Turán type results for distance graphs*

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Abstract

The classical Turán theorem determines the minimum number of edges in a graph on n vertices with independence number α. We consider unit-distance graphs on the Euclidean plane, i.e., graphs $G = (V, E)$ with $V \subset \mathbb{R}^2$ and $E = \{\{\mathbf{x}, \mathbf{y}\} : |\mathbf{x} - \mathbf{y}| = 1\}$, and show that the minimum number of edges in a unit-distance graph on n vertices with independence number $\alpha \leqslant \lambda n$, $\lambda \in \left[\frac{1}{4}, \frac{2}{7}\right]$, is bounded from below by the quantity $\frac{19-50\lambda}{3}n$, which is several times larger than the general Turán bound and is tight at least for $\lambda = \frac{2}{7}$.

Key words: Turán theorem, independence number, distance graphs.

1 Introduction

The classical Turán theorem proved in [8] can be formulated as follows.

Theorem 1. *The minimum number of edges in a graph on n vertices with independence number α is attained on a graph consisting of α pairwise disjoint cliques whose sizes differ at most by one.*

One of the most important classes of graphs arising from combinatorial geometry is that consisting of *distance graphs* $G = (V, E)$, where

$$V \subset \mathbb{R}^n, \quad E = \{\{\mathbf{x}, \mathbf{y}\} : |\mathbf{x} - \mathbf{y}| = 1\}.$$

On the one hand, distance graphs are naturally related to the famous Nelson–Hadwiger problem on the chromatic numbers of spaces, and so their chromatic numbers and their independence numbers are intensively studied (see [1], [6], [7]). On the other hand, multiple questions concerning the edge numbers in distance graphs go back to Erdős (see [1], [3], [6]).

In this paper, we study distance graphs on the plane. Our main goal is to prove a Turán type result for such graphs, that is to find a lower bound for the minimum number of edges in a distance graph in $\mathbb{R}^2$ given a number n of vertices and an independence number α. Before stating our main result it is worth noting that for distance graphs with n vertices, α cannot be arbitrary. It is definitely at least $0.2293n$ (see [2], [4], [5]). Moreover, a strong belief is that it is greater than or equal to $0.25n$. Anyway, given a sequence of graphs with growing sets of vertices, the independence numbers of these graphs are quite far from being constant: they are proportional to the numbers of vertices.

One of the main results of our paper is as follows.

*This work is done under the financial support of the following grants: the grant 15-01-00350 of Russian Foundation for Basic Research, the grant NSh-2964.2014.1 supporting Leading scientific schools of Russia.

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Theorem 2. *The minimum number of edges in a distance graph on n vertices with independence number $\alpha \leqslant \lambda n$, $\lambda \in \left[\frac{1}{4}, \frac{2}{7}\right]$, is bounded from below by the quantity $\frac{19-50\lambda}{3}n$.*

The result of Theorem 2 is much stronger than that of Theorem 1. If, for example, $\lambda = \frac{1}{4}$, then Theorem 1 gives $1.5n$ edges. In the same case, Theorem 2 gives at least $\frac{13}{6}n$ edges. If, in turn, $\lambda = \frac{2}{7}$ and n is divisible by 7, then the classical bound is equal to $\frac{9}{7}n$, and our bound equals $\frac{11}{7}n$. Moreover, in this case, our bound is tight, since one can take disjoint copies of the so-called Moser spindle, which has 7 vertices, 11 edges and independence number 2 (see fig. 1).

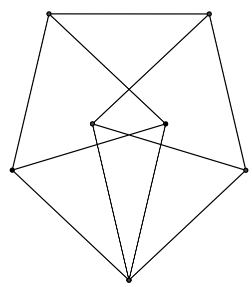

Figure 1: Moser's spindle

Finally, for $\lambda \geqslant \frac{1}{3}$, Turán's bound is trivially tight, since cliques on at most 3 vertices are distance graphs.

The paper is organized as follows. In Section 2, we give a more general setting of the problem and formulate another main result of the paper. In Section 3, we prove a "key lemma". In Section 4, we prove both main theorems of the paper. In Section 5, we give some discussion.

2 More general setting

Consider a graph $\Gamma = (W, E)$. We call *the configuration* of Γ the vector $(|W|, \alpha(\Gamma), |E|)$ and we denote it by $\mathrm{Config}(\Gamma)$.

Let V be a set of vectors (a, b, c) with non-negative integer coordinates a, b, c. Then we call *an extension* of the set V the set of vectors $(a, b+n, c+k)$, where $(a, b, c) \in V$ and n, k are again non-negative integers.

We say that a vector (a, b, c) is *good*, if it belongs to the extension of the set of all linear combinations with non-negative integer coefficients of the following vectors: $(1, 1, 0)$, $(2, 1, 1)$, $(3, 1, 3)$, $(4, 1, 9)$, $(5, 1, 15)$, $(6, 1, 22)$, $(7, 2, 11)$, $(n+1, 1, n(n-1))$, where $n \geqslant 6$.

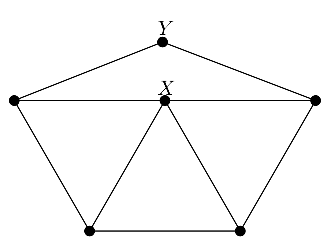

Figure 2: Semi-star graph

We call the graph, which is drawn on Figure 2, *semi-star* graph with *center* X and *top vertex* Y.

The following proposition is quite simple, and we omit its proof here.

