graph G we call the value $\hat{\mu}(G) = \sup\{\frac{h}{g} \colon \langle G, \star h, \star g\rangle \text{ is winning}\}$. As it follows from problem 1.9, $\mathrm{HG}(K_n) = \hat{\mu}(K_n) = n$, $\mathrm{HG}_s(K_n) = sn$. In the general case, $\hat{\mu}(K_n) \geqslant \frac{1}{s}\mathrm{HG}_s(G) \geqslant \mathrm{HG}(G)$.

5.3. Prove that $\hat{\mu}(K_3) = \frac{3+\sqrt{5}}{2}$.

To section 3

5.4. Let $\langle G_2, h_2\rangle$ be a losing game, $H_2 \subset G_2$ be a clique in G_2. Let G_1 be a complete graph. Define a hatness function h_1 on it in such a way that the relation

$$\left(\sum_{u\in G_1}\frac{1}{h_1(u)}\right)\left(\prod_{v\in H_2} h_2(v)\right) < 1$$

holds. Let G be the graph obtained by union of graphs G_1 and G_2 with adding all edges between vertices G_1 and H_2 (fig. 5). Prove that game $\langle G, h\rangle$ is losing if

$$h(v) = \begin{cases} h_1(v), & v \in G_1,\\ h_2(v), & v \in G_2.\end{cases}$$

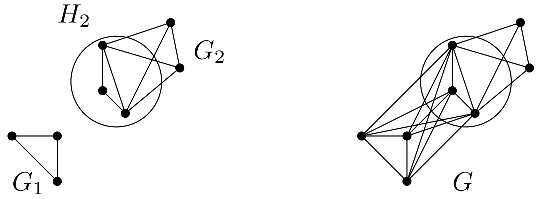

Figure 5. Example to problem 5.4. The number of vertices in G_1 and H_2 should not necessarily coincide

5.5. Let $\mathcal{G} = \langle G, h\rangle$ be a losing game, A be an arbitrary vertex of graph G. Consider graph $G' = (V', E')$ obtained by adding to graph G new pendant vertex B: $V' = V \cup \{B\}$, $E' = E \cup \{AB\}$. Then the sages lose in game $\langle G', h'\rangle$, where $h'(B) = 2$, $h'(A) = 2h(A) - 1$ and $h'(u) = h(u)$ for other vertices $u \in V$.

5.6. Let $\mathcal{G} = \langle G, h, g\rangle$ be a game, a vertex $A \in V(G)$ be connected with all other vertices of G, $h(A) = s + 1$, $g(A) = s$, and $\mathcal{G}' = \langle G \setminus \{A\}, h', (s+1)\cdot g'\rangle$, where $h' = h\Big|_{V(G)\setminus\{A\}}$, $g' = g\Big|_{V(G)\setminus\{A\}}$. Then the games $\mathcal{G}$ and $\mathcal{G}'$ are equivalent.

To section 5

5.7. a) A winning graph G contains a "long bridge" — two-link path ABC, such that after removing this path the graph falls into two components of connectivity: G_1 (containing vertex A) and G_2 (containing vertex C). Let hatness of vertex B equals 5. Prove that at least one of the games $\langle G_1, h\big|_{G_1}\rangle$, $\langle G_2, h\big|_{G_2}\rangle$ is winning.

b) Let graph $\widetilde{G}$ is obtained by subpartition of an arbitrary graph G (i. e. each edge of graph G has been replaced by two-link path). Prove that game $\langle \widetilde{G}, \star 5\rangle$ is losing.

5.8. Let $\mathcal{G} = \langle G, h\rangle$ be a winning game, and it is maximal in the following sense: when hatness function increase in any vertex, the sages lose and, besides that, there does not exist a subgraph of G, on which the sages can win with hatness function h. Suppose that graph G contains edge-bridge AB. Prove that game $\mathcal{G}$ can be represented as the product of games.

5.9. Sages A and B have 1 guess, see each other and all other sages in the graph (and the others see them), $h(A) = 2$, $h(B) = 3$. Prove that if to replace these two sages by one sage C that sees the others, and the others see him, and $h(C) = 6$, $g(C) = 5$, then the result of the game does not change.

5.10. Given natural numbers s and d. Let G be an arbitrary graph, the vertices of which are partitioned into two sets $V(G) = A \cup B$, and each vertex of A has no more than d neighbours from B. Prove that $\mathrm{HG}_s(G) \leqslant \mathrm{HG}_{s'}(G[A])$, where $s' = s(\mathrm{HG}_s(G[B]) + 1)^d$, $G[A]$ and $G[B]$ are induced subgraphs on sets A and B.

Solutions

1.1. Let us call white and red colors light, and blue and green colors dark. Then let the first sage write both dark colors, if he sees on the second a light color, and vice versa; and the second sage write both light colors, if he sees on the first a light color, and vice versa. It is not difficult to understand that at least one of the sages guesses correctly.

1.2. Renumber the hat colors from 1 to 5. Then let the first player write colors under the assumption that the sum of the numbers on the hats of the first and the second sages gives remainder 0 or 1 mod 5, and the second — under the assumption that the same sum gives remainder 2, 3 or 4 mod 5. Since the sum really gives some remainder, the sage with the correct assumption guesses correctly his color.

1.3. First remind how to play the game, when there are only two sages: the hats can be colored in only two colors (denote the colors by 0 and 1) and it is required that somebody necessarily guessed correctly. Here is a strategy: one sage (call him equalizer) checks hypothesis "the hat colors are identical", the other (distinguisher) — "the hat colors are different". Note that by this strategy for any hats placement one of the sages guesses correctly and the other does not.

Consider a visibility graph: the sages are vertices, the pairs of neighbouring sages are edges, then this graph is a cycle on five vertices. Denote its edges by a, b, c, d, e. We assume that the color of sage's hat is a two-digit binary number: 00, 01, 10 or 11; writing this number we will mark its bits by the labels of the edges incident to the vertex. For example, if there are two outgoing edges a and b from the vertex, then we subscribe one of the bits "a", and the other — "b" (the order of labels is not important, all the neighbours of the sage see this labelling).

Let the sages at the endpoints of each edge come to agreement, who on this edge is equalizer, and who is distinguisher. The guessing on each edge x happens as follows: each sage on this edge looks only at bit x of his neighbour's color and calculates the color of his own bit x in accordance with his role on this edge. Therefore, each sage casting a glance to the left and to the right, calculates both bits and names the obtained color as his answer.

It is evident that a sage guesses correctly the color of his hat only if he has guessed correctly both bits. Since the graph contains only 5 edges, only 5 bits have been guessed correctly, and hence, at most two sages have guessed correctly the colors of their hats.

1.4. Put into correspondence to each color of the central sage the sequence of 5 digits, each of which is 1, 2 or 3. The strategy of i-th sage in pit: look at i-th digit of the central sage and name the other two digits. The strategy of the central sage: name the color, i-th digit of which is equal to the digit of i-th sage from pit. If no sage from pit has guessed correctly, then the central guesses his color correctly.

1.5. For each sage v consider $k \cdot h(v)$ hat colors and split them into $h(v)$ groups containing k colors each, we call them megacolors. Then in game $\langle G, k \cdot h, k \cdot g\rangle$ each sage v can understand megacolors of all his neighbours, and, according to the strategy for game $\langle G, h, g\rangle$, name $g(v)$ of his own megacolors. But these $g(v)$ megacolors correspond to $k \cdot g(v)$ usual colors. It is not difficult to see that if the strategy for game $\langle G, h, g\rangle$ is winning, then the obtained strategy is winning too.

1.6. Give arbitrary hats to the sages that are not from K. Then all answers of the sages from K according the strategy are determined. It remains to give a hat to each of them such that he will not guess correctly.

