Definability theory:
Logic. Geometry. Algebra

Alexey Semyonov, Sergey Soprunov, Alexey Kanel-Belov,
Ilya Ivanov-Pogodayev, Roman Isayev, Vladimir Kondratyev,
Boris Rafailovich Frenkin

Summer Conference of the Tournament of Towns
August 10, 2021

An example of definability problem

Problem: to define the unary relation "to be a power of 6"via addition and multiplication relations on the set of natural numbers.
This problem can be solved (and was solved by members of the Project) by 'Arithmetization technique' similar to the Goedel proof of his Incompleteness Theorem. A participant of the project Valery Kozhurkin proposed a different and short solution:

Let $6^k = 2^a 3^b$. It is easy to find 2^a and 3^b as the maximal exponents of 2 and 3 that divide 6.
Let p be a prime number such that $p > 6^k$.
The minimal solutions for the congruences
$p^m + 2^a \equiv 0 \mod (p+2)$
and $p^n + 3^b \equiv 0 \mod (p+3)$
for odd k are $m = a$ and $n = b$.
If a and b are even, we can multiply by 6 in order to work with odd exponents.

In out project by definability problem we mean describing the definability lattice of a given structure. A solution for the problem can be divided in 4 stages.

1. Find as many non-equivalent relations as possible. At some moment you see that nothing new can be obtained. Then you pass to the next stage.
2. For each relation, describe the group of transformations which preserve it. Consideration of these groups enables us to explain why some relations are not definable via others.
3. Prove that there are no other non-equivalent relations. This is the most difficult part of the project for the given structure. Up to now, we have no general way to do it, so we have to invent something specific for each case. In most cases group considerations are helpful.
4. For each pair of relations, indicate their supremum (abbr. sup) least upper bound and infimum (abbr. inf) their greatest lower bound. These are the lattice operations on the closures of relations. .

Rationals with the order (I stage)

1. $(x < y)$
2. $B(x, y, z) \leftrightharpoons (x < y < z) \vee (z < y < x)$
3. $C(x, y, z) \leftrightharpoons (x < y < z) \vee (y < z < x) \vee (z < x < y)$
4. $S(x, y, z, u)$: open intervals (x, y) and (z, u) intersect and do not contain in each other (they are 'linked').
5. $(x = y)$

Rationals with the order (II stage)

Let us indicate the transformation groups for each case.

1. Γ consists of all increasing continuous transformations. All the groups below contain these transformations, so these will not be mentioned explicitly.
2. Γ_B consists of all continuous decreasing transformations.
3. Γ_C contains **transpositions**. Here we use the term «transposition» for a transformation with irrational parameters s, t which maps the intervals $(-\infty, s)$ and $(s, +\infty)$ onto $(t, +\infty)$ and $(-\infty, t)$ respectively and preserves the order of rationals in both cases.
4. Γ_S contains all transformations from the groups Γ_B and Γ_C.
5. $Sym(\mathbb{Q})$ is the group of all transformations of rationals. They preserve the identity relation.

Rational with the order (III stage)

Let us describe the idea of the proof that there are no other non-equivalent relations. For this we require the group-theoretic notion of k-**transitivity**.

A group G is called k-transitive if for every two k-tuples $(a_1, \ldots, a_k); (b_1, \ldots, b_k); a_i \neq a_j; b_i \neq b_j$ there exists a transformation $g \in G$ such that $\forall (i \leqslant k) (g(a_i) = b_i)$.

For instance, the group Γ is 1-transitive but not 2-transitive. And Γ_B is 2-transitive but not 3-transitive.

We will describe all groups including Γ (its supergroups). For this, we will consider all k-transitive but not $(k+1)$-transitive groups for every natural k. It occurs that we will obtain no groups besides the five groups described above. This is the crucial step in the proof for absence of other relations. The proof is difficult but straightforward. Some members of the project succeeded in it

Rationals with the order (IV stage)

Let us represent the above results in the form of an oriented graph. Its vertices are the symbols of relations, and the directed edges indicate definability of some relations via others. Let us recall two notions.

Supremum of the families of relations A and B is the family of relations definable via the relations from the union of A and B.

Infimum of the families of relations A and B is the family of relations definable both via the relations from A and via the relations from B. On this graph the infimum for families A and B, for instance, is the family of all relations such that from each of them there exists directed paths to A and B.

We can represent the results not only as the graph of relations but also as the graph of transformation groups. Its vertices are the transformation groups of relations, and the direction of edges corresponds to the inclusion relation between the groups.

Graph of relations

The lattice for $(\mathbb{Q}, <)$ looks as follows:

$$(\mathbb{Q}, <)$$

$$\infty \; / \qquad \qquad \backslash \; 2$$

$$(\mathbb{Q}, \mathrm{cyc}) \qquad (\mathbb{Q}, \mathrm{betw}) = (\mathbb{Q}, \mathrm{betw})' = (\mathbb{Q}, <)'$$

$$2 \; \backslash \qquad \qquad / \; \infty$$

$$(\mathbb{Q}, \mathrm{sep}) = (\mathbb{Q}, \mathrm{sep})' = (\mathbb{Q}, \mathrm{cyc})'$$

$$| \; \infty$$

$$(\mathbb{Q}, \emptyset)$$

Integers with successor (I stage)

1. $A_n(x, y) \leftrightharpoons y = x + n$
2. $B_n(x, y, z, u) \leftrightharpoons (|x - y| = n) \wedge (x - y = z - u)$
3. $C_n(x, y) \leftrightharpoons |x - y| = n$

Integers with successor (II stage)

Let us define the basic sorts of transformations. To begin with, divide the integers into n classes respective to their remainders modulo n: $r + n\mathbb{Z}$, $0 \leq r < n$. The set consisting of n these classes will be denoted $\mathbb{Z}/n\mathbb{Z}$.

- **A shift** of a class $r + n\mathbb{Z}$ by a value k is a transformation σ of the form $\sigma(r + n \cdot S) = r + n \cdot (S + k)$.
- **A permutation** $\sigma \in S_n$ of clases is a transformation σ of the form $\sigma(r + n \cdot S) = \sigma(r) + n \cdot S$.
- **U-turn** of a class $r + n\mathbb{Z}$ is a transformation σ of the form $\sigma(r + n \cdot S) = r + n \cdot (-S)$.

Then

1. Γ_{A_n} consists of the shifts and the permutations of the classes $\mathbb{Z}/n\mathbb{Z}$.
2. Γ_{B_n} consists of the shifts, the permutations and the U-turns of the classes $\mathbb{Z}/n\mathbb{Z}$.
3. Γ_{C_n} consists of the shifts, the permutations and the simultaneous U-turn of all the classes $\mathbb{Z}/n\mathbb{Z}$.

Unsolved at ToT and Open Problems

Here are some natural problems for usual number sets.

Relations/sets	$\mathbb{Q}$	$\mathbb{Z}$	$\mathbb{N}$
$(x < y)$	solved	unsolved	unsolved
$(y = x + 1)$	try	solved	try
$(z = x + y)$	in process	hard	hard

Two more (from many):
1. "Branching integers:" an infinite non-oriented graph without cycles (an infinite tree) such that every vertex is of degree 3; the relation "to be neighboring vertices".

2. The order on non-negative rationals.