Proposition 1. *Any distance graph is free of* K_4 *(complete graphs with four vertices),* $K_{3,2}$ *(complete bipartite graphs with part sizes 2 and 3), and semi-star graphs.*

We say that a graph is *correct*, if it does not contain either K_4 or $K_{3,2}$, or a semi-star graph as subgraphs. In particular, as we have just mentioned, every distance graph is correct. The second main result of our paper is given below.

Theorem 3. *The configuration of a correct graph is a good vector.*

The proof of Theorem 3 is based on

Key Lemma. *From any correct graph* Γ*, one can delete several vertices together with all the adjacent edges in such a way that for the remaining graph* Γ'*, the vector* $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma')$ *is good.*

In the next section, we prove Key Lemma. In Section 4, we deduce Theorems 2 and 3 from Key Lemma.

3 Proof of Key Lemma

3.1 Some preliminaries

We say that a vector $v = (v_1, v_2, v_3)$ *exceeds* a vector $w = (w_1, w_2, w_3)$, if $v_1 = w_1, v_2 \geqslant w_2, v_3 \geqslant w_3$, and we denote this relation by $v \succeq w$.

Proposition 2. *If* u, v *are vectors of dimension 3 such that* $u \succeq v$ *and* v *is good, then* u *is also good.*

The proposition is straightforward, and thus we prove Key Lemma, provided we show that the vector $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma')$, with an appropriate Γ', exceeds some good vector.

Proposition 3. *The vectors* $(1,1,0)$*,* $(2,1,1)$*,* $(3,1,3)$*,* $(4,1,9)$*,* $(5,1,15)$*,* $(7,2,11)$*,* $(8,2,18)$*,* $(6,1,22)$*,* $(12,3,26)$*,* $(6,2,9)$*,* $(10,2,30)$*,* $(4n,n,10n)$*,* $(4m{+}1,m,10m{+}3)$*,* $(k{+}1,1,k(k{-}1))$ *(*$n,m,k \in \mathbb{Z}$*;* $n,m \geqslant 3$*;* $k \geqslant 6$*) are good.*

Proof. The vectors

$$(1,1,0),\ (2,1,1),\ (3,1,3),\ (4,1,9),\ (5,1,15),\ (7,2,11),\ (6,1,22),\ (k+1,1,k(k-1)) \text{ for } k \geqslant 6$$

are good by definition. Now, we briefly explain what happens with the other vectors:

- $(8,2,18) = 2(4,1,9)$;
- $(12,3,26) = (5,1,15) + (7,2,11)$;
- $(6,2,9) \succeq (6,2,6) = 2(3,1,3)$;
- $(10,2,30) = 2(5,1,15)$;
- $(4n,n,10n) \succeq (4n,n,9n) = n(4,1,9)$;
- $(4m+1,m,10m+3) \succeq (4m+1,m,9m+6) = (m-1)(4,1,9) + (5,1,15)$ for $m \geqslant 3$.

□

Let A be a vertex of the minimum degree in Γ. Consider several cases depending on the value of $\deg A$.

3.2 Case of $\deg A = 0$

Remove the vertex A from Γ. Since A had no neighbours, we get $\text{Config}(\Gamma) - \text{Config}(\Gamma') = (1, 1, 0)$.

3.3 Case of $\deg A = 1$

Remove from Γ the vertex A and its unique neighbour B. Obviously the independence number is reduced by 1 and the number of edges is reduced at least by 1. Therefore, $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (2, 1, 1)$.

3.4 Case of $\deg A = 2$

Remove from Γ the vertex A and both its neighbours B and C. Note that the number of edges is reduced at least by 3, since AB, AC are removed and also some edge adjacent to B and different from AB is removed (2 is the minimum degree in this case). Now it is clear that $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (3, 1, 3)$.

3.5 Case of $\deg A = 3$

3.5.1 Preliminaries

Let B, C, D be the neighbours of A. Since G does not contain K_4, we may assume that the vertices B and D are not adjacent. Below we consider several variants of subgraphs induced on A, B, C, D:

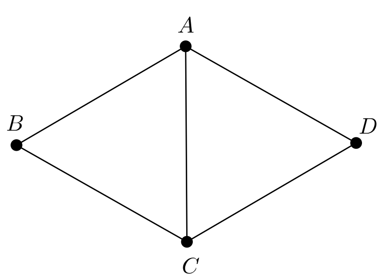

Figure 3: First variant

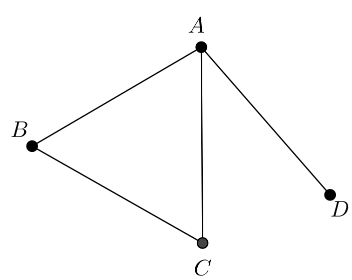

Figure 4: Second variant

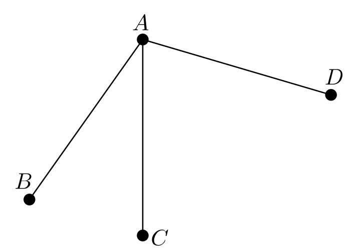

Figure 5: Third variant

3.5.2 Graph from Figure 3

Let us calculate possible total numbers of edges adjacent to B or D. If the number of such edges is 6 or 7, then we remove the vertices B and D and all the vertices adjacent to them. Since the vertices B and D are not adjacent and also they are not adjacent to any of the vertices of the remaining graph Γ', the independence number is reduced at least by 2. Moreover, since Γ is free of $K_{3,2}$, the vertices B and D do not have common neighbours different from A and C. Therefore, the number of vertices that have been removed is 6 or 7. Finally, since the degree of each vertex is at least 3, the total number of edges adjacent to the removed vertices is not less than 9 or 11, respectively. Thus, the vector $\text{Config}(\Gamma) - \text{Config}(\Gamma')$ exceeds the vectors $(6, 2, 9), (7, 2, 11)$, respectively.

