Some features of Miquel's structures.

Solutions

Part 1

1° Well-known.

2° We consider one position of points A_1 and C_1, and another position, which we denote X and Y, respectively. The circles (BC_1A_1) and (BXY) are tangent or intersect at a point $G \neq B$.

In the first case there is a homothety with center B mapping the circle (BC_1A_1) to the circle (BXY). It maps A_1 to X and C_1 to Y. Therefore, $C_1A_1 \parallel XY$. Consider a third position of points A_1 and C_1, which we denote P and Q. Then $\dfrac{A_1X}{A_1P} = \dfrac{C_1Y}{C_1Q}$, hence $PQ \parallel A_1C_1$. Thus the circle (BPQ) is tangent to the circle (BA_1C_1). So all circles (BA_1C_1) are tangent at B.

In the second case the triangles GXA_1 and GYC_1 are similar, since $\angle(GA_1, A_1B) = \angle(GC_1, C_1B)$ and $\angle(GX, XB) = \angle(GY, YB)$. Hence there is a spiral similarity ϕ with center G, mapping A_1 to C_1 and X to Y. Consider a third position of points A_1 and C_1, which we denote P and Q. Then $\phi(P) = Q$ and it follows that $\angle(GP, PB) = \angle(GQ, QB)$, so G lies on the cirle (BPQ).

3-4° Well-known.

5° Denote the intersection point of ℓ_i and ℓ_j by X_{ij}; and the Miquel point of all lines except ℓ_i by A_i. It suffices to prove that A_1, A_2, A_3, and A_4 are concyclic. Using the circles $(A_1A_2X_{35}X_{45})$, $(A_2A_3X_{15}X_{45})$, $(A_3A_4X_{15}X_{25})$, and $(A_4A_1X_{25}X_{35})$ we obtain

$$\begin{aligned}\angle(A_1A_2, A_2A_3) &= \angle(A_1A_2, A_2X_{45}) + \angle(X_{45}A_2, A_2A_3) = \angle(A_1X_{35}, X_{35}X_{45}) + \angle(X_{45}X_{15}, X_{15}A_3)\\ &= \angle(A_1X_{35}, X_{35}X_{25}) + \angle(X_{25}X_{15}, X_{15}A_3) = \angle(A_1A_4, A_4X_{25}) + \angle(X_{25}A_4, A_4A_3) = \angle(A_1A_4, A_4A_3).\end{aligned}$$

6° Below we present an algebraic solution of this problem, which works for parts a) and b) simultaneously. For a synthetic solution, where part a) is easier than part b), see [1], Theorem 2.12.

Let $f(x, y) = 0$ and $g(x, y) = 0$ be the equations of $\mathcal{A}$ and $\mathcal{B}$, respectively, in Cartesian coordinates, where $f(x, y) = x^2 + y^2 + a_1x + a_2y + a_3$ and $g(x, y) = x^2 + y^2 + b_1x + b_2y + b_3$. Note that the powers of point (x, y) with respect to $\mathcal{A}$ and $\mathcal{B}$ are equal to $f(x, y)$ and $g(x, y)$, respectively. So the desired locus is given by equation $f(x, y) = cg(x, y)$ for some constant c. It is easy to see that this equation defines a line if $c = 1$ and a circle $\mathcal{C}$ if $c \neq 1$.

Assume now that $c \neq 1$. The circle $\mathcal{C}$ is given by equation $\dfrac{f(x, y) - cg(x, y)}{1 - c} = 0$. Let (p, q) be a point on the radical axis of $\mathcal{A}$ and $\mathcal{B}$, i.e. $f(p, q) = g(p, q)$. The power of the point (p, q) with respect to $\mathcal{C}$ is equal to $\dfrac{f(p, q) - cg(p, q)}{1 - c} = f(p, q) = g(p, q)$. Therefore, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are coaxial.

7° Let X_b and Y_b be the projections of X and Y onto AC, respectively. Let X_c and Y_c be the projections of X and Y onto AB, respectively. Since X_b, Y_b, X_c, and Y_c are concyclic, we obtain

$$\begin{aligned}\angle(BA, AX) &= \angle(X_cA, AX) = \angle(X_cX_b, X_bX) = \angle(X_cX_b, X_bY_b) + 90^\circ\\ &= \angle(X_cY_c, Y_cY_b) + 90^\circ = \angle(YY_c, Y_cY_b) = \angle(YA, AY_b) = \angle(YA, AC)\end{aligned}$$

Similarly, $\angle(AB, BX) = \angle(YB, BC)$. Hence X is the isogonal conjugate of Y with respect to $\triangle ABC$.

