Paint my hat in 3.5 colors!

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1 Let us introduce: the HATS game!

Let an undirected graph G be given, one sage and one chest with hats of different colors are located in each of its vertices. All the sages are acquainted with each other. Graph G, the location of the sages in the vertices of the graph and the colors of hats in all chests are fixed and known to everybody. In particular, each sage understands, in which vertex each of the others sages is located. The referee performs the following test with the sages. He puts a hat on the head of each sage, the hat is taken from the sage's chest. Each sage sees only the hats of the sages located in the neighbouring vertices of the graph, he does not see his own hat and does not know its color. The sages cannot communicate during the test. At the command of the referee each of the sages writes names of several colors on his paper simultaneously (how many colors the sages has to mention, is determined by the additional rule). We say that the sages have passed the test successfully = «have won», if at least one of them wrote the color of his hat in his paper.

The sages have been informed of the rules of the test before the testing and they have the possibility to hold a meeting, in which they must to define their public strategy. The publicity means that all the participants, including the referee, know this strategy. The strategy of the sages has to be deterministic, that is each sage has to write colors on his paper looking only the colors that he sees on his neighbours. We call the strategy *winning* if for any hats placement at least one sage will guess correctly the color of the hat on his head, i. e. mention this color in the his list of guesses. We say that the sages win, if they have a winning strategy, and that they lose, if they have not.

Therefore, the HATS game is not a game in a sense as it is ordinarily understood. This game lasts only one move.

1.1. The referee puts a hat of white, blue, red or green color on the head of each of two sages. Each of them sees the hat of the other, but does not see his own hat. Each of them writes on his own paper two colors simultaneously. They try to guess correctly the colors of their own hats. Prove that the sages can come to an agreement in the meeting before the test in such a way that at least one of them will guess correctly.

1.2. The referee puts a hat of five possible colors on the head of each of two sages. Each of them sees the hat of the other, but does not see his own hat. Each of them try to guess correctly the color of his own hat. The first sage writes on his own paper two colors and the second — three colors simultaneously. Prove that the sages can come to an agreement in the meeting before test in such a way that at least one of them will guess correctly.

1.3. Five sages stand around the non-transparent baobab. Shah has put red, blue, yellow or green hat the head of each of the sages. Sage does not know the color of his own hat and sees only the two neighbouring sages. As usual, without any communication each sages must makes one assumptions about the colors of his hat. But they fear be too lucky. How they should act to guarantee that for any placement of hats no more than two sages guess correctly the colors of their hats?

1.4. Sultan examines six court sages. By the rule of the examination the sultan locates 5 sages in 5 pits positioned around a circle, and locates the sixth sage in the tower in the center of the circle. The sultan writes one of the numbers 1, 2 or 3 on the forehead of each of the first five sages and writes a number from 1 to 243 on the forehead of the central sage. The sage in the tower sees the numbers of all the other sages, and these sages see his number (but do not see each other). All the sages must simultaneously try to guess correctly their numbers: the sages in the pits must say two numbers and the sage in the tower — one number. The sultan has explained to the sages the rules of the examination beforehand and has given time to communicate before the beginning of the examination. Can the sages act so that at least one of them certainly guess correctly his number?

We identify a vertex of graph G and the sage located in it. We assume that the colors are numbered by 0, 1, 2, 3, … and that the chest of sage v contains hats of colors from 0 to some number $h(v)-1$.

The HATS game is the triple $\langle G, h, g\rangle$, where $G = \langle V, E\rangle$ — a graph, $h\colon V \to \mathbb{N}$ — a function that for each vertex v equals the number of colors of hats keeping in the chest in this vertex, $g\colon V \to \mathbb{N}$ — a function equal to the number of guesses of each sage. We call function h a «hatness», and g — a function of guesses or the number of attempts. For each non negative integer h we denote by $\star h$ the function on V possessing the constant value h. Instead of the notation $\langle G, h, \star 1\rangle$ we will use shorter notation $\langle G, h\rangle$.

1.5. Prove that if the game $\langle G, h, g\rangle$ is winning, then for each non negative integer k the game $\langle G, k\cdot h, k\cdot g\rangle$ is winning, too.

1.6. Game $\langle G, h, g\rangle$ is given. Let $K \subset G$ is anticlique (a set of vertices such that there is no edge connected any pair of them) and for each $v \in K$ $h(v) > g(v)$. Prove that there exists a hats placement, for which none of the sages in K guesses correctly.