If the number of edges adjacent to B or D is at least 8, then we remove the vertices A, B, C, D. The independence number is reduced at least by 1, since the vertex A is not adjacent to any of the remaining vertices. The number of edges is reduced, in turn, at least by 9, for at least 8 edges adjacent to B or D are removed and also the edge AC is deleted. Thus, $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (4, 1, 9)$.

3.5.3 Graph from Figure 4

Let us look at possible total numbers of edges adjacent to B or D. If the number of such edges is 6, then we remove the vertices B and D and all the vertices adjacent to them. Since the vertices B and D are not adjacent and also they are not adjacent to any of the vertices of the remaining graph Γ', the independence number is reduced at least by 2. The number of vertices that have been removed is 6 or 7. Since the degree of any vertex is at least 3, the total number of edges adjacent to the removed vertices is at least 9 or 11, respectively. Thus, the vector $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma')$ exceeds the vectors $(6, 2, 9), (7, 2, 11)$, respectively.

If the number of edges adjacent to B or D is 7 or larger, then we remove the vertices A, B, C, D. The independence number is reduced at least by 1, since the vertex A is not adjacent to any of the remaining vertices. The number of edges is reduced, in turn, at least by 9, for at least 7 edges adjacent to B or D are removed and also at least two more edges adjacent to C are deleted. Thus, $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma') \succeq (4, 1, 9)$.

3.5.4 Graph from Figure 5

Just remove the vertices A, B, C, D. Since the degree of each vertex B, C, D is at least 3 and these vertices are pairwise non-adjacent, the number of removed edges is at least 9. As usual, the independence number is reduced at least by 1, and therefore $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma') \succeq (4, 1, 9)$.

3.6 Case of $\deg A = 4$

3.6.1 Preliminaries

Let B, C, D, E be the vertices adjacent to A. Consider a subgraph induced on B, C, D, E. Note that it cannot have a vertex of degree 3, since otherwise by adding the vertex A we get $K_{3,2}$, which is forbidden. Also, the absence of $K_{3,2}$ yields that among B, C, D, E, there are no 4-cycles. Finally, the absence of K_4 yields, in turn, that among B, C, D, E, there are no 3-cycles (triangles). Thus, only the following 5 variants are possible for a graph on the vertices A, B, C, D, E (see fig. 6–10).

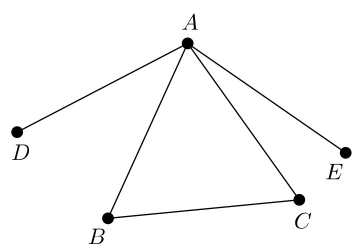

Figure 6: First variant

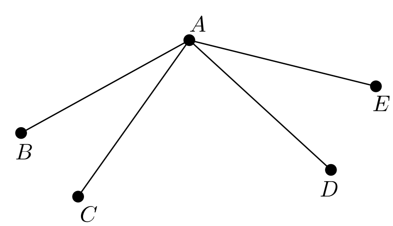

Figure 7: Second variant

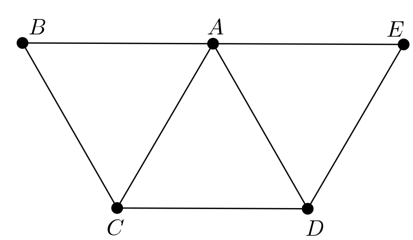

Figure 8: Third variant

3.6.2 Graphs from Figures 6 and 7

Remove the vertices A, B, C, D, E. The independence number is reduced at least by 1. The number of edges is reduced at least by 15, since any vertex among B, C, D, E is of degree at least 4 and at most 1 edge is calculated twice. Therefore, $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma') \succeq (5, 1, 15)$.

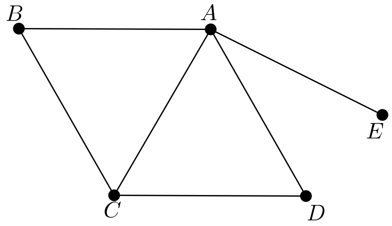

Figure 9: Fourth variant

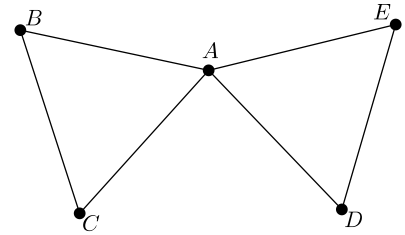

Figure 10: Fifth variant

3.6.3 Graph from Figure 8

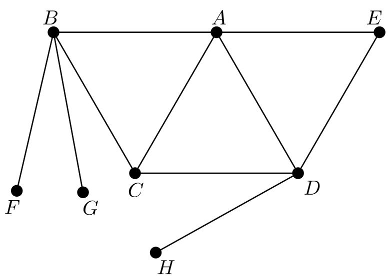

Figure 11:

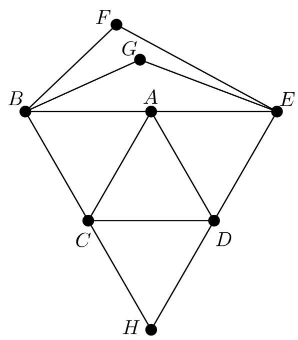

Figure 12:

