DEFINABILITY THEORY

Further problems

B4. All relations definable via sum and product are called arithmetical. In your opinion, do there exist non-arithmetical relations of positive integers? What is a possible way to construct such relations?

C5. Is it true that if a relation R is definable via a relation Q then the set Γ_Q of transformations preserving Q is a subset of the set Γ_R of transformations preserving R?

B5. Consider the set of points of the plane. Is it possible to define the relation $C(x, y, z)$: ``points x, y, z are collinear'' via the relation $D(x, y, z, v) \leftrightharpoons d(x, y) = d(z, v)$?

C6. The relation $R(x, y, z) \leftrightharpoons z = x+y$ is defined on the set of integers $\mathbb{Z}$. Describe the transformation group of this relation. Are the following relations definable via R: the unary relations $x = 0; x = 1$; the binary relation $x < y$?

D2. Present the formula which means that the relation $<$ has no least and greatest elements, and the formula which means that among any two distinct elements there exists an element distinct from these. Are these statements fulfilled for rationals, integers, reals?

Try to prove that there exists an order-preserving bijection between rationals and any countable set with the above properties.

A new structure: Addition of rationals

B6. Given the set of rationals $\mathbb{Q}$ with the relation of sum

$$S(x, y, z) \leftrightharpoons (z = x + y).$$

In the following problems, we consider this set and this relation.

(a) Describe the transformation group of the above relation.

(b) Is it possible to define via S the binary relation

$$M(x, y) \leftrightharpoons (y = 3 * x)?$$

Is the converse true? Describe the transformation group for M.

(c) Try to determine the maximal system of pairwise non-equivalent families of relations.

Cycle E: Problems for research

The cycle E contains the main research problems of the project, they are presented in the table below. Most of them are open (up to now, unsolved) problems. If some of these won't be solved in a few days then we will proceed to collaborate on these problems and will publicate the results.

Relations/sets	$\mathbb{Q}$	$\mathbb{Z}$	$\mathbb{N}$
$(x < y)$	E1	E2	E3
$(y = x + 1)$	E4	E5	E6
$(z = x + y)$	E7	E8	E9

Besides the number structures from the table, we suggest to investigate one more.

E10. ``Branching integers:'' an infinite non-oriented graph without cycles (an infinite tree) such that every vertex is of degree 3; the relation ``to be neighboring vertices''.

Perhaps some of you would be interested in the following structure:

E11. The order on non-negative rationals.

Investigating the issue of definability for every structure (a set with a family of basic relations on it) we distinguish the following stages of research (which in fact may overlap):

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I. The search for relations (families of relations but often consisting of a single element) which are definable via a given basic relation. Proposal of the conjecture that the family of relations is the maximal one, and that these (families of) relations are non-equivalent.

II. For each relation, construction of its transformation group. Perhaps we would have to extend the basic set and to define the relations on the extension. Proof of non-equivalence of the relations found.

III. Proof that the the found family of non-equivalent relations is maximal.

IV. For all (families of) relations found, determine their least upper bound and greatest lower bound. Definitions:

The least upper bound (the supremum, sup) of families of relations A and B is the family of relations definable via the relations from the union of families A and B.

The greatest lower bound (the infimum, inf) of families of relations A and B is the family of relations that are definable both via the relations from A and via the relations from B.