1.7. Let $r' = \frac{p}{q}$. If $G = \langle G, \star h, \star g\rangle$ is winning, then game $\langle G, \star ph, \star pg\rangle$ is also winning. And since $\frac{p}{q} \leqslant \frac{h}{g}$, then $pg \leqslant qh$, whence game $\langle G, \star ph, \star qh\rangle$ is also winning, because the increasing of the number of attempts cannot destroy the working strategy.Then $h' = ph$, $q' = qh$ are the desired.

1.8. Let $G = \langle G, h, g\rangle$ be a winning game, $r'\colon V \to \mathbb{Q}$ be a function such that $0 < r'(v) \leqslant \frac{h(v)}{g(v)}$ for all v. Prove that there exist functions h' and g', for which $\frac{h'(v)}{g'(v)} = r'(v)$ for all v, and game $\langle G, h', g'\rangle$ is winning.

Proof. Let $r'(v) = \frac{p(v)}{q(v)}$. Denote $P = \prod\limits_v p(v)$. Then game $\langle G, P \cdot h, P \cdot g\rangle$ is winning by problem 1.5. Since $\frac{p(v)}{q(v)} \leqslant \frac{h(v)}{g(v)}$, then by increasing the number of attempts in some vertices we obtain a winning game $\left\langle G, P \cdot h, \frac{P}{p} \cdot qh\right\rangle$. Therefore, $h' = P \cdot h$, $g'(v) = \frac{P}{p(v)} \cdot q(v)h(v)$ are the desired.

1.9. *"If" case.* Note that sage v guesses correctly in $\frac{g(v)}{h(v)}$ fraction of placements of all hats. And since in each placement at least one sage guesses correctly, the sum of these fractions is at least 1.

"Only if" case. We will show that if the inequality holds, then the sages have a winning strategy.

Let $H = \prod\limits_v h(v)$. Code the hat colors by numbers from 0 to $H - 1$: for each sage v let the possible color $i \in \{0, 1, \dots, h(v) - 1\}$ of his hat correspond to the remainder $\frac{iH}{h(v)}$ modulo H. In other words the set of reminders

$$0, \quad \frac{H}{h(v)}, \quad \frac{2H}{h(v)}, \quad \dots, \quad \frac{(h(v)-1)H}{h(v)} \pmod{H}. \qquad (*)$$

is a set of possible hats colors of sage v. When the hats placement is given, let S be the sum of numbers of all hats modulo H. The sages do not know the value of S, but each sage v can calculate value S_v that is the sum of the numbers of the hats, that the devilkin has given to the other sages, modulo H.

The strategy of the sages is the following: each sage v checks the hypothesis $S \in [a_v, b_v)$, where $[a_v, b_v)$ is an interval of length $\frac{Hg(v)}{h(v)}$, containing $\frac{Hg(v)}{h(v)}$ consecutive remainders modulo H, these intervals the sages choose in their meeting before the test. In order to check the hypothesis sage v has to solve the "inequality": he finds, for which reminder x the inclusion $S_v + x \in [a_v, b_v)$ is satisfied. After solving this problem he obtain a list $\frac{Hg(v)}{h(v)}$of possible values of x, but this list contains only $g(v)$ remainders of the form $(*)$. Then sage v will name $g(v)$ corresponding to them colors.

The given inequality in the problem statement is equivalent to the fact that the sum of lengths of all segments $[a_v, b_v)$ is at least H. If it holds, then, evidently, we can assign to each sage one segment in such a way that each remainder modulo H belongs to at least one of the segments. It guarantees the victory of the sages: whatever the sum of hats is equal, at least one of the sages will make right assumption and will guess correctly.

2.1. The sages cannot win even in game $\langle P_3, \star 3k, \star k\rangle$. It follows from the inequality of problem 5.1.

2.2. a) Denote the sages by A, B and C: $\underset{A}{\overset{\frac{3}{10}}{\bullet}}\!\!-\!\!\underset{B}{\overset{\frac{3}{10}}{\bullet}}\!\!-\!\!\underset{C}{\overset{\frac{3}{5}}{\bullet}}$. We will demonstrate the winning strategy for the sages. Let sage C name colors $[\frac{c_B}{2}]$, $[\frac{c_B}{2}] + 1$, $[\frac{c_B}{2}] + 2 \bmod 5$, and sage A name colors c_B, $c_B + 3$, $c_B + 6 \bmod 10$. Sage B, casting glances at the neighbours, suspects that they both do not guess correctly only if

$$c_B \notin S = \{c_A, c_A - 3, c_A - 6, 2c_c, 2c_C + 1, 2c_C + 2, 2c_C + 3, 2c_C + 4, 2c_C + 5\}.$$

It remains to note that for the remainders modulo 10 the inclusion

$$\{c_A, c_A - 3, c_A - 6\} \subset \{2c_c, 2c_C + 1, 2c_C + 2, 2c_C + 3, 2c_C + 4, 2c_C + 5\}$$

is impossible for any c_A and c_C. Therefore, set S contains at least 7 elements, and sage B can name in his answer the three (or less) remainders, not belonging to the set.

b) For every possible color c_B sage A makes three guesses, i. e. he names 30 answers of 11-element set of A's colors. Therefore, he names some color 1 or 2 times. Give the hat of this color to sage A. Then sage B will suspect, for which 8 colors of his hat sage A do not guess correctly. But the game $\underset{B}{\overset{\frac{3}{8}}{\bullet}}\!\!-\!\!\underset{C}{\overset{\frac{3}{5}}{\bullet}}$ is losing, and any strategy that is used by our sages in the above situation, is immediately reduced to the strategy in this losing game.

c) It is sufficient to verify that game $\underset{A}{\overset{\frac{s}{t(s)}}{\bullet}}\!\!-\!\!\underset{B}{\overset{\frac{s}{s+1}}{\bullet}}\!\!-\!\!\underset{C}{\overset{\frac{s}{t(s)}}{\bullet}}$ is losing. Apply the statement of the constructor "removing a half-edge" (problem 3.4) for vertex B, making this vertex invisible for A and C. As a result vertices A and C see nobody, have hatness s^2+s+1 and s^2+s guesses. Therefore, the referee can give them such hats that they will not guess correctly. After that the strategy of sage B is completely determined, he has hatness $s+1$ and s guesses, so he will not guess correctly too.

Another solution can be obtained by standard "probabilistic" observations: the number of hats placements, for which sage v guesses correctly, does not exceed fraction $g(v)/h(v)$ of the total number of placements. We need only note that for our graph $\frac{3s}{s^2+s+1} < 1$ for $s > 1$.

2.3. For each natural we s prove the statement by induction on n. Base case $n=1$, i. e. the loss in game $\underset{A}{\overset{\frac{s}{2s}}{\bullet}}\!-\!\!\!-\!\underset{B}{\overset{\frac{s}{4s-1}}{\bullet}}$ follows from the statement of problem 1.9.

Induction step. Consider three leftmost vertices A, B, C. Consider all possible hat color assignments to sage B. Sage A names $s(4s-1)$ colors from set $\{0,1,2,\ldots,2s-1\}$ in total. Therefore some color c_A occurs in his answers at most $[\frac{s(4s-1)}{2s}] = 2s-1$ times. Give to sage A the hat of this color. Then sage B sees color c_A and knows, for which $2s-1$ colors of his hat sage A names color c_A. So sage B may assume that the color of his own hat is taken from set C_B consisting of $4s-1-(2s-1)=2s$ colors. At that moment the devilkin (the other name of the referee) declares that in current hats placement the color of B's hat belongs to C_B and inform the other sages what is the set C_B. Then the game from induction step takes place on the remained graph and it is losing.