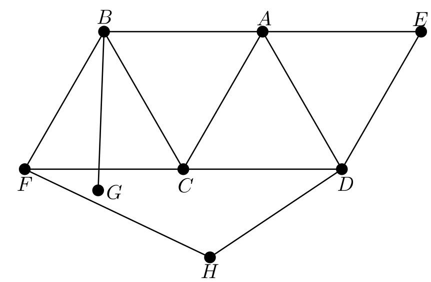

Figure 13:

First, assume that the vertices B, D are both of degree 4. Denote by F, G the vertices adjacent to B and different from A and C. Also, denote by H the fourth vertex adjacent to D. The vertex H does not coincide either with F or with G, since otherwise B and D share three neighbours and we obtain a $K_{3,2}$. Remove the 8 vertices A,B,C,D,E,F,G,H (see fig. 11). The independence number is reduced at least by 2, since the vertices B and D are neither adjacent one to the other, nor adjacent to any of the remaining vertices.

Let us prove that the number of removed edges is greater than or equal to 18. The sum of the degrees of the vertices A,B,C,D,E,F,G,H is at least 32. If we show that the number of edges in a subgraph on the vertices A,B,C,D,E,F,G,H is at most 14, then we are done.

Some 10 edges are drawn on fig. 11. Moreover, all the edges adjacent to A, B, D are indicated there. Let us prove that among the vertices C, E, F, G, H, there are at most 4 edges. Since the vertices B, C have no more than 2 common neighbours, the edges CF, CG cannot appear simultaneously. Without loss of generality, assume that there is no CG.

Since C and E have at most 2 common neighbours, the edges CH and EH cannot appear simultaneously.

If the edge $EF(EG)$ is present as on fig. 12, then the vertices $A, B, C, D, E, F(G)$ form a semi-star graph with center A and top vertex $F(G)$. Therefore, the graph Γ does not have edges EF and EG.

If in Γ, the edges CF and FH appear simultaneously (see fig. 13), then the vertices A, B, C, D, F, H form a semi-star graph with center C and top vertex H.

The edge CE is absent due to the construction of the subgraph on the vertices A, B, C, D, E. So only the pairs of vertices $(F, G), (G, H)$ remain, which can form the third and the fourth edges of the subgraph on the vertices C, E, F, G, H. Thus, we really get the bound 14 for the number of edges in the subgraph on the vertices A, B, C, D, E, F, G, H, and we eventually have that $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (8, 2, 18)$.

Recall that we assumed that the vertices B, D were both of degree 4. Of course, if the same is true for C, E, then again $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (8, 2, 18)$.

Thus, assume that there exists a vertex of degree at least 5 both among B, D and C, E. In this case, remove the vertices A, B, C, D, E. The independence number is reduced at least by 1. The number of edges is, in turn, reduced at least by 15, since the sum of the degrees of the vertices B, C, D, E is at least 18 and there are only 3 edges between these vertices. Finally, $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (5, 1, 15)$.

3.6.4 Graph from Figure 9

Divide the argument into two parts roughly in the same way as it was done in the previous case. Namely, either the degrees of both B and D equal 4, or at least one among B, D has at least 5 neighbours. The second situation is much simpler, as before, so let us start here with it. Indeed, remove the vertices A, B, C, D, E. The independence number is reduced at least by 1. The number of edges is, in turn, reduced at least by 15, since the total number of edges adjacent to the vertices B, C, D, E is not less than 17 and only 2 of them were calculated twice. Thus, $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (5, 1, 15)$.

Now, assume that both B, D are of degree 4. We proceed like in Subsection 3.6.3. Since the vertices B and D cannot have 3 common neighbours (due to the absence of $K_{3,2}$), they have exactly 2 such neighbours — A and C. So we can denote by F, G the two other vertices adjacent to B and by H, I — the two other vertices adjacent to D (see fig. 14).

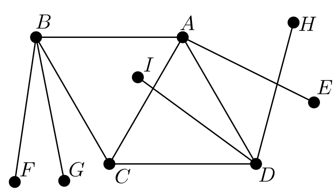

Figure 14:

Let us prove, as in Subsection 3.6.3, that removing some 8 vertices (namely, A, B, C, D, F, G, H, I) gives us the bound $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (8, 2, 18)$. Of course, we just need to show that here again the number of edges is reduced at least by 18, and to this end we need to analize the structure of a subgraph on the vertices A, B, C, D, F, G, H, I and to see that the number of edges in this subgraph is at most 14. This seems to be very similar to what was done earlier. However, there are important subtleties: actually, either that is true, or we come back to a previously considered situation.

Since the graph Γ is free of $K_{3,2}$, among CF, CG as well as among CH, CI, at most one edge is present in Γ. Without loss of generality, assume that the edges CG, CI are absent.

If among CF, FG both edges are drawn, then we come back to the situation from fig. 8 with the vertices B, A, C, F, G. Analogously, if among CH, HI both edges are drawn, then we come back to the situation from fig. 8 with the vertices D, A, C, H, I. Therefore, we may assume that among CF, FG, CH, HI at most two edges are present.

Furthermore, Γ is free of $K_{3,2}$ and thus among FH, FI, GH, GI we have at most 3 edges.

Summing up all the above inequalities, we see that a subgraph on the vertices C, F, G, H, I has at most 5 edges, which means that we do really have the bound by 14 for the number of edges in a subgraph on the vertices A, B, C, D, F, G, H, I. The case is complete.