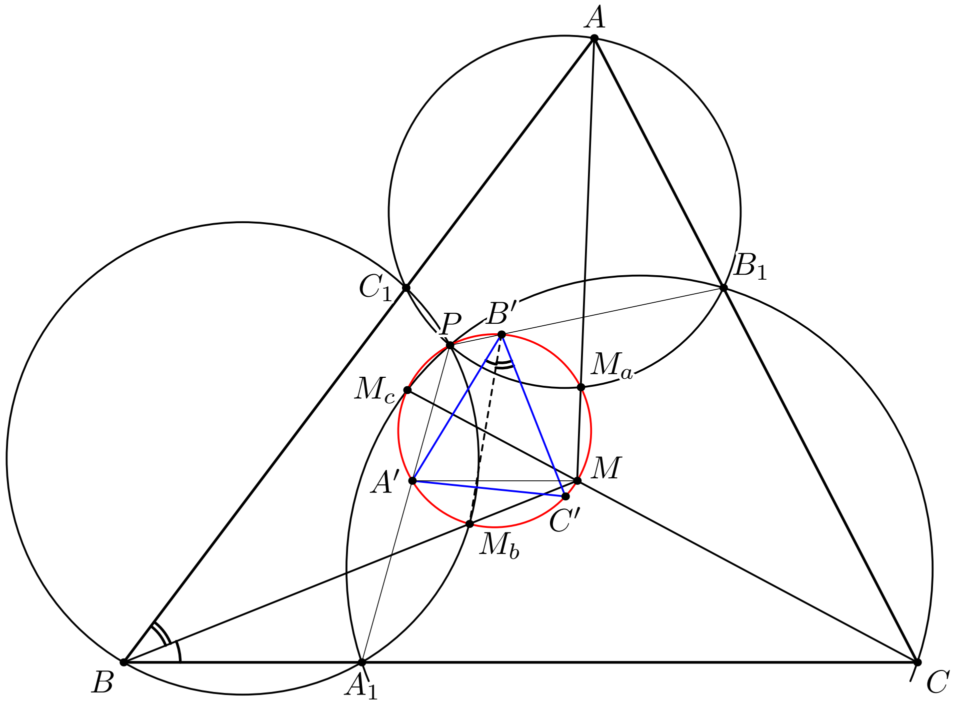
8. Observe that

$$\begin{aligned}\angle(MM_a, M_aP) &= \angle(AM_a, M_aP) = \angle(AB_1, B_1P) = \angle(CB_1, B_1P)\\ &= \angle(CA_1, A_1P) = \angle(CM_c, M_cP) = \angle(MM_c, M_cP).\end{aligned}$$

So M_c lies on the circle (MPM_a). Similar argument shows that M_b also lies on this circle.

9. The following equalities imply that $MA' \parallel BC$.

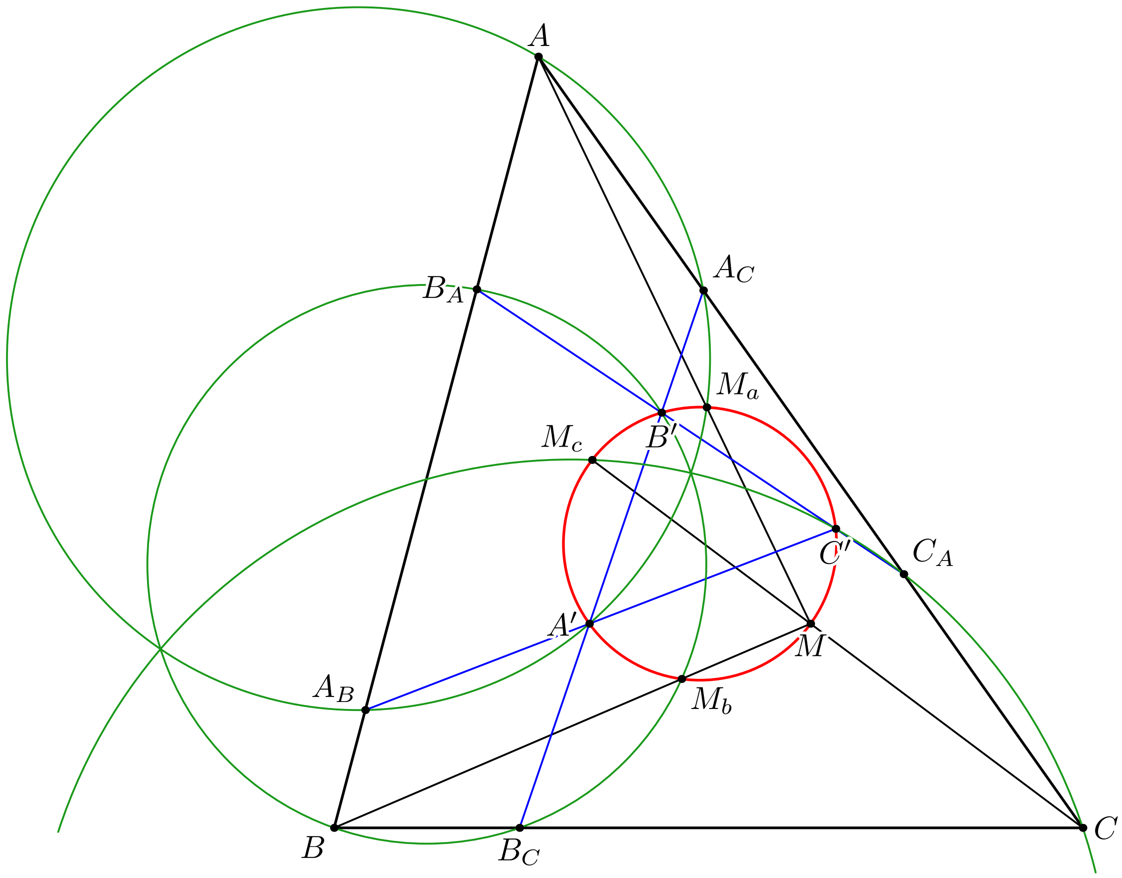
$$\angle(MA', A'P) = \angle(MM_c, M_cP) = \angle(CM_c, M_cP) = \angle(CA_1, A_1P) = \angle(BC, A'P),$$



