Some features of Miquel's structures.

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The symbol ° denotes some well-known facts, without which, however, the solution of further problems will be difficult. An asterisk ⋆ indicates a problem that is suspected to be difficult.

Part 1

1° *(Miquel's theorem)* In a triangle ABC, points C_1, A_1, B_1 are chosen on the sides AB, BC, CA, respectively. Prove that the circumcircles of $\triangle AB_1C_1, \triangle A_1BC_1, \triangle A_1B_1C$ have a common point.

2° Let an angle ABC be given. Points C_1, A_1 move along the lines AB, BC with constant (not necessarily equal) speeds. Prove that all circles BC_1A_1 pass through another fixed point other than B. When is it wrong?

3° *(Trigonometric form of Ceva's theorem)* In a triangle ABC, points C_1, A_1, B_1 are chosen on the sides AB, BC, CA, respectively. Prove that lines AA_1, BB_1, CC_1 meet at one point or are parallel if and only if

$$\frac{\sin\angle\overrightarrow{ABB_1}\cdot\sin\angle\overrightarrow{BCC_1}\cdot\sin\angle\overrightarrow{CAA_1}}{\sin\angle\overrightarrow{B_1BC}\cdot\sin\angle\overrightarrow{C_1CA}\cdot\sin\angle\overrightarrow{A_1AB}}=1$$

4° *(Miquel's point)* Let ℓ_1, ℓ_2, ℓ_3, and ℓ_4 be four lines in general position. Excluding one line, one gets three lines forming a triangle, four triangles in total. Prove that the circumcircles of these four triangles have a common point.

5° *(Miquel's circle)* Let $\ell_1, \dots \ell_5$ be 5 lines in general position. Prove that Miquel's points of all five possible quadruples of these lines are concyclic.

6° Given two circles $\mathcal{A}, \mathcal{B}$. Prove that the locus of points X such that

$$\frac{\text{power of } X \text{ with respect to } \mathcal{A}}{\text{power of } X \text{ with respect to } \mathcal{B}} = const$$

is a circle, in the case

a) when $\mathcal{A}, \mathcal{B}$ intersect

b) for arbitrary position $\mathcal{A}$ and $\mathcal{B}$.

7° In a triangle ABC, the pedal circles of points X and Y coincide. Prove that X and Y are isogonal conjugate with respect to $\triangle ABC$.

8. Inside a triangle ABC, a point M is selected, and points C_1, A_1, B_1 are chosen on the sides AB, BC, CA, respectively. Lines AM, BM, CM intersect for the second time the circumcircles of triangles AB_1C_1, A_1BC_1, A_1B_1C at points M_a, M_b, M_c, respectively. Let P be the intersection point of the circles AB_1C_1, A_1BC_1, A_1B_1C. Prove that the points M, M_a, M_b, M_c, and P are concyclic. (From now on, we denote the corresponding circle by $\mathcal{M}$).

9. Let, in the notation of problem 8, the line PA_1 intersect $\mathcal{M}$ again at A'. Prove that $MA' \parallel BC$.

10. Prove that the lines M_aA', M_bB', M_cC' are concurrent or parallel.

11⋆ Prove that the circumcircles of the triangles AM_aA', BM_bB', CM_cC' are coaxial.

12⋆ Let a, b, c, d be lines in general position and let X_{ab}, X_{ac}, X_{ad}, X_{bc}, X_{bd}, X_{cd} be their intersection points. Let $\mathcal{K}$ be a circle with a point K on it. Let Y_{ij} be the intersection point of $X_{ij}K$ with $\mathcal{K}$. Prove that the lines $Y_{ab}Y_{cd}$, $Y_{ac}Y_{bd}$, $Y_{ad}Y_{bc}$ are concurrent or parallel.

13. In a triangle ABC arbitrary points C_1, C_2 on the side AB, points A_1, A_2 on the side BC, points B_1, B_2 on the side CA are selected. The lines A_1B_1 and A_2B_2 intersect at L_c, points L_a, L_b are defined similarly. Circumcircles of $\triangle A_1A_2L_c$ and $\triangle B_1B_2L_c$ intersect at points L_c and N_c. Points N_b, and N_a are defined similarly.

a) Prove that the lines AN_a, BN_b, CN_c meet at one point (let's call it N)

b) Prove that N, N_a, N_b, N_c lie on a circle (let's call it $\mathcal{N}$).

Let the circles AB_1C_1, A_1BC_1, A_1B_1C intersect at P, and let the circles AB_2C_2, A_2BC_2, A_2B_2C intersect at Q.

c) Prove that P and Q lie on $\mathcal{N}$.

d) Prove that the intersection point A' of lines PA_1 and QA_2 lies on $\mathcal{N}$.

Part 2

In this section, the hyperbola icon)(will denote some problems. In these problems, your goal will be to prove the original statement, and then formulate and prove a similar statement for a hyperbola.