3.6.5 Graph from Figure 10

If the degree of a vertex among B, C, D, E is at least 5, then we remove A, B, C, D, E. It is already clear that $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma') \succeq (5, 1, 15)$. Let us discuss the opposite case.

We need some new definitions. Let a vertex of a graph satisfy the three following conditions: it is of degree 4; each of its neighbours is of degree 4; the configuration of the neighbours is the same as the one of the vertex A on fig. 10. We call such vertex *a key vertex*. If all the vertices of a graph are key vertices, then we call *key graph* the graph itself.

Proposition 4. *If in a connected graph, there is a vertex of degree 4 and all the vertices of degree 4 are key ones, then the graph is key.*

All the cases, in which a graph Γ has a non-key vertex of degree 4, are already considered. Thus, it remains to analize the case of a key graph.

Lemma 1. *Any key graph contains a cycle of length at least 4.*

Proof. Take a vertex A in a key graph and suspend the graph on A. Let the level of A be 0. Let U be a vertex of the maximum level and V be a vertex of the previous level adjacent to U. Let W be a common neighbour of U and V. Let X be a vertex adjacent to V and different from U and W. Consider paths from U to A and from X to A, in which the level of any vertex is by 1 smaller than the level of the preceeding vertex. Since obviously the level of V is greater than 1, the vertices U and X do not coincide with A. Let B be the first common point of the paths UA, XA. Then since U and X are not adjacent and their levels differ at most by 1, they do not coincide with B. Therefore, the cycle $UBXV$ (UB, BX denote paths, whereas UV, VX denote edges) consists of at least 4 edges, which completes the proof. □

Take a key graph Γ. Consider its shortest cycle of length greater than 3. Note that if two vertices in the cycle are not consecutive, then they cannot be adjacent. Indeed, otherwise, if the length of the cycle exceeds 4, then we would get a cycle, which is shorter than the initial one, although its length would be still greater than 3; if the length of the initial cycle is, in turn, exactly 4, then the existence of an edge inside the cycle would contradict our assumption that all the vertices are key ones.

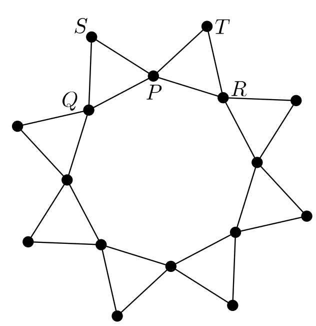

Figure 15: A minimum cycle of length greater than 3 in a key graph

Let us analize the vertices, which are adjacent to the cycle. Let P be a vertex of the cycle. Denote by Q and R its neighbours in the cycle. Let S, T be the two other neighbours of P. Clearly among Q, R, S, T we have two pairs of adjacent vertices and they are not among $(Q, R), (S, T)$. Without loss of generality, we assume that they are $(Q, S), (R, T)$. Now, consider the vertices adjacent to Q. These are of course P, S and two more vertices that are also adjacent one to the other, but not adjacent to P, S: one of these vertices belongs to the cycle. Taking the next vertex of the cycle and proceeding the same way we see that all the edges coming out from the vertices of the cycle look like on fig, 15 (an example with 8 vertices). Here any two vertices adjacent to some two different vertices of the cycle do not coincide, since otherwise either they are not key ones, or there is a shorter cycle of lenght exceeding 3.

Consider different cases as on fig. 16–19.

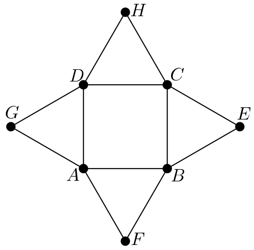

Figure 16:

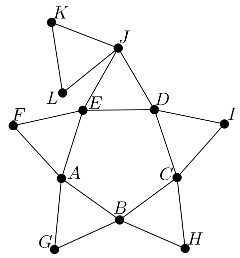

Figure 17:

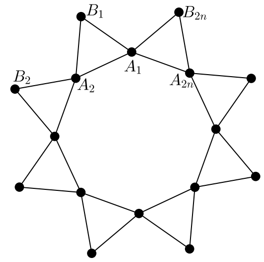

Figure 18:

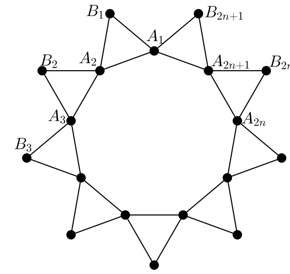

Figure 19:

Cycle of length 4 (fig. 16) Among the edges, which are not drawn on the picture, only the edges EG and FH might belong to the graph Γ. Therefore, the number of edges in a subgraph on the vertices A, B, C, D, E, F, G, H is at most 14. Remove the vertices A, B, C, D, E, F, G, H. The number of edges is reduced at least by 18, since, as usual, the total number of edges adjacent to the removed vertices is 32 and at most 14 edges are counted twice. The independence number is reduced at least by 2, since the vertices A and C are not adjacent one to the other as well as they are not adjacent to any of the remaining vertices. Thus, $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (8, 2, 18)$.

Cycle of length 5 (fig. 17) Among the edges, which are not drawn on the picture, only some two edges from K and some two edges from L may belong to the graph, since otherwise a cycle of length 4 appears. Remove the 12 vertices $A, B, C, D, E, F, G, H, I, J, K, L$. The number of removed edges is at least $48 - 22 = 26$. The independence number is reduced at least by three due to the vertices A, C, J. Thus, $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (12, 3, 26)$.