10. We have

$$\angle(A'B', B'M_b) = \angle(A'M, MM_b) = \angle(CB, BM).$$

Similarly $\angle(C'B', B'M_b) = \angle(AB, BM)$, and so on. It follows that the triangles ABC and $A'B'C'$ are similar and have different orientations. Moreover, $A'M_a$, $B'M_b$, and $C'M_c$ pass through the point, corresponding to the isogonal conjugate of M in $\triangle ABC$.



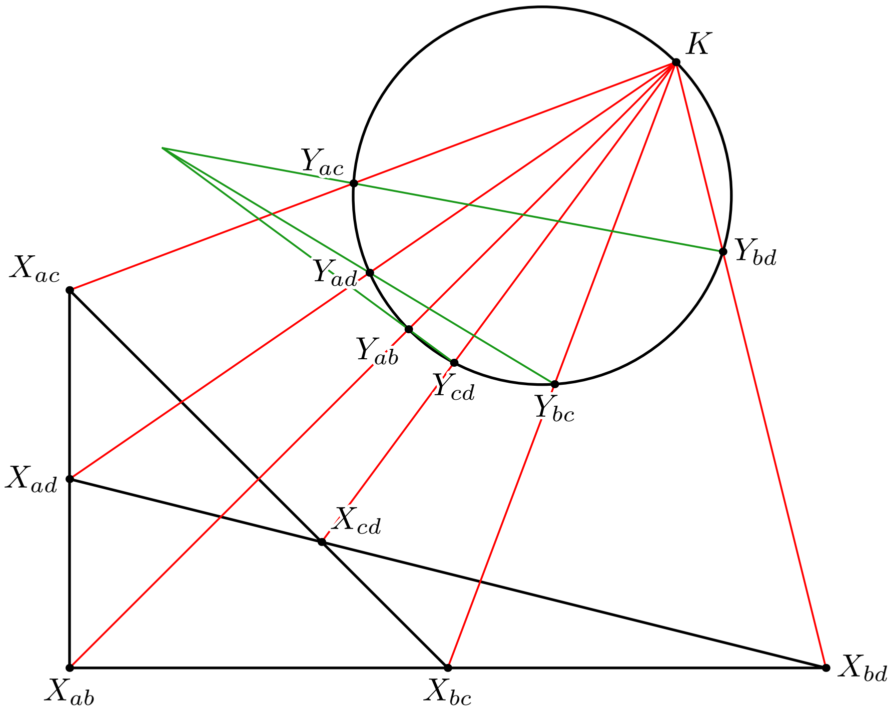
11⋆ Denote the circles (AM_aA'), (BM_bB'), and (CM_cC') by ω_a, ω_b, and ω_c, respectively. Let $A_B = AB \cap A'C'$, $B_A = BA \cap B'C'$, $A_C = AC \cap A'B'$, $C_A = CA \cap C'B'$, $B_C = BC \cap B'A'$, $C_B = CB \cap C'A'$.

By angle chase, ω_a passes through A_B and A_C. Similarly $B_A, B_C \in \omega_b$ and $C_A, C_B \in \omega_c$. Then

$$\frac{AB \cdot AB_A}{AC \cdot AC_A} = \frac{\sin\angle ACB \cdot \sin\angle AC_AB_A}{\sin\angle ABC \cdot \sin\angle AB_AC_A} = \frac{\sin\angle A_CCB_C \cdot \sin\angle A_CC_AB'}{\sin\angle A_CB'C_A \cdot \sin\angle A_CB_CC} = \frac{A_CB_C \cdot A_CB'}{A_CC_A \cdot A_CC}$$

which implies that the ratio of powers of A with respect to ω_b and ω_c is equal to the ratio of powers of A_C with respect to ω_b and ω_c. Similarly, this ratio is the same for A_B. The result now follows from problem 6.