1.7. Let h and g be natural numbers, $G = \langle G, \star h, \star g\rangle$ be a winning game, $r' \leqslant \frac{h}{g}$ be a rational number. Prove that there exist natural numbers h' and g' such that $\frac{h'}{g'} = r'$ and game $\langle G, \star h', \star g'\rangle$ is winning.

1.8. Formulate and prove the analogue of the previous statement for non-constant functions of hatness and guessing.

1.9. Denote by K_n a complete graph on n vertices. Prove that the game $\langle K_n, h, g\rangle$ is winning if and only if

$$\sum_{v\in K_n} \frac{g(v)}{h(v)} \geqslant 1.$$

2 Paths and trees

The theory of HATS game on the complete graph K_3 is given by the problem statement 1.9. Now consider a path P_3 which is less complicated graph.

2.1. Prove that the sages lose in the game $\langle P_3, \star 3, \star 1\rangle$.

2.2. a) Prove that the game $\overset{\frac{3}{10}}{\bullet}\!-\!\!-\!\overset{\frac{3}{10}}{\bullet}\!-\!\!-\!\overset{\frac{3}{5}}{\bullet}$ is winning (the numerator is the number of guesses, and the denominator is the hatness).

b) Prove that the game $\overset{\frac{3}{11}}{\bullet}\!-\!\!-\!\overset{\frac{3}{10}}{\bullet}\!-\!\!-\!\overset{\frac{3}{5}}{\bullet}$ is losing.

c) Prove that the game $\overset{\frac{s}{t(s)}}{\bullet}\!-\!\!-\!\overset{\frac{s}{t(s)}}{\bullet}\!-\!\!-\!\overset{\frac{s}{t(s)}}{\bullet}$ is losing, where $t(s) = s^2 + s + 1$.

Let G be a graph and s be a non negative integer. Denote by $\mathrm{HG}_s(G)$ the s-hat number of G, i.e. the maximum number of hats h for which the game $\langle G, \star h, \star s\rangle$ is winning. For $s = 1$ this number is called hat number of G and is denoted by $\mathrm{HG}(G)$.

2.3. Prove that for any non negative integers n and s the game $\overset{\frac{s}{2s}}{\underset{A}{\bullet}}\!-\!\!-\!\bullet\!-\!\!-\!\bullet\cdots$ on path P_n is losing. Here all vertices except the leftmost vertex A have hatness $4s-1$ and s guesses.

2.4. Prove that one can find n such that $\mathrm{HG}_2(P_n) = 6$, $\mathrm{HG}_3(P_n) = 10$, $\mathrm{HG}_4(P_n) = 14$.

2.5. Prove that for any non negative integer s the game $\langle P_n, \star(4s-2), \star s\rangle$ is winning for $n \geqslant 2s$.

2.6. a) Let $t(s) = s^2 + s + 1$. Prove that for each tree T the game $\langle T, \star t(s), \star s\rangle$ is losing.

b) Len $K_{1,n}$ be "a star" graph (i.e. a tree consisting of a root and n leaves). Prove that for sufficiently large n the game $\langle K_{1,n}, \star(s^2+s), \star s\rangle$ is winning.

c) Prove that for any non negative integer h there exists integer n such that the game on graph $K_{1,n}$ is winning if all the sages in pendant vertices have one guess and hatness 3, and the central sage has 2 guesses and hatness h.

3 Constructors

3.1. Let $\langle G, h, g\rangle$ be a winning game, A_1 and A_2 be vertices of G, not connected with an edge and such that $h(A_1) = h(A_2)$. We glue vertices A_1 and A_2 of G into one new vertex A, denote by $\widetilde{G}$ the obtained graph. Let functions $\widetilde{h}$ and $\widetilde{g}$ defined on the set of vertices of $\widetilde{G}$ coincide with h and g in all vertices except A_1 and A_2, and $\widetilde{h}(A) = h(A_1)$, $\widetilde{g}(A) = g(A_1) + g(A_2)$. Then the game $\langle \widetilde{G}, \widetilde{h}, \widetilde{g}\rangle$ is also winning. (And then if the game $\langle \widetilde{G}, h, \widetilde{g}\rangle$ is losing, then the game $\langle G, h, g\rangle$ is losing too.)