14. Let A, B be two fixed points and let X be a point moving along a line. Examine the function f for intervals of monotonicity. Construct an extremum point with a compass and a ruler if

a) $f(X) = XA + XB$

b) $f(X) = XA - XB$

15° *(Fagnano's problem)* In a triangle ABC, points C', A', B' are chosen on the sides AB, BC, AC, respectively, which do not coincide with the vertices of $\triangle ABC$. It is known that the triangle $A'B'C'$ has the smallest possible perimeter among all triangles inscribed in $\triangle ABC$. Prove that AA_1, BB_1 and CC_1 are altitudes of $\triangle ABC$.

16)(*(Optical Property)* Let A be a point on an ellipse with foci F_1 and F_2. Prove that the outer bisector of the angle F_1AF_2 is tangent to the ellipse (has exactly one common point with it).

17)(An ellipse with foci F_1 and F_2 is tangent to the sides of an angle ABC. Prove that $\angle ABF_1 = \angle CBF_2$.

18)(An ellipse with focus F is fixed, and a line ℓ is tangent to it. Let P be the projection of F onto ℓ. Prove that if ℓ is moving, then P is moving along a circle tangent to the ellipse at two points.

19. Let $\mathcal{K}$ be an ellipse with foci F_1 and F_2. A circle ω with center O is tangent to $\mathcal{K}$ at points X and Y (the ellipse lies inside the circle). Prove that

a) $OF_1 = OF_2$.

b) XF_1OF_2Y is an inscribed pentagon.

c))(Let a point P move along ω. Then the angle between PF_1 and one of the tangents from P to the ellipse is constant.

d) Redefine ω so that ω does not have to touch $\mathcal{K}$ twice.

e) The line through O and the center of $\mathcal{K}$ meets $\mathcal{K}$ at Z. Prove that the circumcircle of $\triangle OZF_1$ is tangent to ω.

f) Let circles α and β touch ω internally, pass through F_1, and intersect for the second time at the point E. Prove that from the two intersection points α and $\mathcal{K}$ you can choose a point I, and from the two intersection points β and $\mathcal{K}$ you can choose a point J, so that E will lie on the line IJ.

g)* The line through O and the center of $\mathcal{K}$ intersects $\mathcal{K}$ at points Z and T, and the circle at points A and B. Point U is chosen on the line ZT so that $\angle UF_1O = 90°$. Prove that the cross-ratio of the points A, Z, U, B is equal to the cross-ratio of the points B, T, U, A (in the order indicated).

h) Show that if we take ω as an absolute for the Klein model of hyperbolic plane, then $\mathcal{K}$ is a circle or an equidistant curve.

20)(Let circles α and β intersect at points X and Y. An ellipse $\mathcal{K}$ is inscribed in the "slice" of their intersection, twice tangent to each of the circles. A line ℓ_X is tangent to $\mathcal{K}$, separates the point X from $\mathcal{K}$, and intersects the "slice" at points S and T. Also, ℓ_X intersects the circle α outside the segment ST at A_1, and intersects the circle β outside the segment ST at B_1. Similarly, chose a line ℓ_Y and define points A_2 and B_2. Prove that $A_1A_2 \parallel B_1B_2$.

21* Points N and M move along two circles with the same angular velocities. Find the envelope (curve touching all) of lines NM.

22* Two points N and M move along two lines with constant speeds. Find the envelope of lines NM.

23)(Given two intersecting circles, consider all ellipses lying inside both circles and touching each of the circles twice. Find the locus of their foci.

Part 3

24° *(Orthologic triangles)* Let A, B, C, A_1, B_1, C_1 be points in general position. Let the perpendiculars from A, B, and C to the lines B_1C_1, A_1C_1, and A_1B_1, respectively, intersect at one point. Prove that perpendiculars from A_1, B_1, and C_1 to the lines BC, AC, and AB, respectively, also intersect at one point.

25⋆ Let A, B, C, A_1, B_1, C_1 be points in general position. Suppose that there exists a point P such that $\angle(AP, B_1C_1) = \angle(BP, A_1C_1) = \angle(CP, A_1B_1) = \alpha$. Prove that there exists a point Q such that $\angle(A_1Q, BC) = \angle(B_1Q, AC) = \angle(C_1Q, AB) = -\alpha$.

We use the notation of Problem 13. Suppose additionally that the points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle $\mathcal{R}$ with center R.

26. Prove that P and Q are isogonal conjugate with respect to $\triangle ABC$.

27. Prove that:

a) $R \in \mathcal{N}$.

b) RN is a diameter of $\mathcal{N}$.

c) $PR = QR$.

28. Prove that an ellipse $\mathcal{K}$ with foci P and Q can be inscribed into the triangle ABC.

29. Lines PA' and QA' meet $\mathcal{R}$ again at points X and Y. Prove that XY is tangent to $\mathcal{K}$.

30. Prove that $\mathcal{K}$ is tangent to $\mathcal{R}$ if and only if $\mathcal{N}$ intersects $\mathcal{R}$, in which case the tangency points coincide with the intersection points.

31. Prove that in a triangle:

a) The Lemoine point, two Brocard points and the circumcenter form a deltoid (i.e. a kite) with two right angles.

b) An ellipse with foci at Brocard's points touches the sides at the bases of the symmedians.

32⋆ Suppose that the lines AA_1, BB_1, CC_1 meet at a point L. Prove that L lies on the radical axis of $\mathcal{N}$ and $\mathcal{R}$.