12* By the Desargues involution theorem, there exists an involution on the pencil of lines through K, which swaps KY_{ab} with KY_{cd}, KY_{ac} with KY_{bd}, and KY_{ad} with KY_{bc}. So there exists an involution on $\mathcal{K}$, which swaps Y_{ab} with Y_{cd}, Y_{ac} with Y_{bd}, and Y_{ad} with Y_{bc}. Such involution must map every point $P \in \mathcal{K}$ to the second intersection point of PU with $\mathcal{K}$, where $U = Y_{ab}Y_{cd} \cap Y_{ac}Y_{bd}$. Therefore, $Y_{ab}Y_{cd}$, $Y_{ac}Y_{bd}$, $Y_{ad}Y_{bc}$ are concurrent.



13. Observe that N_c is the second intersection point of the circles (A_1B_1C) and (A_2B_2C).

a) Follows from trigonometric Ceva's theorem for the triangle ABC, since

$$\frac{\sin\angle BCN_c}{\sin\angle ACN_c} = \frac{A_1A_2}{B_1B_2}, \qquad \frac{\sin\angle CAN_a}{\sin\angle BAN_a} = \frac{B_1B_2}{C_1C_2}, \qquad \frac{\sin\angle ABN_b}{\sin\angle CBN_b} = \frac{C_1C_2}{A_1A_2}.$$

b, c) Follows from problem 8.

d) Follows from problem 9.

Part 2

14. See [1], pp. 6-7.

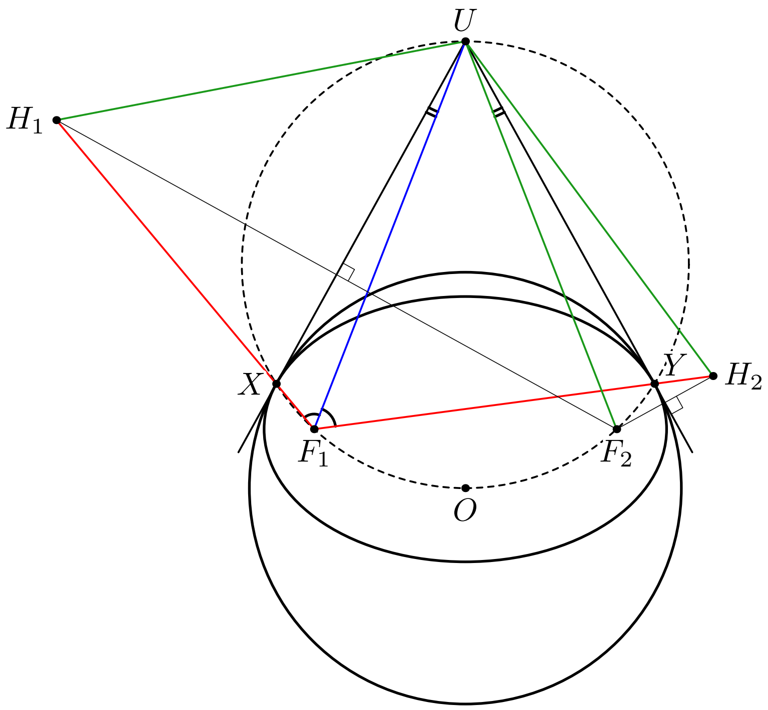
15° We fix B', C', and move A'. Since $B'A' + A'C'$ is minimal, BC is the external bisector of $\angle B'A'C'$. Similarly, AC and AB are the external bisectors of $\angle A'B'C'$ and $\angle A'C'B'$, respectively. So A, B, and C are excenters of $\triangle A'B'C'$. Thus AA', BB', and CC' are altitudes of $\triangle ABC$.

16)(See [1], Theorem 1.1 and the discussion after its proof.

17)(See [1], Theorem 1.3 and the discussion after its proof.

18)(Let $\mathcal{K}$ be the ellipse, let G be its second focus, and let M be the midpoint of FG. Let A be the reflection of F in ℓ and let $B = \ell \cap \mathcal{K}$. Then $MP = GA/2 = (FB + BG)/2$ is constant, so P moves along a circle with center M tangent to $\mathcal{K}$.

19. **a) b)** Construct tangents to the ellipse at X and Y. Let them meet at U.



First, let us prove an auxiliary fact — the bisector of the angle XF_1Y passes through U. Reflect F_2 in both tangents, denote reflections by H_1 and H_2. Triangles UH_1F_1 and UH_2F_1 are congruent, by SSS. Therefore, $\angle XF_1U = \angle YF_1U$.

