Distance graphs and Turán's theorem

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1 Definitions

Let $G = (V, E)$ be a graph without loops, multiple edges and orientation. *A clique* in G is any complete subgraph. Single vertex or single edge are also cliques. *The clique number* of graph G denoted by $\omega(G)$ is the maximal integer k such that G contains a clique on k vertices. *An independent set* is a set of vertices in G such that no two of the vertices form an edge. It is an "anticlique" in a sense. Single vertex is not only a clique but an independent set too. Accordingly, *an independence number* of graph G is the maximal integer k such that G contains an independent set of k vertices. It is denoted by $\alpha(G)$. And finally, *the chromatic number* of graph G is the minimal number $\chi(G)$ of colors for which one can color vertices of graph in these colors so that the endpoints of any edge have different colors.

2 Problems, I

2.1 Exercises

Problem 1. Prove that $\chi(G) \geqslant \omega(G)$.

Problem 2. Prove that $\chi(G) \geqslant \frac{|V|}{\alpha(G)}$.

Problem 3. Let $\Delta(G)$ be the maximum degree of vertices of graph G. Prove that $\chi(G) \leqslant \Delta(G) + 1$.

Brooks' theorem (without proof). *If connected graph G is neither a complete graph nor a simple cycle (non self-intersecting) of odd length, then $\chi(G) \leqslant \Delta(G)$.*

2.2 Turán's theorem

Problem 4. Let $G = (V, E)$and $|V| = n$. Prove that if $\omega(G) < 3$ (in other words, the graph does not contain triangles) then the number of edges in G is at most $\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil$. Prove that this upper bound is sharp (i.e. can not be increased).

Problem 5. Prove that problem 4 is equivalent to the following statement. Let $G = (V, E)$ and $|V| = n$. Prove that if $\alpha(G) < 3$ then the number of edges in G is at least

$$C_n^2 - \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil,$$

and this lower bound is sharp.

Problem 6 (Turán's theorem). Let $G = (V, E)$ and $|V| = n$. Prove that if $\alpha(G) \leqslant k$ then the number of edges in G is at least

$$n \cdot \left[\frac{n}{k}\right] - k \cdot \frac{\left[\frac{n}{k}\right]\left(\left[\frac{n}{k}\right] + 1\right)}{2},$$

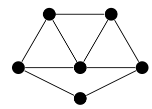
and this lower bound is sharp.

2.3 Distance graphs in the palne

A *distance graph on the plane* or *graph of distances on the plane* is a graph such that its vertices are some points of the plane and edges are all pairs of points at distance 1.

Problem 7. Prove that distance graphs do not contain subgraphs K_4 (complete graphs on 4 vertices).

Problem 8. Prove that distance graphs do not contain subgraphs $K_{2,3}$ (complete bipartite graphs with parts of 2 and 3 vertices).

Problem 9. Prove that distance graphs do not contain subgraphs $W =$  .

Problem 10. Do note confuse distance graphs and planar graphs (the latter can be drawn on the plane in such a way that its edges intersect only at their endpoints). Give examples of non-planar distance graph and planar but non-distance graph. You may use Kuratowski's criterion without proof.

2.4 Turán's theorem for distance graphs on the plane

Problem 11. Let $G = (V, E)$ have $4n$ vertices and $\alpha(G) \leqslant n$. In this case $|E| \geqslant 6n$ by Turán's theorem. Prove that if G is a distance graph on the plane then the stronger inequality $|E| \geqslant 7n$ holds. Use the result of problem 7.

Next problems strengthen the inequality of problem 11 by applying the result of problem 7 only.

Problem 12. Let graph $G = (V, E)$ (not necessarily being a distance graph) has $4n$ vertices. Assume that $\alpha(G) \leqslant n$, $\omega(G) \leqslant 3$ (that means G does not contain K_4) and *mimimum* vertex degree in G is at most 3. Prove that it is possible to remove at most 4 vertices with all its edges from G in such a way that in the new graph $G' = (V', E')$ we have $\alpha(G') \leqslant \alpha(G) - 1$ and $|E'| \leqslant |E| - 8$ (by removing of at most 4 vertices we delete at least 8 edges).