2.4. The inequality $\mathrm{HG}_2(P_4) \geqslant 6$ holds because the product of games $\overset{\frac{2}{6}}{\bullet}\!-\!\!\!-\!\underset{v_1}{\overset{\frac{2}{3}}{\bullet}} \times_{v_1} \underset{v_1}{\overset{\frac{1}{2}}{\bullet}}\!-\!\!\!-\!\underset{v_2}{\overset{\frac{1}{2}}{\bullet}} \times_{v_2} \underset{v_2}{\overset{\frac{2}{3}}{\bullet}}\!-\!\!\!-\!\overset{\frac{2}{6}}{\bullet}$ is winning by theorem of product.

The inequality $\mathrm{HG}_3(P_6) \geqslant 10$ holds since the game $\overset{\frac{3}{10}}{\bullet}\!-\!\!\!-\!\overset{\frac{3}{10}}{\bullet}\!-\!\!\!-\!\underset{v_1}{\overset{\frac{3}{5}}{\bullet}} \times_{v_1} \underset{v_1}{\overset{\frac{1}{2}}{\bullet}}\!-\!\!\!-\!\underset{v_2}{\overset{\frac{1}{2}}{\bullet}} \times_{v_2} \underset{v_2}{\overset{\frac{3}{5}}{\bullet}}\!-\!\!\!-\!\overset{\frac{3}{10}}{\bullet}\!-\!\!\!-\!\overset{\frac{3}{10}}{\bullet}$ is winning (the leftmost and rightmost multipliers are winning by problem 2.2).

Finally, the inequality $\mathrm{HG}_4(P_{10}) \geqslant 14$ holds as a result of the fact that game $G(u) \times_u \underset{u}{\overset{\frac{1}{2}}{\bullet}}\!-\!\!\!-\!\underset{w}{\overset{\frac{1}{2}}{\bullet}} \times_w G(w)$ is winning, where $G(u) = \overset{\frac{4}{15}}{\bullet}\!-\!\!\!-\!\underset{v_1}{\overset{\frac{4}{5}}{\bullet}} \times_{v_1} \underset{v_1}{\overset{\frac{1}{3}}{\bullet}}\!-\!\!\!-\!\underset{v_2}{\overset{\frac{2}{3}}{\bullet}} \times_{v_2} \underset{v_2}{\overset{\frac{2}{5}}{\bullet}}\!-\!\!\!-\!\overset{\frac{4}{14}}{\bullet}\!-\!\!\!-\!\underset{u}{\overset{\frac{4}{7}}{\bullet}}$.

In view of the statement of problem 2.3 any game in the form $\underset{A}{\overset{\frac{s}{4s-1}}{\bullet}}\!-\!\!\!-\!\overset{\frac{s}{4s-1}}{\bullet}\!-\!\!\!-\!\overset{\frac{s}{4s-1}}{\bullet}\cdots$ is losing (as compared to problem 2.3 here the hatness of vertex A has been increased). For $s=2$, 3, 4 this gives, by the way, for all n the inequalities $\mathrm{HG}_2(P_n) \leqslant 6$, $\mathrm{HG}_3(P_n) \leqslant 10$, $\mathrm{HG}_4(P_n) \leqslant 14$.

2.5. Consider a hatness function h on graph P_s:

$$h(v_i) = \begin{cases} 4s-2 & \text{for } 1 \leqslant i < s, \\ 2s-1 & \text{for } i = s. \end{cases}$$

To prove the statement of the problem it is sufficient to verify that game $\langle P_s, h, \star s\rangle$ is winning. For construction of sages' strategy we need the following auxiliary statement — a theorem about game with hint.

Let game $\mathcal{G} = \langle G, h, g\rangle$ be winning under condition that the devilkin makes the following hint during the game. For one vertex $B \in V(G)$ a natural number $w_B \leqslant h(B)$ is fixed and it is known that the devilkin will come to sage B during the game and will tell him a set of w_B consecutive remainders (i. e. the set of remainders in the form x, $x+1$, …, $x+w_B \bmod h(B)$), containing the color of his hat; the other sages will not hear this hint. Vertex B, number w_B and the rule of proclaiming of the hint are known to the sages beforehand. Denote a game with hint by $\langle G, h, g, B, w_B\rangle$.

For example, game $\langle G, h, g, B, w_B\rangle$ is certainly winning in the case $w_B \leqslant g(B)$.

T h e o r e m. Let graph G contain vertex B, and graph $\tilde{G}$ be obtained from graph G by appending new vertex A and edge AB. Let hatness function $\tilde{h}$ and function of the number of guesses $\tilde{g}$ be given on graph $\tilde{G}$, and let $h = \tilde{h}\Big|_{V(G)}$, $g = \tilde{g}\Big|_{V(G)}$. Let for some natural numbers w_A, w_B such that $g(A) \leqslant w_A \leqslant h(A)$ and $g(B) \leqslant w_B \leqslant h(B)$, the conditions hold:

The colors of sage B

The colors of sage A

	0	1	2		w_B				$\dots$					$h(B)$
0	L	L	L	L	L									
1						L	L	L	L	L				
2	L										L	L	L	L
$\dots$		L	L	L	L	L								
							L	L	L	L	L			
w_A	L	L										L	L	L
			L	L	L	L	L							
								L	L	L	L	L		
	L	L	L										L	L
$\dots$				L	L	L	L	L						
									L	L	L	L	L	
	L	L	L	L										L
					L	L	L	L	L					
$h(A)$										L	L	L	L	L

Figure 6. The strategy of sage A. Here $h(A) = 14$, $h(B) = 14$, $w_A = 6$, $w_B = 5$. In the construction of the table it is not required, but to complete the picture one can assume that $g(A) = 4$, $g(B) = 4$.

(i) game with hint $\langle G, h, g, B, w_B\rangle$ is winning,
(ii) $w_B \cdot h(A)$ is divisible by $h(B)$,
(iii) $w_A w_B \geqslant (w_A - g(A))h(B)$.

Then game with hint $\langle \tilde{G}, \tilde{h}, \tilde{g}, A, w_A\rangle$ is winning.

Proof. To describe the strategy of sage A, construct table $h(A) \times h(B)$, in which some squares are empty, and the others contain letters "L" by the following rule. Number the rows of the table by numbers from 0 to $h(A) - 1$, we identify the numbers of rows with possible colors of A's hat. Number the columns of the table by numbers from 0 to $h(B) - 1$, we identify the numbers of columns with possible colors of B's hat. For each i ($0 \leqslant i \leqslant h(A) - 1$) we put letters "$L$" in the cells of i-th row in columns with numbers

$$iw_B, \quad iw_B + 1, \quad \dots, \quad iw_B + w_B - 1 \pmod{h(B)} \tag{1}$$

(i. e. w_B letters "L" in total), see fig. 6. One may consider the obtained table as toroidal: calculations modulo $h(B)$ in rule (1) allow to identify $h(B)$-th column with zeroth column, and condition (ii) allows to identify $h(A)$-th row with zeroth row.

Lemma. Consider arbitrary w_A consecutive rows of this table (taking into account its toroidal nature, i. e. one can take several lower rows and the corresponding number of upper rows). Then each column of the table contains at most $g(A)$ empty cells in these rows.