It follows that U lies on the bisector of the angle XF_1Y (and similarly, the angle XF_2Y), and on the perpendicular bisector of XY (since the segments of tangents are equal). Hence X, F_1, F_2, Y U are concyclic. Since $XOYU$ is inscribed, O lies on this circle, too. XO is the bisector of the angle F_1XF_2 (since it is perpendicular to the tangent to the ellipse), therefore, $F_1O = OF_2$.

c)$^{)(}$ By Problem 18, the locus of projections of F_1 onto tangents to the ellips is a circle. Applying spiral similitude with center F_1, angle $\pi/2 - \alpha$ and ratio $1/\sin\alpha$, we obtain that the locus of points P such that the oriented angle between PF_1 and the tangent to the ellipse through P is equal to α, is also a circle. For $\alpha = \angle YXU$ this locus in the circle ω.

d) By the previous item, ω could be defined as the locus of P such that the oriented angle between PF_1 and the tangent to the ellipse through P is equal to α.

Note. If the circles (OF_1F_2) and ω do not intersect, the tangent points of $\mathcal{K}$ and ω are not real (complex).

e) Let the perpendicular to OF_1 through F_1 intersects ω at U_1, let Z' be the projection of U_1 onto OZ. Points F_1 and Z' lie on the circle with diameter OU_1 touching ω. Moreover, $\angle(F_1U_1, U_1Z') = \angle(F_1O, OU) = \angle(F_1X, XU)$, since $U_1Z'OF_1$ and F_1XOU are inscribed. Hence U_1Z touches $\mathcal{K}$ (since the circle ω is the locus of points such that the angle between the tangent and the segment joining with the focus, is constant), and Z' coincides with Z.

f) Follows from the following Lemma that generalizes the statement of the previous item (by a spiral similitude with center F_1).

Lemma. Let P be an arbitrary point of the ellips $\mathcal{K}$, let the tangent to $\mathcal{K}$ through P intersect ω at A and B. It follows that the circle APF_1 touches ω.

g)* From e) it follows that Z and T are projections onto AB of the endpoints of the chord of ω having the midpoint F_1. Hence in cross ratios one can replace Z and T by the endpoints of this chord, and replace A and B by the intersection points of these tangents with UF_1. If U lies outside the circle, then perform a projective transformation that maps the circle to itself and takes U to infinity. If U lies inside the circle, then perform a projective transformation that maps U to the center. IN both cases the statement is obvious.

h) Directly follows from the previous. (About models of Lobachevsky plane one can read in [2].)

20$^{)(}$ Let F_1 and F_2 be the foci of the ellipse, where F_1 is closer to Y than F_2. From 19c it follows that $F_1A_1B_1$ and $F_2A_2B_2$ are similar, by equal angles. Let $S = A_1B_1 \cap A_2B_2$. from the optic property we have

$\angle A_1SF_1 = \angle A_2SF_2$. Hence there exists a product of a dilation with center S and the reflection in the bisector of the angle A_1SA_2, which maps $A_1F_1B_1$ to $A_2F_2B_2$. Hence $\frac{A_1S}{A_2S} = \frac{B_1S}{B_2S}$, therefore $A_1A_2 \parallel B_1B_2$.

21* Let O be the center of the spiral similitude taking one of the circle to the other and taking N to M. Since all the triangles ONM are similar to each other, the projection H of O onto MN lie on a certain circle ω. Thus we can reformulate the problem in the following way: let ω and O be a circle and a point; we need to find a curve touching lines passing through a point $H \in \omega$ and perpendicular to OH.

If O lies on ω, then all such lines pass through the antipodal point. Otherwise, this curve ia a conic with one its focus at O. This conic is an ellipse, if O lies inside ω, and a hyperbola, if it lies outside ω (It could be derived from the Problem 18.)

22* **Answer:** A parabola tangent to given lines.

Proof. Let A and B be points moving linearly along two given lines intersecting at X. Consider a point $F \neq X$, which is a common point of all circles (ABX) (it is known that such point exists, if A and B do not pass X simultaneously. If they pass X simultaneously, then AB has a constant direction). The Miquel point of four lines is the focus of the parabola touching these four lines (e.g., see [1], Theorem 4.10). this argument completes the proof.

23$^{)(}$ **Answer:** The circle passing through the intersection points of two given circles, which as the Apollonius circle for their centers.

24° It is known that both conditions are equivalent to $AB_1^2 + BC_1^2 + CA_1^2 = BA_1^2 + AC_1^2 + CB_1^2$.

