# Theory of definability: Logic. Algebra. Geometry

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## Introduction

Truth and provability belong to central concepts of mathematics. But mathematics includes not only theorems and conjectures but also definitions. For instance, using the ternary relation of product $xy = z$ we can define the binary relation of divisibility $x|y$ and the unary relation «to be a prime» $Prime(x)$. We will deal with the *definability theory* which now has perhaps more unsolved problems with simple and clear formulation than the proof theory and the model theory. We will consider and try to solve these open problems in the second part of the project. The first part contains some quite simple problems (exercises) as well as more difficult problem such that you may fail in solving them but any advancement and discussion would be useful. These problems are starred. Some problems, especially in the second part, are more similar not to olympiad but to research problems: you have to specify the condition and construct your own plan of research.

We will study *definitions of relations* starting with some non-mathematical example.

Here is the first example: «A brother in law is a brother of the husband.» [In English a brother in law may also mean a brother of the wife but we don't consider this case here.] In more detail, without any abbreviations:

By definition, «a person $A$ is a brother in law of a person $C$» means that there exists a person $B$ such that $B$ is the husband of the person $C$ and $A$ is a brother of the person $B$.

We have defined the binary relation $D(A, C)$ «to be a brother in law» via the relation $H(C, B)$ «to be the husband» and the relation $F(A, B)$ «to be a brother». Using the language of mathematics we write:

$$D(A, C) \risingdotseq (\exists B)(H(C, B) \wedge F(A, B)).$$

Here $\risingdotseq$ is read "is by definition"; to the left from this sign we indicate the name of the relation that is defined, and to the right we indicate its definition via given relations; $(\exists B)$ is read «there exists $B$ such that»; $\wedge$ is read «and».

A further example: «A perfect square is the product of some integer by itself.»

In more detail: «An integer $x$ is a perfect square if there exists an integer $y$ such that $x = y \cdot y$.» We have defined the unary relation «to be a perfect square» via the ternary relation «$x$ is the product of $y$ by $z$».

In the mathematical language: «$x$ is a perfect square» $\risingdotseq (\exists y)(x = y \cdot y)$.

In this project we restrict the possible form of definitions. All the above definitions do fit but this is not the case with the definitions of the form «a person $A$ is an ancestor of a person $B$ if there exists a sequence of persons which starts by $A$ and finishes by $B$ and such that each subsequent person is a parent of the preceding one». In the definitions, it is allowed to mention not sets or sequences but only elements, usually numbers. In other words, in a definition it is forbidden to say «for any set of numbers» or «there exists a set of numbers» but it is allowed to say «for any number», «there exists a number», «the number $x$ equals the number $y$». In the sequel, we assume that the binary relation of equality «x=y» is always admissible.

## A. Introductory cycle

**A1** Define the following relations via the ternary relation of positive integers «product» $xy = z$:
- a) the binary relation «to be divisible by»;
- b) the unary relation «to be the unit»;
- c) the unary relation «to be a prime».

**A2** Define the following relations via the ternary relations «product» and «sum» of positive integers:
- a) the unary relations «to be 2», «to be 3»;
- b) the unary relations «to be a power of 2», «to be a power of 4».

**A3** (*) Define the unary relation «to be a power of 6» via the ternary relations «product» and «sum» of positive integers.

**A4** Define the ternary relation «product» of positive integers via the ternary relation «sum» of positive integers and the unary relation «to be the square of a positive integer».

**A5** Define the following relations via the relation «less» («order», $<$) of rationals:
   a) the binary relations «greater or equal», «greater»;
   b) the ternary relation «to lie between».

## B. Equivalence of relations

Two relations are *equivalent* if the first of them is definable through the second one, and conversely, the second of them is definable through the first one.

**B1** For the relations definable via the order of rationals, try to find the maximum possible set of non-equivalent relations; for this, consider unary, binary, ternary etc. relations.

**B2** (*) Prove that for each $n$ there exists only a finite number of non-equivalent $n$-ary relations for the order of rationals.

**B3** Among the relations definable via the binary relation «consecution» $y = x + 1$ for integers, find the maximum possible set of non-equivalent ones.

## C. Transformations and invariants

*A transformation* is a one-to-one mapping of a set We say that a transformation *preserves a relation* if fulfilment of the relation for arbitrary elements of the domain is equivalent to its fulfilment for their images. In other words, the relation is *an invariant of the transformation.*

The collection of all transformations preserving a given relation is called *the transformation group of this relation.* Similarly for a family of relations.

In the problems below we consider only relations definable via the order of rationals.

**C1** Construct the transformation group for each of the relations found above which may occur non-equivalent.

**C2** For any two non-equivalent relations find a transformation which preserves one of these and doesn't preserve the other one.

**C3** (*) Prove that there exists only a finite number of non-equivalent relations. Try to find all of these.

**C4** (*) Try to construct a plan for search for non-equivalent relations and for constructing their transformation groups.

## D. Non-standard models

**D1** Let $S$ is the set of relations definable via «consecution» $y = x + 1$ for integers. Suppose $S$ contains two relations for which we want to prove non-equivalence but we fail to construct a transformation which «distinguishes» them. Try to extend the set of integers (for instance, add one more «copy» of integers and define the relation of consecution on the joint of two «copies») in such a way that the extension possesses a transformation which distinguishes these relations.