Proof. In view of toroidal nature of the table it is sufficient to verify this statement for the set of first w_A rows. Consider j-th column. It is evident that this column contains letter "L" in the entry at i-th row ($0 \leqslant i \leqslant w_A - 1$) if and only if

$$0 \leqslant (j - iw_B) \bmod h(B) \leqslant w_B - 1. \tag{2}$$

In the integer sequence $d_i(j) = j - iw_B$ the distance between $d_0(j)$ and $d_{w_A-1}(j)$ is equal to

$$(w_A - 1)w_B.$$

By condition (iii) the inequality holds:

$$(w_A - 1)w_B \geqslant (w_A - g(A))h(B) - w_B,$$

which means that for each j inequality (2) has at least $w_A - g(A)$ solutions for variable i, i. e. each column of the table contains at least $w_A - g(A)$ letters "L" in the chosen w_A rows. Thus, it contains at most $g(A)$ empty squares. $\square$

The hint that sage A receive from the devilkin is actually a set of w_A consecutive rows of the table. Then the strategy of sage A is to name the colors, corresponding to the numbers of the rows with empty cells in j-th column of the table, where j is the color of B's hat. Sage A can do it, because by lemma the rows indicated in the devilkin's hint contain at most $g(A)$ empty cells in j-th column.

Describe the strategy of sage B. He sees color i of the hat of sage A and concludes that A does not guess correctly only in the cases, when B's color corresponds to the columns containing letter "L" in i-th row. Therefore, B may think that his own color is given by the set of these w_B columns, and, receiving this hint, he plays with this hint by the strategy for graph G.

The theorem is proven.

Turn to the problem solution. We present the strategy for the sages.

For $k = 1, 3, \ldots, s-1$ denote by P_k a path on vertices $v_1, \ldots, v_k$ (it is a subgraph of P_s). Function h allows us to define the hatnesses of vertices $v_1, \ldots, v_k$. Check by induction on k ($1 \leqslant k \leqslant s$) that game with hint $\langle P_k, h, \star s, v_k, s+k-1\rangle$ is winning (remind that in this game the devilkin pointed to sage v_k the range of $s+k-1$ consecutive colors containing the color of his hat).

Base case $k = 1$: in game $\langle P_1, h, \star s, v_1, s\rangle$ the only player v_1 wins thanks to hint.

Inductive step $k \to k+1$, $k \leqslant s-2$. Let game with hint $\langle P_k, h, \star s, v_k, s+k-1\rangle$ be winning. Then by the proven theorem game $\langle P_{k+1}, h, \star s, v_{k+1}, s+k\rangle$ is winning too: here $B = v_k$, $w_B = s+k-1$, $G = \langle P_k, h, \star s, v_k, s+k-1\rangle$, $A = v_{k+1}$, $w_A = s+k$, $\tilde{G} = \langle P_{k+1}, h, \star s, v_{k+1}, s+k\rangle$. Condition ii) of theorem holds because $h(A) = h(B)$, and the condition iii) is provided by the inequality

$$w_A w_B = (s+k-1)(s+k) \underset{(*)}{\geqslant} k(4s-2) = (w_A - g(A))h(B),$$

where inequality $(*)$ is reduced to evident inequality $(s-k)^2 \geqslant s-k$.

The last step $k = s-1 \to s$ also holds by the proven theorem. It is verified similarly with the only difference that condition ii) holds due to the fact that number $w_B = 2s-2$ is even, and therefore $w_B \cdot h(A) = (2s-2)(2s-1)$ is divisible by $h(B) = 4s-2$.

Thus, we have proved that game with hint $\langle P_s, h, \star s, v_s, 2s-1\rangle$ is winning. But then game $\langle P_s, h, \star s\rangle$ is evidently winning too.

2.6. a) Solution 1. Induction by the number of vertices of the tree. Inductive step. The adding to the losing tree next pendant vertex can be interpreted as gluing of two losing games by constructor of problem 3.3, where one of the games is the game on tree $\langle T, \star t(s), \star s\rangle$, and another one is $\overset{\frac{s}{s+1}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\frac{s}{s^2+s+1}}{\bullet}$.

Solution 2. We prove the statement by induction on the number of tree vertices. Base case $n = 1$ is trivial. Prove the inductive step.

Let the sages choose some strategy f in game $\langle T, \star t(s), \star s\rangle$. The following two propositions hold.

Proposition I. For each sage A at least $t(s) - s$ colors of his hat can be used for construction of "disproving" hats placements. By the other words, one can choose $t(s) - s$ colors and for each of them construct a hats placement, in which A's hat has the chosen color and none of the sages guesses.

Proposition II. For any sage A and any set C of $s+1$ colors (of his hats) one can define the hat colors on set $N(A)$ (the set of neighbours of A) in such a way that after appending any hat $\alpha \in C$ for sage A to this placement one can supplement the obtained partial hats placement to the hats placement on the whole tree T so that none of the sages on $T \setminus \{A\}$ guesses correctly.

It is clear that proposition I follows from II. Besides that, from II the inductive step immediately follows: take any sage A and any set C of $s+1$ colors, then proposition II provides the hats placement on set $N(A)$, which uniquely determines what s colors are named by sage A in this game. Give to sage A a unnamed color from set C, then A will not guess correctly. By proposition II one can make so that the other sages will not guess correctly.

Proof of proposition II.

Take an arbitrary sage A and an arbitrary set C of $s+1$ colors (of his hats). Conduct an *experiment*: give any hat $\alpha \in C$ to sage A and remove him (in our mind) from tree T. The tree falls into connected components, for which the inductive step holds. It is evident that each component contains one sage from $N(A)$. Let $B \in N(A)$ be one of these sages and T_B be his connectivity component. Since we have already defined the color of A's hat, strategy f determines the strategy of sage B for game on T_B, the other sages from T_B can use strategy f too. By proposition I this strategy can be disproved, when one gives to sage B a hat of some set containing $t(s) - s$ colors.

The above experiment can be conducted in $s+1$ ways. The obtained games on T_B differ by the strategy of sage B and for each of these games we have a "disproving" set consisting of $t(s)-s$ colors of sage B. It remains to note that the intersection of these $s+1$ sets contains at least $t(s) - (s+1)s = 1$ elements, i. e. it is non empty. Assign the color from this intersection to sage B. Similarly, we will treat with the other connectivity components. As a result we have built a hats placement on set $N(A)$, for which proposition II holds.

b) This solution is reported to us by S. Berlov. We prove that for $n = (s^2+s)!$ the sages win. Let A be the central sage. Consider an $(s+1) \times s$ table. Invite $(s^2+s)!$ sages to play in our game and put into correspondence each arrangement of numbers from 0 to s^2+s-1 in this table to a separate sage. The strategy of the sages is the following. Each "peripheral" sage finds in his table the row containing number c_A, and names all the numbers of this row. Further on, sage A for each number i from 1 to $(s^2+s)!$ checks whether somebody of the sages wins if $c_A = i$ (it can be easily checked, because A sees the colors of all hats and knows the sages' tables). Let i_1, i_2, …, i_k be the list of "bad" values of c_A, for which none of "peripheral" sages wins. If $k \leqslant s$, then A just names these values, and the sages have won. Suppose that $k \geqslant s+1$. Since all possible tables are occurred among the tables of sages, there exists sage B, for which the numbers i_1, i_2,…, i_{s+1} are placed in different rows of his table. But then one of the rows contains the number c_B, and if this row contains number i_ℓ, then sage B wins when $c_A = i_\ell$. Therefore color i_ℓ is not bad. A contradiction.

c) Similarly to p. b). By a "scrap-heap" we mean three heaps of stones containing h stones in total (the stones are numbered from 0 to $h-1$, and the heaps are numbered by 0, 1, 2, i. e. by possible hat colors of peripheral sages, empty heaps are allowed). Let n be the number of all possible scrap-heaps. Define the strategy of the sages on graph $K_{1,n}$. Give a unique scrap-heap to each sage B_i. The strategy of B_i is to name the number of heap containing the stone c_A. The strategy of sage A is to enroll those colors of his own hat, for which none of B_i has guessed correctly, and to name all listed colors. This is possible because the list contains at most two colors. Indeed, if the list contains colors c_1, c_2, c_3, then consider any scrap-heap, in which stones c_1, c_2, c_3 lie in the first, second and third heap respectively. Without loss of generality one can assume that the owner of the scrap-heap has received a hat of the first color. But then he certainly guesses correctly his own color, if sage A has received hat of color c_1 that contradicts the definition of c_1.