25* Let $A_2B_2C_2$ be the image of the triangle ABC under the rotation about the point P through the angle $\alpha - 90°$. Then the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are orthologic, so there exists a point Q such that $A_1Q \perp B_2C_2$, $B_1Q \perp A_2C_2$, and $C_1Q \perp A_2B_2$. Since $\angle(BC, B_2C_2) = \alpha - 90°$, we obtain $\angle(A_1Q, BC) = \angle(A_1Q, B_2C_2) - \angle(BC, B_2C_2) = -\alpha$. Similarly $\angle(B_1Q, AC) = \angle(C_1Q, AB) = -\alpha$, as required.

Below we sketch a different solution, which does not use problem 24.

Let us call a triple of lines a', b', c' harmonic to a triple of lines a, b, c, if in a triangle whose sidelines are parallel to a, b, c, the corresponding cevians parallel to a', b', c' are concurrent.

Lemma. The relation 'harmonic' is symmetric, i.e., if a', b', c' is harmonic to a, b, c, then a, b, c is harmonic to a', b', c'.

Proof. Let ℓ be a line. Through a point O (O not in ℓ) let us construct lines parallel to a, b, c, a', b', c'. Let these lines intersect ℓ at A, B, C, A', B', C', respectively. Using Ceva theorem in the sine form, rewrite the condition that a', b', c' is harmonic to a, b, c as $\frac{\vec{AB'}}{\vec{B'C}} \cdot \frac{\vec{CA'}}{\vec{A'B}} \cdot \frac{\vec{BC'}}{\vec{C'A}} = -1$. We see that this condition is invariant under replacement of a, b, c by a', b', c'. Lemma is proved.

Now let a, b, c and a', b', c' be the sidelines of triangles ABC and $A_1B_1C_1$. By t_φ denote t rotated by angle φ. The condition of the problem means that a, b, c is harmonic to a'_φ, b'_φ, c'_φ. Using rotation by $-\varphi$, we get that $a_{-\varphi}$, $b_{-\varphi}$, $c_{-\varphi}$ is harmonic to a', b', c', and the statement follows.

We use the notation of Problem 13. Suppose additionally that the points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle $\mathcal{R}$ with center R.

26. *(Generalized pedal triangles)* Let P' be the isogonal conjugate to P. Let P_a, P_b, P_c be projections of P onto BC, CA, AB, respectively; similarly denote projections of P'. We know that the midpoint R_0 of PP' is the center of the circle $(P_aP_bP_c) = (P'_aP'_bP'_c)$. Let $\angle(PA_1, CA_1) = \angle(PB_1, AB_1) = \angle(PC_1, BC_1) = \varphi$. Let R_1 be the circumcenter of $(A_1B_1C_1)$. Triangle $P_aP_bP_c$ maps to $A_1B_1C_1$ by some spiral similitude with center P, hence $PP_aA_1 \sim PR_0R_1$, thus R_1 is a point on the perpendicular bisector of PP' such that $\angle(PR_1, R_1R_0) = \varphi$. Hence $\angle(R_1R_0, R_1P') = \varphi$. Perform the spiral similitude with center P' taking R_0 to R_1. It maps $P'_aP'_bP'_c$ to $A'_2B'_2C'_2$ so that $P'R_0R_1 \sim P'P'_aA'_2 \sim P'P'_bB'_2 \sim P'P'_cC'_2$, so A'_2, B'_2, C'_2 are the points of BC, CA, AB such that $\angle(P'A'_2, CA'_2) = \angle(P'B'_2, AB'_2) = \angle(P'C'_2, BC'_2) = -\varphi$. Since R_0 is the center of the circle $(P'_aP'_bP'_c)$, R_1 is the center of the circle $(A'_2B'_2C'_2)$. Radii of circles $(A_1B_1C_1)$ and $(A'_2B'_2C'_2)$ are both equal to $R(P_aP_bP_c)/\sin\varphi$, hence these circles coincide. It follows that $A'_2 = A_2$, $B'_2 = B_2$, $C'_2 = C_2$, $Q = P'$, and $R_1 = R$.

In addition, note that triangles ABC and $A_1B_1C_1$ satisfy the condition of the Problem 25 with P and Q as points of concurrency.

27. From the previous proof we have $\angle(PR, QR) = 2\varphi = \angle(PA_1, QA_2)$. By problem 13, we obtain $A' = PA_1 \cap QA_2 \in \mathcal{N}$, hence $R \in \mathcal{N}$.

Note that $\angle(PN_c, N_cN) = \angle(PN_c, N_cC) = \angle(PA_1, A_1C) = \varphi$. Similarly, $\angle(QN_b, N_bN) = -\varphi$. This means that the arcs NP and NQ of $\mathcal{N}$ are equal. $PR = QR$ follows from the proof of Problem 26. So RN is the perpendicular bisector of PQ, i.e. RN is a diameter of $\mathcal{N}$.