Let $\mathcal{G}_1 = \langle G_1, h_1, g_1\rangle$, $\mathcal{G}_2 = \langle G_2, h_2, g_2\rangle$ be two games such that $V_1 \cap V_2 = \{v\}$. Let $G = G_1 \underset{v}{+} G_2$ be the union of graphs G_1 and G_2, in which both vertices v are glued into one new vertex. Define functions $h, g\colon V_1 \cup V_2 \to \mathbb{N}$:

$$h(u) = \begin{cases} h_i(u), & u \in V_i \setminus \{v\},\ (i = 1, 2), \\ h_1(v)h_2(v), & u = v, \end{cases} \qquad g(u) = \begin{cases} g_i(u), & u \in V_i \setminus \{v\},\ (i = 1, 2), \\ g_1(v)g_2(v), & u = v. \end{cases}$$

We say that the game $\mathcal{G} = \langle G, h, g\rangle$ is a product of games $\mathcal{G}_1$ and $\mathcal{G}_2$ and denote it by $\mathcal{G}_1 \times_v \mathcal{G}_2$ (fig. 1).

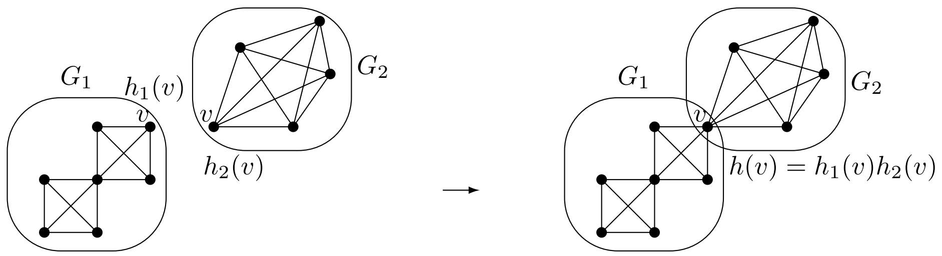

Рис. 1. The product $G_1 \times_v G_2$

3.2. The theorem about the product. If the sages win in games $\mathcal{G}_1$ and $\mathcal{G}_2$, then they also win in game $\mathcal{G}_1 \times_v \mathcal{G}_2$.

3.3. Let $G = G_1 +_A G_2$, where G_1 and G_2 are graphs, for which $V(G_1) \cap V(G_2) = \{A\}$. Let games $\mathcal{G}_1 = \langle G_1, h_1, g_1\rangle$ and $\mathcal{G}_2 = \langle G_2, h_2, g_2\rangle$ be losing, and the following conditions hold:

$$g_1(A) = g_2(A) = s, \qquad h_1(A) \geqslant h_2(A) = s + 1.$$

Then game $\mathcal{G} = \langle G_1 +_A G_2, h\rangle$ is losing, where

$$h(x) = \begin{cases} h_i(x), & x \in V_i \setminus \{A\}\ (i = 1, 2), \\ h_1(A), & x = A, \end{cases} \qquad g(x) = \begin{cases} g_i(x), & x \in V_i \setminus \{A\}\ (i = 1, 2), \\ s, & x = A. \end{cases}$$

3.4. A half-edge removal. Let $\langle G, h, g\rangle$ be a winning game, AB be an edge of graph G, $\widetilde{G}$ be the graph obtained from G by replacing edge AB by directed edge $B \to A$ (i.e. sage A does not see sage B, but B sees A). Let function $\widetilde{g}$ on the vertices of graph G coincide with g in all vertices except A, and $\widetilde{g}(A) = h(B)g(A)$. Then game $\langle \widetilde{G}, h, \widetilde{g}\rangle$ is winning too. (And therefore, if game $\langle \widetilde{G}, h, \widetilde{g}\rangle$ is losing, then $\langle G, h, g\rangle$ is also losing.)

3.5. Substitution theorem. Let $\mathcal{G}_1 = \langle G_1, h_1, g_1\rangle$ be $\mathcal{G}_2 = \langle G_2, h_2, g_2\rangle$ be winning games. Let A be an arbitrary vertex of graph G_2. Consider the new graph G obtained from G_2 by substitution of graph G_1 on the place of vertex v (each vertex G_1 is connected with former neighbours of vertex A by new edges, see fig. 2). Then game $\langle G, h, g\rangle$ is winning, where

$$h(u) = \begin{cases} h_2(u), & u \in V(G_2) \setminus \{A\}, \\ h_1(u)h_2(A), & u \in V(G_1), \end{cases} \qquad g(u) = \begin{cases} g_2(u), & u \in V(G_2) \setminus \{A\}, \\ g_1(u)g_2(A), & u \in V(G_1). \end{cases}$$