3.1. It is evident: on graph $\widetilde{G}$ sage A at first has to name $g(A_1)$ colors by the strategy of vertex A_1 in graph G (taking into account the colors of the neighbours of A_1 only), and then $g(A_2)$ colors by the strategy of vertex A_2 (looking only at the neighbours of A_2). The sages, who see on graph G only one of A_i, play as if A is this A_i. As for those sages, who saw in graph G both sages A_1 and A_2 and now see only one sage A, they must play assuming that the hats of A_1 and A_2 have the same color.

3.2. The hatness of sage v is equal to $h_1(v)h_2(v)$. So one can assume that the hat of sage v has "composite color", i. e. its color is an ordered pair (c_1, c_2), where c_i is the color of v's hat in game $\mathcal{G}_i$. Fix winning strategies for games $\mathcal{G}_1$ and $\mathcal{G}_2$ and build strategy for game $\mathcal{G}_1 \times_v \mathcal{G}_2$. Let all the sages from graph $G_i \setminus \{v\}$ play by the winning strategy for game $\mathcal{G}_i$ (the neighbours of v from graph G_i look only at component c_i of the color of sage v). As for sage v, he plays by both strategies at once: looking only at his neighbours in graph G_1, sage v finds $g_1(v)$ first components of his color by the winning strategy for game $\mathcal{G}_1$, and by the winning strategy for game $\mathcal{G}_2$ he finds $g_2(v)$ second components. Taking all possible pairs of the founded colors, he makes $g_1(v)g_2(v) = g(v)$ guesses.

The constructed strategy is winning, because either somebody in $G_1 \setminus \{v\}$ or in $G_2 \setminus \{v\}$ guesses correctly, or v guesses correctly both components of his color.

3.3. Assuming the contrary let f be a winning strategy in game $\mathcal{G}$. Denote by N_1 the set of neighbours of vertex A in graph G_1. For any hats placement φ on the vertices of graph G_1 the answers of all the sages from set $V(G_1) \setminus A$ are determined by strategy f. We will show that there exist $s+1$ hats placements φ_i $(i = 1, \ldots, s+1)$ on graph G_1, such that for $i \neq j$

$$\varphi_i\big|_{N_1} = \varphi_j\big|_{N_1}, \qquad \varphi_i(A) \neq \varphi_j(A),$$

and such that if the sages from G_1 play according strategy f, then for all these placements none of the sages from $V(G_1) \setminus A$ guesses correctly.

For each hats placement α on vertices of N_1 denote by $C(\alpha)$ the set of hat colors of sage A, such that for all placements β on G_1, for which

$$\beta\big|_{N_1} = \alpha, \qquad \beta(A) \in C(\alpha),$$

none of the sages from set $V(G_1) \setminus A$ guesses correctly by strategy f. Suppose that the statement from the previous paragraph does not hold. Then each set $C(\alpha)$ contains at most s colors. Consider then the following strategy for game $\mathcal{G}_1$: let all the sages from G_1, except A, play by strategy f, and sage A name the colors from set $C(\alpha)$ (supplementing them by arbitrary colors, if $C(\alpha)$ contains less than s elements). This strategy is winning, because if nobody in $V(G_1) \setminus A$ has guessed correctly, then a hat from $C(\alpha)$ is on the head of A, and he guesses correctly. Contradiction.

Consider these $s+1$ placements φ_i. Fix a hats placement $\alpha = \varphi_i\big|_{N_1}$ on N_1 and restrict ourselves to only those hats placements on G_2, where sage A receives a hat of one of $s+1$ colors $\varphi_i(A)$, $i = 1, \ldots, s+1$. Then strategy f defines the actions of the sages on graph G_2, i. e. in losing game $\mathcal{G}_2$ subject with the only restriction that in the case $h_1(A) > s+1$ sage A by this strategy can name more than $s+1$ colors, i. e. more than his hatness in game G_2. But in this case mention of "outsider" colors does help to win. Therefore there exists disproving placement ψ on G_2. If $\psi(A) = \varphi_j(A)$, then $\psi \cup \varphi_j\big|_{V(G_1)\setminus A}$ is a disproving hats placement for strategy f in game $\mathcal{G}$.

3.4. It is evident. At first, sage A on graph $\widetilde{G}$ has to name the $g(A)$ colors, which he names by the strategy for graph G, when B's hat is painted in the first color. After that sage A names the $g(A)$ colors, which he names when B's hat is painted in the second color and so on.

3.5. For each natural N denote the set $\{0, 1, \ldots, N-1\}$ by $[N]$ for short.

For each vertex u of substituted graph G_1 define its color in game $\langle G, h, g\rangle$ as a pair from set $[h_1(u)] \times [h_2(A)]$. Let sage u look for the first component of his color by the strategy of game $\mathcal{G}_1$, and the second component by the strategy of A in game $\mathcal{G}_2$. The neighbours of vertex A from graph G_2 in new graph G see the whole subgraph G_1, and therefore can determine, who has guessed correctly the first component. Let B be the first of these sages (in lexicographical order). Then the vertices of graph $G_2 \setminus \{A\}$ can play by the strategy of game $\mathcal{G}_2$, using the second component of color B as a color of A. Since $\mathcal{G}_2$ is a winning game, some of the vertices win. If it is vertex from $G_2 \setminus \{A\}$, then it guesses correctly its color in graph G too. And if the winner of $\mathcal{G}_2$ was vertex A, then B correctly found both components of its color.

3.6. For proof we modify the strategy from the previous solution. In view of this construction vertex v after substitution gets a composite color from set $[h(v)] \times [h'(A)]$, and the strategy of sage v consists in calculating both components of his color, i. e. he chooses $s = g(v)$ colors $c_1, \ldots, c_s \in [h(v)]$, calculates G'-component of his color, i. e. chooses $g'(A)$ colors $e_1, \ldots, e_{g'(A)} \in [h'(A)]$, and after that he names all the pairs of colors (c_i, e_j).

We will change the construction of substitution and describe how the sages play in changed situation. The change affects only the sages $v \in G$, we assign for these sages new hatness and number of guesses: $h^*(v) = h(v)$ and $g^*(v) = g'(A)$. Therefore now v's hat has a color from set $[h(v)]$ (instead of a composite color), that is interpreted by his neighbours from $N_G(v)$ and $N_{G'}(A)$ as G-component of his color as before.

The strategy of each sage $v \in G$ consists of two phases. The first phase: casting glances at the neighbours in G, sage v calculates "G-component" of his color, i. e. a set consisting of s colors $c_1, \ldots, c_s \in [h(v)]$. Further, sage v identifies the obtained set and $[h'(A)]$ (by the rule $c_i \mapsto i$; remind that

$h'(A) = s$). After that the second phase begins: he looks at his neighbours in graph G' and apply the strategy of sage A naming $g'(A)$ colors from his newfound set $[h'(A)]$.

It remains to describe strategy of the sages from $N_{G'}(A)$. They all see the whole graph G, so they know, what set of colors each sage v has identified with set $[h'(A)]$. Besides that, they all know, who from $V(G)$ has guessed correctly G-component of his color. Let w be the first of these sages in lexicographical order. Since sage w has guessed correctly G-component of his color, the color of his hat belongs to the set $[h'(A)]$, that he has constructed in the first phase. Then during the second phase the sages of graph $G' \setminus \{A\}$ simply play the winning strategy of game $\mathcal{G}'$, substituting w with its constructed set $[h'(A)]$ in the place of A, and sage w actually plays by this strategy too, as explained above. As a result, somebody of them will guess correctly.