28. Let Q_a, Q_b, and Q_c be the reflections of Q in BC, CA, and AB, respectively. Let PQ_a, PQ_b, PQ_c intersect BC, CA, AB at A^*, B^*, C^*, respectively. Then P is the circumcenter of Q_a, Q_b, Q_c, hence $PA^* + QA^* = PB^* + QB^* = PC^* + QC^*$. So there is an ellipse with foci P and Q passing through A^*, B^*, C^*. It is tangent to the sides of $\triangle ABC$ by problem 16.

29. Since $\angle(PA_1, BC) = \angle(BC, QA_2) = \varphi$, A_1A_2YX is inscribed in $\mathcal{R}$ and A_1A_2YX is symmetric in the common perpendicular bisector of XY and A_1A_2 (in particular, $XY \parallel A_1A_2$. To prove that XY is tangent to $\mathcal{K}$ it suffices to show that the center of $\mathcal{K}$ (that is the midpoint of PQ lies on the midline of A_1A_2YX), or, equivalently, to show that $PA_1 = QY$. From previous we know that $\angle(PR, QR) = 2\varphi$ and $RP = RQ$. Since $\angle(A_1R, RY) = 2\angle(A_1A_2, A_2Y) = 2\varphi$, we have $\angle(PR, A_1R) = \angle(QR, YR)$, and by SAS, triangles PRA_1 and QRY are congruent. Thus $PA_1 = QY$ follows.

30. For the circle $\mathcal{R}$, point P and angle φ consider the Brocard ellipse that is a conic touching the lines PZ rotated by φ around $Z \in \mathcal{R}$ (see the Problem 19c, inverse statement). For 6 positions of Z (A_1 and X from Problem 29 and 3 analogous pairs) the corresponding tangents to the Brocard ellipse also touch $\mathcal{K}$. Thus $\mathcal{K}$ is the Brocard ellipse. Now the statement follows from the Problem 19b and its inverse.

31.

a) We put this situation into a general case from the solution of Problem 26 (generalized pedal triangles).

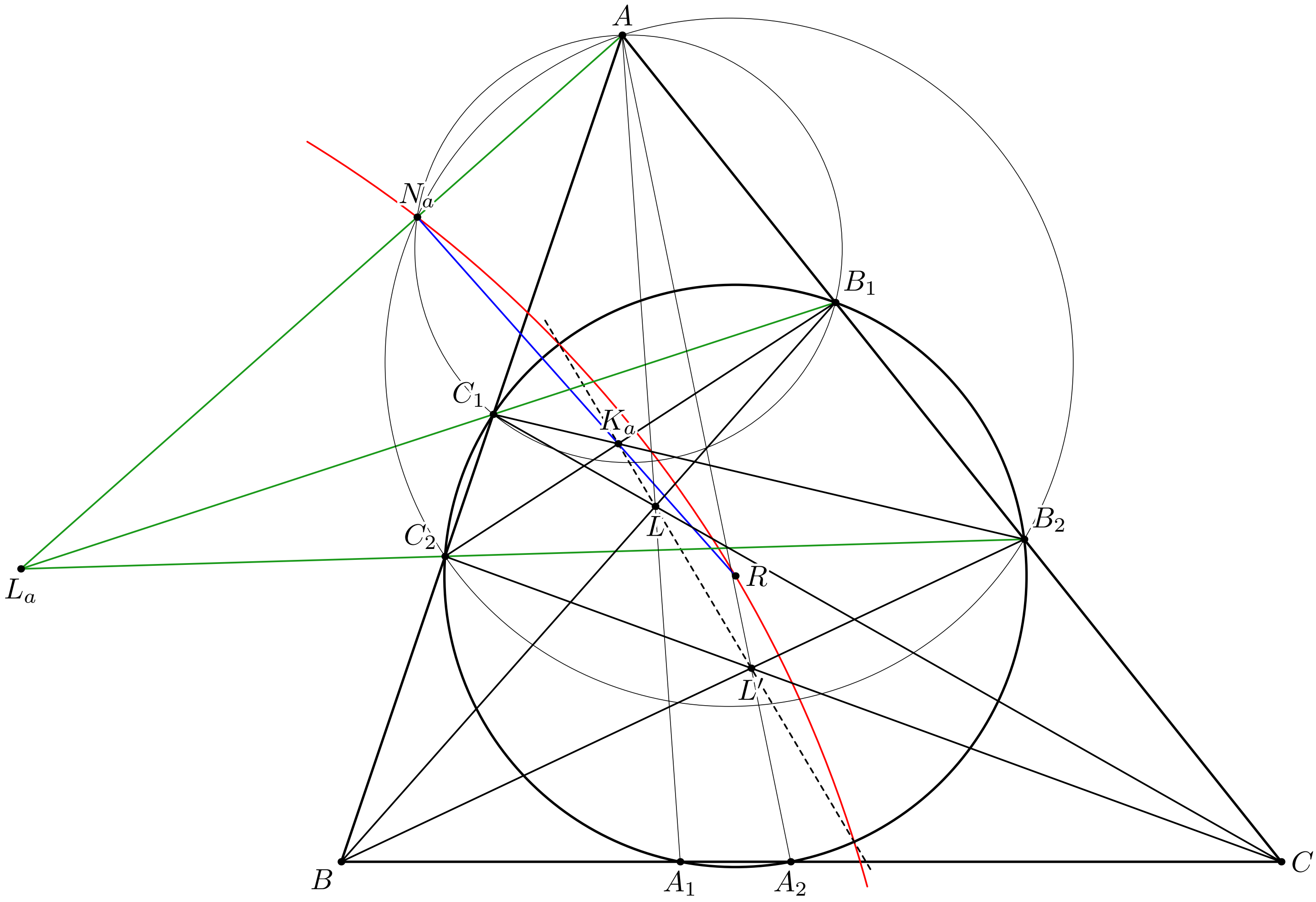
For P and Q being the Brocard points, ABC is a particular case of generalized pedal triangles with $B = A_1$, $C = B_1$, $A = C_1$ and $B = C_2$, $C = A_2$, $A = B_2$. Thus in this case $O = R$, and we know that $PO = OQ$ with $\angle(PO, OQ) = 2\varphi$, where $90^\circ - \varphi$ is the Brocard angle.