3.6. Substitution with reducing. Let $\mathcal{G} = \langle G, h, \star s\rangle$, $\mathcal{G}' = \langle G', h', g'\rangle$ be winning games. Let A be a vertex of graph G', and $h'(A) = s$. Let $\langle \widetilde{G}, \widetilde{h}, \widetilde{g}\rangle$ be the winning game obtained by the substitution of

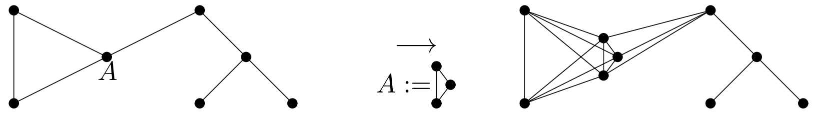

Рис. 2. Substitution of a graph on the place of vertex A.

game $\mathcal{G}$ on the place of vertex A to game $\mathcal{G}'$ (as in problem 3.5). By the rule of the substitution for all substituting vertices v

$$\widetilde{h}(v) = h(v)h'(A) = s \cdot h(v), \qquad \widetilde{g}(v) = g(v)g'(A) = s \cdot g'(A).$$

Consider new functions h^*, g^* on graph $\widetilde{G}$, which differ from $\widetilde{h}$, $\widetilde{g}$ only by the values in substituting vertices v, and this difference is the cancellation by s:

$$h^*(v) = h(v), \qquad g^*(v) = g'(A).$$

Then game $\langle \widetilde{G}, h^*, g^* \rangle$ is also winning.

3.7. Blowing up of a vertex. Let $\mathcal{G} = \langle G, h, g \rangle$ be winning game, $A \in V(G)$, $\widetilde{G}$ be the graph obtained from G by the substitution of clique B consisting of $g(A)$ vertices on the place of vertex A. Then game $\langle \widetilde{G}, \widetilde{h}, \widetilde{g} \rangle$ is also winning, where

$$\widetilde{h}(v) = \begin{cases} h(v), & v \in V(G) \setminus \{A\}, \\ h(A), & v \in B, \end{cases} \qquad \widetilde{g}(v) = \begin{cases} g(v), & v \in V(G) \setminus \{A\}, \\ 1, & v \in B. \end{cases}$$

4 “Petals” and “petunias”

We define a *petal graph* to be a graph G obtained from a path by adding a vertex v adjacent to every vertex of this path, see fig. 3, we say that v is the *stem* of G.

Then, we define a *petunia* to be a graph constructed in the following way. Take two petals L_1 and L_2, denote one vertex in each of them by v_1, and construct a graph $M_2 = L_1 \underset{v_1}{+} L_2$. After that consider graph M_2 and a new petal L_3 denote one vertex in each of them by v_2, and construct a graph $M_3 = M_2 \underset{v_2}{+} L_3$ and so on.

A *royal petunia* is a petunia (рис. 4), for which the vertex v_i in each step of its construction were chosen as the stem of petal L_{i+1}.

4.1. Let G be a petal of n vertices, see fig. 3, let the stem has hatness 2 and the other vertices have hatness 7. Prove that the sages lose in the game $\langle G, h \rangle$.

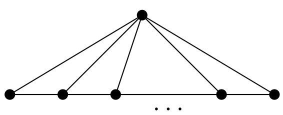

Рис. 3. A petal of n vertices

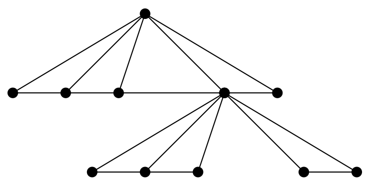

Рис. 4. A royal petunia

4.2. Let G be a petal of n vertices, $f(s) = s^2 + s$. Prove that $\mathrm{HG}_s(G) \leqslant f(f(s))$.

4.3. Let M be a petunia, h_s be maximum integer such that the game $\langle M, \star h_s, \star s \rangle$ is winning. Prove that $h_s \leqslant f(f(f(s)))$.

4.4. a) Prove that $\mathrm{HG}_s(G) = 4s(s+1) - 2$, where G is a petal of $n \geqslant 2s+1$ vertices (fig. 3).

b) Prove the same equality if G is a royal petunia.