3.7. Each neighbour of A in $V(G) \setminus \{A\}$ now sees the whole set B, computes a "virtual color of sage A"

$$c_A = \sum_{v \in B} c_v \pmod{h(A)}$$

and plays by the strategy from game $\mathcal{G}$. As for the sages from B, they take for themselves one answer a_i each from the strategy of sage A, and sage v_i names color

$$a_i - \sum_{v \in B, v \neq v_i} c_v \pmod{h(A)}$$

(therefore, i-th sage verifies hypothesis $c_A = a_i$).

4.1. Let B be the vertex of hatness 2. Apply for vertex B the statement of constructor "removing half-edge" (problem 3.4) making this vertex invisible for the other vertices. Then the other sages do not see B, have two attempts, and lose by the statement of problem 2.6 b). Giving them a disproving hats placement, the referee will make so that sage B will not guess correctly too.

4.2. Fix number s. Consider the case, when the upper vertex has hatness $s+1$, and the other vertices have hatness h. Acting as in the previous problem, we make the upper vertex invisible for the others, as a result, the number of guesses of the vertices on path P_{n-1} becomes equal to $s(s+1) = f(s)$. By the statement of problem 2.6 b) the sages lose for $h > f(f(s))$. Thus $t_s \leqslant f(f(s))$.

4.3. We will prove by induction on the number of petals that for $h > f(f(f(s)))$ the game is losing. The base case, one petal G, follows from the previous problem: game $\langle G, \star h, \star s\rangle$ is already losing for $h > f(f(s))$.

But we need one more relative statement – a "modified base of the induction": game $\langle G, \star\bar{h}, \star s\rangle$ is losing for $h > f(f(f(s)))$, where $\star\bar{h}$ denotes the hatness function that is equal to h in all the vertices of petal G, except one vertex B for which $\bar{h}(B) = s+1$.

Proof of the statement. If B is the upper vertex of petal G, then this is again the statement of the base. Let B be an arbitrary vertex of petal on path P_{n-1} (fig. 7, left). Using constructor "removal of half-edge" (problem 3.4), make vertex B invisible for the other vertices. As a result the number of guesses of its neighbours becomes equal to $f(s)$. Now vertex B can be deleted from G, because nobody sees it, it has hatness $s+1$ and s guesses, so it is fated to fail in guessing. It can be easily seen that the remained graph is a union of two petals with common upper vertex (or one petal, but this case is trivial) and its vertices have s or $f(s)$ guesses (fig. 7, center). Add to graph horizontal edge between former neighbours of B (fig. 7, right), this edge can help to the sages to win. As a result, we have obtained a petal, which vertices have hatnesses at least $f(f(f(s)))$ and at most $f(s)$ guesses. By the statement of the base the sages nevertheless lose.

Now prove the inductive step. Consider petunia $M_n = M_{n-1} +_{v_n} L_n$. In view of modified statement of the base game $\langle L_n, \star\bar{h}, \star s\rangle$ is losing for $h > f(f(f(s)))$, where we denote by $\star\bar{h}$ the hatness function that is equal to h in all the vertices of petal G, except v_n, and hatness of v_n is equal to $s+1$. Game $\langle M_{n-1}, \star h, \star s\rangle$ is losing by the induction hypothesis. It remains to note that game $\langle M_n, \star h, \star s\rangle$ can be obtained by constructor of problem 3.3 from losing games $\langle M_{n-1}, \star h, \star s\rangle$ and $\langle L_n, \star\bar{h}, \star s\rangle$ and therefore is losing too.

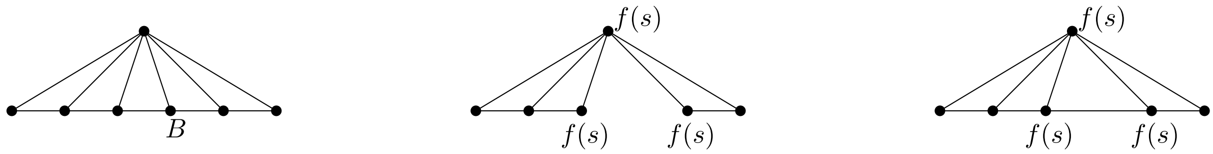

Figure 7. Removing of vertex B from petal G

4.4. a) Estimation $\mathrm{HG}_s(G) < 4s(s+1)-1$. Let A be the stem of the petal, $h(A) = s+1$, and the other vertices v have hatness $h(v) = 4s(s+1)-1$. It is sufficient to verify that game $\langle G, h, \star s\rangle$ is losing. By the statement of problem 5.6, this game is equivalent to the game on path P_n, where all vertices have hatness equal to $4s(s+1)-1$ and $s(s+1)$ guesses. But this game is losing by problem 2.3.

Now we will prove that game $\langle G, \star 4s(s+1)-2, \star s\rangle$ is winning for sufficiently large n.

Let $\mathcal{G}_0 = \langle P_k, \star 4s(s+1)-2, \star s(s+1)\rangle$. For sufficiently large k this game is winning by problem 2.5.

Similarly to the problem 2.6 c), one can prove that for any natural h there exists such natural n, that game on graph $K_{1,n}$ is winning, if the hatnesses of all peripheral sages are equal to $s+1$ and they all have one attempt for guessing, and the hatness of the central sage is equal to h and he has s attempts. Set $h = 4s(s+1)-2$ and choose suitable n. Substitute with reducing game $\mathcal{G}_0$ in the place of each peripheral sage[1]. We obtain a winning game, where the hatnesses of all vertices are equal to $4s(s+1)-2$, the numbers of attempts are equal to s, and the graph is a subraph of a large petal.

b) As we have checked in p. a), game $\langle G, h, \star s\rangle$ is losing, where G is a petal with large number of vertices, and h is the function defining hatness of the stem by $s+1$, and the hatnesses of the other vertices by $4s(s+1)-1$. By the statement of problem 3.3 gluing of stem of such petal to a vertex of another losing game with s guesses gives again a losing game. But royal petunia by definition is constructed by consecutive stem's glueings of petals! Therefore a game on a royal petunia with large petals, where all the vertices have hatness $4s(s+1)-1$ and s guesses, except the first (rooted) stem with hatness $s+1$ and s guesses, is losing. It gives the estimation $\mathrm{HG}_s(G) < 4s(s+1)-1$.

Then $\mathrm{HG}_s(G) = 4s(s+1)-2$, since this hatness is realized already on petals that are royal petunias too though not very branchy.

5.1. *"If" case.* Fix one of h_0 colors of the central sage's hat. Then the strategies of the other vertices are determined, and in $\prod\limits_{i=1}^{n}(h_i - g_i)$ cases none of pendant vertices guesses correctly. Therefore the central sage must guess. But the central sage can do this only in $g_0\prod\limits_{i=1}^{n} h_i$ cases in total. We obtain an inequality that is equivalent to the inequality from the condition.

"Only if" case. We will show that for $N = \prod\limits_{i=1}^{n} h_i$ game $\langle K_{1,n}, (N\cdot h_0; h_1, \ldots, h_n), (N\cdot g_0; g_1, \ldots, g_n)\rangle$ is winning.

Encode $h_0 \cdot h_1 \cdot \ldots \cdot h_n$ colors of the central sage by sets $(c_0; c_1, \ldots, c_n)$, where $0 \leqslant c_i < h_i$. Let i-th sage, when he sees color $(c_0; c_1, \ldots, c_n)$, name colors $c_i, c_i+1, \ldots c_i+g_i-1 \pmod{h_i}$. And let the central sage look at the others and name all variants, in which none of them guesses correctly. How many are there such variants? There are h_0 variants for zeroth component and $h_i - g_i$ variants for each of the others. But the inequality from the condition is equivalent to the inequality $h_0\prod\limits_{i=1}^{n}(h_i - g_i) \leqslant N\cdot g_0$,

[1]We need here the following more general version of the constructor than that in problem 3.6.