Now let L be Lemoine point. Though L draw a line B_2C_1 ($B_2 \in AC$, $C_1 \in AB$) so that B, C, B_2, C_1 are concyclic. From $ABC \sim AB_2C_1$ it follows that AL is the median in AB_2C_1, hence $B_2L = C_1L$. Similarly construct C_2A_1, A_2B_1. We have $\angle(LB_2, AC) = \angle(AB, BC) = \angle(AC, LB_1)$. It follows $LB_2 = LB_1$. Thus all 6 segments LA_1, LA_2, LB_1, LB_2, LC_1, LC_2 are equal, and $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle (known as Taylor circle) centered at $R = L$. Now we will show that points P and Q (from general construction of Problem 26) are Brocard points. (Note that $LP = LQ$). Since $LA_1 = LA_2 = LC_2$, We have $A_2C_2 \perp BC$. Similarly, $B_2A_2 \perp CA$, and $C_2B_2 \perp AB$. Now from the circles (AB_2C_2Q), (BC_2A_2Q), (CA_2B_2Q) we have: $y = \angle(C_2A_2, A_2Q) = \angle(C_2B, BQ) = 90^\circ - \angle(QC_2, C_2B) = \angle(B_2C_2, C_2Q) = \angle(B_2A, AQ)$, and similarly, $y = \angle(A_2C, CQ)$. This means that Q is the Brocard point (with y as the Brocard angle), and $(A_2B_2C_2)$ is its generalized pedal circle corresponding to $\varphi = \angle(QC_2, AB) = 90^\circ - y$. Hence $\angle(PL, LQ) = 2\varphi$. Thus P, Q, L, O are concyclic with P and Q symmetric in OL.

b) Let AS be the symmedian, so that $BS : CS = c^2 : b^2$. It suffices to show that $PBS \sim QCS$, or $PB : QC = c^2 : b^2$; from this it follows that $\angle(PS, SB) = \angle(CS, SQ)$.

Now from the sine law, $\frac{PB}{\sin y} = \frac{c}{\sin APB} = \frac{c}{\sin B}$. Similarly, $\frac{QC}{\sin y} = \frac{b}{\sin C}$. Dividing the first equality by the second one, we get the required similarity.

32* It is not hard to see that the lines AA_2, BB_2, and CC_2 are concurrent at some point L'. Let $K_a = B_1C_2 \cap B_2C_1$, points K_b and K_c are defined similarly. By the Pappus theorem the points L, L', and K_a are collinear. It suffices to show that K_a lies on the radical axis of $\mathcal{R}$ and $\mathcal{N}$ (a similar argument then implies that K_b and K_c also lie on this radical axis). The lines B_1C_1, B_2C_2, and AN_a are concurrent, since they are the radical axes of circles $(AB_1C_1N_a)$, $(AB_2C_2N_a)$, and $\mathcal{R}$. So AN_aL_aN is the polar line of K_a with respect to $\mathcal{R}$. Moreover $N_a \in \mathcal{N}$ and NR is the diameter of $\mathcal{N}$, hence $RN_a \perp AN_a$. Thus the inversion in $\mathcal{R}$ maps K_a to N_a. This inversion maps the line $B_2C_1K_a$ to the circle $RB_2C_1N_a$. Therefore $K_aC_1 \cdot K_aB_2 = K_aR \cdot K_aN_a$. It follows that K_a lies on the radical axis of $\mathcal{R}$ and $\mathcal{N}$.



References

[1] A.Akopyan, A.Zaslavsky, *Geometry of Conics.*

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