Cycle of length $2n$, $n \geqslant 3$ (fig. 18) Let the cycle consist of vertices $A_1, \dots, A_{2n}$, and let $B_1, \dots, B_{2n}$ be the vertices outside the cycle adjacent to the vertices of the cycle. Note that all possible edges are drawn on the picture, since otherwise there is a cycle of length strictly greater than 3, but strictly smaller than $2n$. Remove the vertices $A_1, \dots, A_{2n}, B_1, \dots, B_{2n}$. The independence number is reduced at least by n due to the vertices $A_2, A_4, \dots, A_{2n}$. The number of edges is reduced at least by $16n - 6n = 10n$. Thus, $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (4n, n, 10n)$.

Cycle of length $2n+1$, $n \geqslant 3$ (fig. 19) Let the cycle consist of vertices $A_1, \dots, A_{2n+1}$, and let $B_1, \dots, B_{2n+1}$ be the vertices outside the cycle adjacent to the vertices of the cycle. Note that, as in the previous case, all possible edges are drawn on the picture. Remove the vertices $A_1,\dots,A_{2n+1},B_1,\dots,B_{2n}$. The independence number is reduced at least by n due to the vertices $A_2, A_4, \dots, A_{2n}$. The number of edges is reduced at least by $(16n+4) - (6n+1) = 10n+3$. Thus, $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (4n+1, n, 10n+3)$.

3.7 Case of $\deg A = 5$

Let B, C, D, E, F be the vertices adjacent to A. If a subgraph on the vertices B, C, D, E, F contains a 3-cycle, then, with the addition of the vertex A, a K_4 appears. In case of a 4-cycle, we get a $K_{3,2}$. Finally, with a 5-cycle, we obtain a semi-star graph. Therefore, there are no cycles on the vertices B, C, D, E, F, which means that the number of edges in this subgraph is at most 4. Also, the absence of $K_{3,2}$ yields that in the subgraph on the vertices B, C, D, E, F there are no vertices of degree 3. Thus, 4 edges can be drawn only as on fig. 20.

If the number of edges in a graph on the vertices B, C, D, E, F is bounded by 3, then the subgraph on the vertices A, B, C, D, E, F has at most 8 edges. Remove these vertices. As usual, the number of the removed edges is at least $30 - 8 = 22$. Thus, $\text{Config}(\Gamma) - \text{Config}(\Gamma') \succeq (6, 1, 22)$.

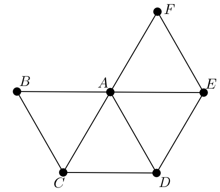

Figure 20:

If the number of edges in a graph on the vertices B, C, D, E, F is exactly 4, then the subgraph on the vertices A, B, C, D, E, F has 9 edges (see fig. 20). Call the vertex A *a support vertex*, if each of the vertices B, C, D, E, F is of degree 5.

If A is not a support vertex, then remove the vertices A, B, C, D, E, F. Clearly in this case, the sum of the degrees of the removed vertices is at least 31. Thus, the number of the removed edges is not less than 22 and we have $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma') \succeq (6, 1, 22)$.

Let A be support. Since Γ is $K_{3,2}$-free, the vertices C and E have no other common neighbours than A and D. Since the vertices C, E are of degree 5, let G, H be the vertices adjacent to C and let I, J be the vertices adjacent to E (see fig. 21).

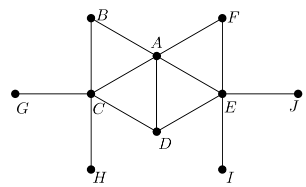

Figure 21:

Let us prove that the number of edges in a subgraph on the vertices on fig. 21 does not exceed 20.

On Figure 21, 13 edges are drawn. Moreover, for the vertices A, C, E, all the adjacent edges are indicated there. So it remains to show that a subgraph on the vertices B, D, F, G, H, I, J has at most 7 edges.

Since in the graph on fig. 20 all the edges between the vertices A, B, C, D, E, F are present, the edges BD, BF, DF do not belong to Γ. Furthermore, since Γ does not contain a semi-star, it does not have any of the edges BI, BJ, FG, FH. Also Γ is $K_{3,2}$-free, which means, in particular, that Γ cannot contain more than one edge in each of the following pairs: $(BG, BH), (DG, DH), (DI, DJ), (FI, FJ)$. Since the edges BG, DG cannot be present in Γ simaltaneously, we may assume without loss of generality that Γ does not contain the edges BH and DG. Similarly, let us assume that Γ does not contain the edges DJ and FI.

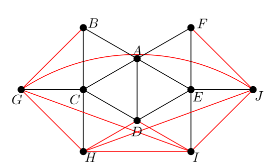

Figure 22:

Only 10 edges remain that are colored red on fig. 22. Suppose that, in contrast to what we want to prove, one can keep some 8 red edges in such a way that the subgraph on the vertices

$$A, B, C, D, E, F, G, H, I, J$$

stay correct. Since Γ does not contain a semi-star graph, among the edges BG, GH, DH, only at most 2 can be drawn. Also, at most 2 edges are among DI, IJ, FJ. Therefore, if among the red edges, at least 8 are in Γ, then Γ contains the edge HI. If Γ contains the edge DH, then the vertices A, C, D, E, H, I form a semi-star graph with center D and top vertex I. Similarly, the edge DI is not in Γ. Once again, since we have at least 8 edges in Γ, we have in Γ the edges GH, GI, GJ, HJ, IJ. This eventually gives us a K_4 on the vertices G, H, I, J leading to a contradiction.