Let $\mathcal{G} = \langle G, h, \star s g_0\rangle$, $\mathcal{G}' = \langle G', h', g'\rangle$ be winning games. Let A be a vertex of graph G', and $h'(A) = s$. Let $\langle \widetilde{G}, \widetilde{h}, \widetilde{g}\rangle$ be the winning game obtained by the substitution of game $\mathcal{G}$ on the place of vertex A to game $\mathcal{G}'$ (as in problem 3.5). By the rule of the substitution for all substituting vertices v

$$\widetilde{h}(v) = h(v)h'(A) = s\cdot h(v), \qquad \widetilde{g}(v) = g(v)g'(A) = sg_0\cdot g'(A).$$

Consider new functions h^*, g^* on graph $\widetilde{G}$, which differ from $\widetilde{h}$, $\widetilde{g}$ only by the values in substituting vertices v, and this difference is the cancellation by s:

$$h^*(v) = h(v), \qquad g^*(v) = g_0\cdot g'(A).$$

Then game $\langle \widetilde{G}, h^*, g^*\rangle$ is also winning.

thus the central sage has enough attempts.

5.2. Denote the sages by A, B and C: $\underset{A}{\overset{\frac{g}{h}}{\bullet}}\!\!-\!\!\!-\!\!\underset{B}{\overset{\frac{g}{h}}{\bullet}}\!\!-\!\!\!-\!\!\underset{C}{\overset{\frac{g}{h}}{\bullet}}$. We present a winning strategy for the sages. Let sage A name colors $c_B, c_B+1, \ldots, c_B+(g-1) \bmod h$, and sage C $c_B, c_B+[\frac{h}{g}], \ldots, c_B+[(g-1)\frac{h}{g}] \bmod h$, where $[x]$ denotes the rounding to the nearest integer. Therefore, if B's hat is from the set

$$I_A = \big(c_A, c_A-1, \ldots, c_A-(g-1)\big) \bmod h,$$

then sage A will guess correctly, and if B's hat is from the set

$$I_C = \left(c_C, c_C - \left[\frac{h}{g}\right], \ldots, c_C - \left[(g-1)\cdot\frac{h}{g}\right]\right) \bmod h,$$

then sage C will guess correctly. It remains to prove that for sage B there are at most g colors not covered by set $I_A \cup I_C$ or, equivalently, that

$$h - |I_A| - |I_C| + |I_A \cap I_C| \leqslant g.$$

Since $|I_A| = |I_C| = g$, it is equivalent to the inequality $|I_A \cap I_C| \leqslant 3g - h$.

Suppose that this statement is wrong and $|I_A \cap I_C| > 3g - h$. Then there exists k such that both numbers $c_C - [k \cdot \frac{h}{g}]$ and $c_C - [(k+3g-h)\cdot\frac{h}{g}]$ belong to $I_A \cap I_C$ (the elements of $I_A \cap I_C$ can be written in two forms: $c_A - i = c_C - \left[\ell \cdot \frac{h}{g}\right]$; the number ℓ that corresponds to the minimal possible i, can be taken as k). Since both numbers belong to set I_A, consisting of consecutive remainders, the distance between them does not exceed $g-1$:

$$\left(c_C - \left[k\cdot\frac{h}{g}\right]\right) - \left(c_C - \left[(k+3g-h)\cdot\frac{h}{g}\right]\right) \leqslant g-1,$$

that is equivalent

$$\left[(k+3g-h)\cdot\frac{h}{g}\right] - \left[k\cdot\frac{h}{g}\right] \leqslant g-1.$$

Getting rid of rounding, we obtain the corollary:

$$(k+3g-h)\cdot\frac{h}{g} - 0.5 - k\cdot\frac{h}{g} - 0.5 \leqslant g-1.$$

The last is equivalent to the inequality $(3g-h)\cdot\frac{h}{g} \leqslant g$, i.e. $0 \leqslant g^2 - 3gh + h^2$, that contradicts the condition.

5.3. By the statement of problem 5.1 the existence of k, for which game $\langle P_3, \star kh, \star kg\rangle = \langle K_{1,2}, \star kh, \star kg\rangle$ is winning, is equivalent to the condition

$$\left(1-\frac{g}{h}\right)\left(1-\frac{g}{h}\right) \leqslant \frac{g}{h}.$$

For non-negative h it is equivalent to the inequality $g^2 - 3gh + h^2 \leqslant 0$, that for $1 \leqslant g \leqslant h$ is equivalent to the inequality

$$\frac{h}{g} \leqslant \frac{3+\sqrt{5}}{2}.$$

Now the problem statement is evident.

5.4. Consider an arbitrary strategy of the sages in game $\langle G, h\rangle$. The product $\prod\limits_{v\in H_2} h_2(v)$ enumerates hats placements on H_2. When we choose each of these hats placements, we fix strategy of the sages on G_1. For the fixed hats placement on H_2 consider hats placements on G_1. The sum $\sum\limits_{u\in G_1}\frac{1}{h_1(u)}$ estimates from above the fraction of those placements on G_1, where at least one sage from G_1 guesses correctly. Then the product $\left(\sum\limits_{u\in G_1}\frac{1}{h_1(u)}\right)\prod\limits_{v\in H_2} h_2(v)$ estimates from above the maximum fraction of those placements on G_1, where at least one sage from G_1 guesses correctly provided that each sage makes $\prod\limits_{v\in H_2} h_2(v)$ guesses. Therefore the inequality from the problem condition means that there exists hats placement α on G_1, for which none of sages from G_1 guesses correctly, whatever hats were given to the sages from H_2. Therefore after assigning hats placement α to sages from G_1, none of them guesses correctly and the strategies of the other sages on graph G_2 are completely determined and now are suitable for game $\langle G_2, h_2\rangle$. Since this game is losing, hats placement α can be enlarged to hats placement on G_2, for which nobody from G_2 guesses correctly too.

5.5. Let the sages have fixed a strategy on graph G'. We will construct a disproving hats placement for this strategy. The strategy of sage A for each of $2h(B)-1$ possible colors of hat of B prescribes to name one of two colors. Some of these two colors is named at most $h(B)-1$ times. Give to sage A the hat of this color, this will fix the strategy of sage B on the remained graph G. Now, to prevent correct guessing of A, we give to B a hat of one of at least $h(B)$ remained colors. Since game $\mathcal{G}$ is losing, we can construct hats placement on graph G so that nobody on G will guess correctly.

5.6. If $\mathcal{G}$ is a winning game, we remove by problem 3.4 all half-edges $\overrightarrow{vA}$ and obtain the winning game. In this game A has s guesses, $s+1$ colors and no information, therefore we can assign A's color such that A does not guess. But now the remaining sages play the game $\mathcal{G}'$. Hence $\mathcal{G}'$ is winning.

If the game $\mathcal{G}'$ is winning, substitute $\mathcal{G}'$ in the winning game $\overset{\frac{s}{s+1}}{\underset{A}{\bullet}}\!\!-\!\!\!-\!\!\overset{\frac{1}{s+1}}{\underset{B}{\bullet}}$ in place of vertex B by problem 3.6. We obtain a winning game $\mathcal{G}$.