Thus, we have finally shown that the number of edges on the vertices $A, B, C, D, E, F, G, H, I, J$ is at most 20. Remove these vertices. The number of the removed edges is at least $50 - 20 = 30$. The independence number is reduced at least by 2, since the vertices C, E are not adjacent one to the other. So $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma') \succeq (10, 2, 30)$.

3.8 Case of $\deg A = n \geqslant 6$

Let $B_1, \ldots, B_n$ be the vertices adjacent to A. If in a subgraph on the vertices $B_1, \ldots, B_n$, there is a vertex of degree at least 3, then we obtain a $K_{3,2}$. Therefore, the maximum degree of a vertex in this subgraph is bounded by 2. So this subgraph has at most n edges. Then the number of edges in the subgraph on the vertices $A, B_1, \ldots, B_n$ does not exceed $2n$.

Remove the vertices $A, B_1, \ldots, B_n$. Clearly the independence number is reduced at least by 1 and the number of edges is reduced at least by $(n+1)n - 2n = n(n-1)$. Thus, $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma') \succeq (n+1, 1, n(n-1))$.

4 Proofs of the two main theorems

4.1 Proof of Theorem 3

Let us proceed by induction in the number of vertices.

Base of induction. Note that in cases 3.6–3.8 of Key Lemma definitely not all the vertices were being removed from the corresponding graphs Γ. And in cases 3.2–3.5 at most 7 vertices were being removed. So we may consider here all the graphs on at most 7 vertices.

Let us call the graph from Theorem 1 the *α, n-Turán graph.* Note that for $\alpha \geqslant \frac{1}{3}n$, the α, n-Turán graph is the disjoint union of K_3, K_2 and K_1, and so it is correct and its configuration is good.

Consider all possible pairs (α, n), where $\alpha \leqslant n \leqslant 7$. For all such pairs, but

$$(1, 4), (1, 5), (1, 6), (1, 7), (2, 7),$$

we have $\alpha \geqslant n/3$, which has been just discussed. For the pairs $(1, 4), (1, 5), (1, 6), (1, 7)$, the only corresponding graphs are the complete graphs on 4,5,6,7 vertices. They are of course not correct.

Only one case of $\alpha = 2, n = 7$ remains. Consider a vertex of the minimum degree in any such correct graph. Remove it and all its neighbours. The new graph is correct, and its independence number is at most 1. Therefore, it has no more than 3 vertices. This means that at least 4 vertices were removed, and so the above-considered vertex had at least 3 neighbours. Thus, each vertex in the graph has degree greater than or equal to 3, and consequently the number of edges is bounded from below by $\frac{7 \cdot 3}{2}$, that is, it is at least 11.

The base of induction is proved.

Inductive step. Apply Key Lemma and remove from graph Γ some of its vertices in such a way that the vector $\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma')$ is good. Then by the induction hypothesis, the vector $\mathrm{Config}(\Gamma')$ is good. Since the sum of good vectors is good, the vector $\mathrm{Config}(\Gamma) = (\mathrm{Config}(\Gamma) - \mathrm{Config}(\Gamma')) + \mathrm{Config}(\Gamma')$ is good, too. $\square$

4.2 Proof of Theorem 2

Lemma 2. *If a vector (u, v, w) is good, then it exceeds the vector $(u, v, \frac{19}{3}u - \frac{50}{3}v)$*

Proof. Let us check the lemma for the "basis" vectors:

$$(1,1,0) \succeq (1,1,-31/3), (2,1,1) \succeq (2,1,-12/3), (3,1,3) \succeq (3,1,7/3),$$

$$(4,1,9) \succeq (4,1,26/3), (5,1,15) \succeq (5,1,15), (6,1,22) \succeq (6,1,64/3),$$

$$(7,2,11) \succeq (7,2,11), (n+1,1,n(n-1)) \succeq (n+1,1,19/3n-31/3),\ n \geqslant 6.$$

The last series of inequalities holds true, since for $n = 6$, we have $(7,1,30) \succeq (7,1,83/3)$ and if n increases by 1, then the third coordinate in the left-hand side increases by $2n$ and the third coordinate in the right-hand side increases by 19/3.

Suppose the lemma is true for some vectors u, v. Of course the relations $a \succeq c, b \succeq d$ yield the relation $a + b \succeq c + d$. Then for $u + v$, the lemma is also true. The same type of argument can be used for any λu, where λ is a positive constant. Finally, the relation "$\succeq$" is transitive. Thus, the lemma is true for all good vectors. $\square$

It follows from the lemma that the configuration of our graph Γ exceeds the vector $(n, \lambda n, \frac{19-50\lambda}{3}n)$, and, therefore, the number of edges in our graph is really greater than or equal to $\frac{19-50\lambda}{3}n$.

5 Some comments

In order to prove the main results, we used the fact that in any distance graph on the plane, there are no K_4, $K_{3,2}$ and semi-stars. A natural question arises: maybe one could use only one or two of these forbidden graphs and get the same result?

First, assume that only K_4 and semi-stars are forbidden. In this case, one can prove the following result.

Theorem 4. *The minimum number of edges in a graph on n vertices with independence number $\alpha \leqslant \lambda n$, $\lambda \in \left[\frac{1}{4}, \frac{2}{7}\right]$, and without K_4 and semi-stars is bounded from below by the quantity $\frac{17-43\lambda}{3}n$.*

This result is a bit worse than the one of Theorem 2. For example, if $\lambda = \frac{1}{4}$, then Theorem 4 gives the bound by $\frac{25}{12}n$ instead of $\frac{26}{12}n$ following from Theorem 2.