5.7. a) Let $\mathcal{G}$ be a losing game on path ABC where $h(A)=h(C)=2$, $h(B)=5$. If both games were losing, then the initial game would be obtained from these games and game $\mathcal{G}$ by constructor of problem 3.3 (where $s=1$, $g_1=g_2=\star 1$) and were losing too.

b) Set $V(\widetilde{G})$ consists of the vertices of graph G and the set of new vertices V_1 (that are situated in the middles of two-link paths). Define a function on $V(\widetilde{G})$:

$$h(v)=\begin{cases}2, & v\in V(G),\\ 5, & v\in V_1.\end{cases}$$

It is sufficient to verify that game $\langle\widetilde{G},h\rangle$ is losing. It is evident. Indeed, each sage from V_1 has two neighbours in $\widetilde{G}$ with hatness 2, so he names by his strategy at most four colors. Then we give him a hat of the color, that he does not name, and he will not guess correctly. Now all the answers of the sages from $V(G)$ are determined, and we give to each sage a hat of the color that he does not name, too.

5.8. Let after deletion of the bridge graph G fall into components G_1 (containing vertex B) and G_2 (containing vertex A). Define hatness functions h_1 and h_2 on these graphs by the rule

$$h_1(x)=\begin{cases}h(x), & x\in V(G_1)\setminus\{B\},\\ \lceil\frac{h(B)}{2}\rceil, & x=B.\end{cases}\qquad h_2(x)=\begin{cases}h(x), & x\in V(G_2)\setminus\{A\},\\ \lceil\frac{h(A)}{2}\rceil, & x=A.\end{cases}$$

Let $\mathcal{G}_1'=\langle G_1, h\big|_{G_1}\rangle$, it is a losing game due to properties of function h.

If games $\mathcal{G}_1=\langle G_1,h_1\rangle$ and $\mathcal{G}_2=\langle G_2,h_2\rangle$ are both winning, then game $\mathcal{G}_1\times_B \overset{2}{\underset{B}{\bullet}}\!\!-\!\!\!-\!\!\overset{2}{\underset{A}{\bullet}}\times_A\mathcal{G}_2$ is also winning. If the values $h(A)$ and $h(B)$ are even, we obtain the desired decomposition in the product of games. But if at least one of the numbers $h(A)$, $h(B)$ is odd, then the hatness function of the obtained game majorizes h and by properties of function h such game cannot be winning, a contradiction.

It remains to consider the case, when at least one of games game $\mathcal{G}_1=\langle G_1,h_1\rangle$ or $\mathcal{G}_2=\langle G_2,h_2\rangle$ is losing, let it be game $\mathcal{G}_2$. Let us apply constructor of problem 5.5 to loosing game $\mathcal{G}_2$: take a new vertex B of hatness 2 connected with A, let the hatness of vertex A become equal to

$$2\left\lceil\frac{h(A)}{2}\right\rceil - 1 \leqslant h(A)$$

and the hatnesses of other vertices be defined by function h. Denote the obtained game by $\mathcal{G}_2'$. Apply now constructor of problem 3.3 to games $\mathcal{G}_1'$ and $\mathcal{G}_2'$ (we assume that $s = 1$, $g_1 = g_2 = \star 1$). We will obtain a losing game on graph G, in which hatness function does not exceed h. It is impossible.

5.9. Denote the initial game by $\mathcal{G}_1 = (G_1, h_1, g_1)$, where $g_1 \equiv 1$. Denote by G_2 the subgraph of graph G_1 obtained from G_1 by removing vertex A, denote by G_3 the subgraph of G_2 obtained from G_2 by removing vertex B, and denote by G_4 the graph obtained from G_3 by adding vertex C, which is connected to all other vertices, i. e. in fact G_4 is obtained from G_2 by renaming vertex B to C. Consider games $\mathcal{G}_2 = (G_2, h_2, g_2)$, $\mathcal{G}_3 = (G_3, h_3, g_3)$, $\mathcal{G}_4 = (G_4, h_4, g_4)$, where

$$h_2(v) = \begin{cases} h_1(v) & v \in G_3, \\ 3 & v = B \end{cases}, \qquad h_3(v) = h_1\big|_{G_3}, \qquad h_4(v) = \begin{cases} h_1(v) & v \in G_3, \\ 6 & v = C \end{cases},$$

$$g_2 \equiv 2 \qquad g_3 \equiv 6 \qquad g_4(v) = \begin{cases} 1 & v \in G_3, \\ 5 & v = C \end{cases}.$$

Suppose that game $\mathcal{G}_1$ is winning. When we remove by problem 3.4 all the half-edges leading from vertices A and B, we obtain a winning game, in which the number of guesses of all other sages (i. e. the sages from G_3) become equal to 6, and the strategies do not depend of colors of A's and B's hats. Assume that there exists a hats placement on G_3, for which nobody from G_3 guesses correctly. This hats placement determines the strategies of sages A and B, playing on edge AB in game $\overset{\frac12}{\underset{A}{\bullet}}\!\!-\!\!\!-\!\!\overset{\frac13}{\underset{B}{\bullet}}$, and as a result all the sages lose. That is impossible. Therefore the restriction of the game to graph G_3, i.e. game $\mathcal{G}_3$, is winning. Making by problem 3.6 substitution of this game with reducing in game $\overset{\frac56}{\underset{C}{\bullet}}\!\!-\!\!\!-\!\!\overset{\frac16}{\underset{v}{\bullet}}$ in the place of vertex v, we obtain winning game $\mathcal{G}_4$.

Conversely, let game $\mathcal{G}_4$ be winning. We wish to replace player C by two players A (with hatness 2) and B (with hatness 3). Let us demonstrate the winning strategy of the players in the obtained game. All players from G_3 will use the strategy of game $\mathcal{G}_4$, interpreting pair (color A, color B) as composite color of player C. Show how one can "convert" the strategy of player C to a pair of strategies of A and B.

The color of player C is an element of the set $\mathcal{C} = \{0,1\} \times \{0,1,2\}$. Let for the current hats placement of his neighbours C must name all the colors from set $\mathcal{C}$, except $(1,2)$. Then the actions of players A and B consist in that A names the color with the same parity as B's color and B names color 0 or 1 of opposite parity to A's color. Player C has guessed correctly, if one of colors $(0,0)$, $(0,2)$, $(1,1)$ or $(0,1)$, $(1,0)$ was on his head. By our rule in the first three cases A guesses correctly, in the other two B guesses correctly.

The roles of A and B are assigned similarly for the other sets of five guesses of C.

5.10. Show that game $\langle G, \star\mathrm{HG}_{s'}(G[A]) + 1, \star s\rangle$, where $s' = s(\mathrm{HG}_s(G[B]) + 1)^d$, is losing. For this we construct hats placement such that the sages will lose. Since $\mathrm{HG}_{s'}(G[A]) \geqslant s' \geqslant \mathrm{HG}_s(G[B]) + 1$, i. e. the hatness in the game under consideration is larger than $\mathrm{HG}_s(G[B]) + 1$, it is sufficient to consider the case, when the hatness of the sages from B is equal to $\mathrm{HG}_s(G[B]) + 1$. Applying problem 3.4, remove all half-edges from A in B, making set B invisible for set A. Since during this action we erase at most d half-edges for each sage from A, the number of sages' guesses will increase, but will not exceed s'. And the strategy of the sages from A now does not depend on the hats placement on B. Therefore we can assume that they play a game $\langle G[A], \star\mathrm{HG}_{s'}(G[A]) + 1, \star s'\rangle$ on graph $G[A]$. By the definition of s-hat number this game is losing. Thus, there exists a hats placement on A, such that nobody from A guesses correctly. Give to the sages from A this placement, then the strategy of the sages from B of game on $G[B]$ is determined. Since the hatnesses of the sages from B are greater than $\mathrm{HG}_s(G[B])$, we can assign colors on B so that the sages will lose.