The proof of Theorem 4 is very close to the proof of Theorem 2. We do not present it in this paper because of its complete similarity to the above-given argument. We only list here a set of "good" vectors, which plays, in a proof, the same role as it was in Proposition 3:

$$(1,1,0), (2,1,1), (3,1,3), (4,1,9), (5,1,14), (6,1,20), (7,2,11), (8,2,17), (n+1,1,\frac{3n^2}{4}),\ n \geqslant 6,$$

$$(5,2,8), (6,2,9), (6,2,12), (7,2,14), (7,3,14), (8,3,16), (9,3,18), (10,3,20), (11,3,22).$$

Note that we do not claim that Theorem 4 cannot be improved further. However, for our proofs, $K_{3,2}$ appears to be important.

Now, assume that only K_4 is exluded. For simplicity, consider again the illustrative case of $\alpha \leqslant n/4$. We claim that in this case, the bound for the number of edges is $2n$ and this bound is *tight* for $n \equiv 0 \pmod 4$. If we are right, then of course semi-stars appear to be important as well: $2n$ is smaller than $\frac{25}{12}n$. So let us prove the claim. On the one hand, the graphs on fig. 23 show that $2n$ is the best possible bound under the current conditions.

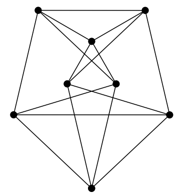

Figure 23: Graphs for the $2n$ bound

It is worth noting that the graphs on fig. 23 are not only K_4-free, but also $K_{3,2}$-free. Thus, $K_{3,2}$ is important only together with both K_4 and the semi-star graph. Of course, we see a semi-star graph on fig. 23.

On the other hand, let us show that the lower bound for the number of edges in a K_4-free graph with independence number at most one fourth of the number n of vertices is indeed $2n$. For more transparence, let us switch to the case when the number of vertices is $4n$ and the independence number is at most n. In this notation, we have to show that the number of edges is at least $8n$. As usual, we proceed by induction on n.

The case of $n = 1$ is obviously impossible: there are no graphs on 4 vertices without K_4, but with $\alpha = 1$. So let $n = 2$. Either each of the 8 vertices of a given graph is of degree at least 4, in which case the number of edges is indeed at least 16 (and thus the base of induction is proved), or there is a vertex of degree at most 3, and we will show below that in this case, one can remove 4 vertices from the graph reducing the independence number at least by 1 and the number of edges at least by 8: for $n = 2$, that is impossible, as we would again obtain a graph on 4 vertices without K_4, but with independence number 1. Therefore, we get the base of induction. To make this argument complete and to provide the induction step, we need the following lemma.

Lemma 3. *Let Γ be a graph with $4n$ vertices ($n \geqslant 2$), without K_4 and with $\alpha(\Gamma) \leqslant n$. Let A be a vertex of the minimum degree in Γ. Suppose $\deg A \leqslant 3$. Then one can remove 4 vertices from the graph reducing the independence number at least by 1 and the number of edges at least by 8.*

The induction step is obvious, so that it remains to prove the lemma.

Proof. Let us consider all possible values of $\deg A$.

Case of $\deg A = 0$. Remove the vertex A from Γ. Obviously in the new graph Γ' the independence number is smaller. However, we have not yet removed 8 or more edges. Consider Γ'. It has $4n-1$ vertices and $\alpha(\Gamma') \leqslant n-1$. Consequently, the chromatic number $\chi(\Gamma')$ is bounded from below by $\frac{4n-1}{n-1} > 4$. In other words, $\chi(\Gamma') \geqslant 5$. Of course this means that the maximum degree of a vertex in Γ' is greater than 3. It cannot be exactly equal to 4, since by Brook's theorem (we do not forget that Γ' is K_4-free) the chromatic number would be bounded by 4 from above. Thus, we have a vertex B of degree at least 5 in Γ'. Remove it. In the new graph Γ'', the number of vertices is $4n-2$, the independence number is at most $n-1$, and the number of edges is by at least 5 smaller than in the initial graph Γ. Since $\frac{4n-2}{n-1} > 4$, we apply once again the above argument and find a vertex C of degree at least 5. Removing C, we already get even more than we needed: the number of vertices is reduced by 3 (we promised 4). The number of edges is reduced by 10 (we promised 8). The independence number is reduced by 1 or more. The case is complete.

Cases of $\deg A \in \{1, 2\}$**.** Here the same procedure as in the first case applies. Let us consider only the case of degree 2. Remove the vertex A and both its neighbours B, C. We removed 3 vertices and at least 3 edges (2 is the *minimum* degree of a vertex). The independence number is already reduced. In the new graph, we have $4n - 3$ vertices, and since $\frac{4n-3}{n-1} > 4$, we find a vertex D of degree 5. We remove it, and we are done.

Case of $\deg A = 3$**.** Let B, C, D be the neighbours of A. We do not forget that the degrees of these vertices are at least 3 each. Since K_4 is forbidden, we may assume that BD is not in our graph. One can easily check that if in addition some of the edges BC, CD is absent or the degree of at least one vertex among B, C, D is strictly greater than 3, then the total amount of edges adjacent to A, B, C, D is at least 8. Thus, it suffices to remove the vertices A, B, C, D.

It remains to consider the case when the degrees of the vertices B, C, D are all exactly equal to 3 and both edges BC and CD are in the graph. In this case, the vertex B has one more neighbour E. Remove from the graph the vertices A, B, C, E, and we are done.

□

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