You may use the following approach to problem 12. Let A be a vertex of minimal degree in G. The possible the values of this degree are from 0 to 3. For the first three values apply problem 2 plus Brooks' theorem in order to prove that the remaining graph has a vertex of big degree. For the last value investigate possible cases.

Problem 13. Let $G = (V, E)$ be a distance graphs on the plane, $|V| = 4n$ and $\alpha(G) \leqslant n$. Using induction and problem 12 prove that $|E| \geqslant 8n$.

Problem 14. Let graph $G = (V, E)$ (not necessarily being a distance graph) have $4n$ vertices, $\alpha(G) \leqslant n$ and $\omega(G) \leqslant 3$. Prove that the estimation $|E| \geqslant 8n$ can not be strengthened.

We can improve the bound better by using additional "forbidden" subgraphs.

Problem 15*. Applying results of problems 7, 8 and 9 prove that if a distance graph has $4n$ vertices and $\alpha(G) \leqslant n$, then $|E| \geqslant \frac{26}{3}n$.

Problem 16 (open problem). Improve the bound of problem 15.

2.5 Distance graphs in high-dimensional spaces

If you already know what is n-dimensional space usually denoted by $\mathbb{R}^n$, you are extremely smart, but this knowledge is not obligatory right now. We will give all necessary definitions later. And now we tend to avoid the word "'space". Consider graph $G(n, 3, 1)$. Its vertices are all 3-element subsets of the set $\{1, 2, \ldots, n\}$, so it has $\binom{n}{3}$ vertices. And the edges correspond to the pairs of subsets which has 1-element intersection. See example of graph $G(5, 3, 1)$ in fig. 2.

Problem 17. Find the number of edges in graph $G(n, 3, 1)$.

Problem 18. Find the number of triangles in graph $G(n, 3, 1)$.

Problem 19. Prove that $\alpha(G(n, 3, 1)) = n$, $n - 1$ or $n - 2$ depending on the remainder n mod 4.

Problem 20. Find $\omega(G(n, 3, 1))$.

Problem 21*. Prove that if $n = 2^k$, then $\chi(G) = \frac{|V|}{\alpha(G)} = \frac{(n-1)(n-2)}{6}$.

Let f and g be two functions defined on the set of non negative integers and having no zero values. We remind that f and g are called *asymptotically equal* (or *equivalent*) if $\frac{f(n)}{g(n)} \to 1$ for $n \to \infty$. It is written as $f \sim g$. For example $n^4 \sim n^4 + 100n^2$. Function f is said to be *infinitesimal* with respect to g if $\frac{f(n)}{g(n)} \to 0$ for $n \to \infty$. It is denoted as $f = o(g)$. For example $n^3 = o(n^4)$.

Problem 22. For each integer $n \geq 3$ let W_n be a subset of the set of vertices of graph $G(n, 3, 1)$. Denote by $r(W_n)$ the number of edges with both endpoints in W_n. Let $n = o(|W_n|)$ for $n \to \infty$. Prove that Turán's theorem implies that $r(W_n) \geqslant f(n)$, where f is a function that is asymptotically equal to $\frac{|W_n|^2}{2\alpha(G(n,3,1))} \sim \frac{|W_n|^2}{2n}$.

Now we will give a formal definition of the space $\mathbb{R}^n$. It is just a set of "points" $\mathbf{x}$, where each of points is a sequence of n real numbers: $\mathbf{x} = (x_1, \ldots, x_n)$. For any two points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ we define a distance between them by formula

$$|\mathbf{x} - \mathbf{y}| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

In particular, for $n = 1$ this definition gives us the usual line, for $n = 2$ the usual plane and for $n = 3$ the usual space.

Further, the *scalar product* of vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is the expression

$$(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \ldots + x_n y_n.$$

It easy to check that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x} - \mathbf{y}|^2 = (\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}) - 2(\mathbf{x}, \mathbf{y}).$$

Problem 23. Prove that graph $G(n, 3, 1)$ is isomorphic to graph (V, E)

$$V = \{\mathbf{x} = (x_1, \ldots, x_n) : \ x_i \in \{0, 1\}, \ x_1 + \ldots + x_n = 3\}, \quad E = \{\{\mathbf{x}, \mathbf{y}\} : \ (\mathbf{x}, \mathbf{y}) = 1\}.$$

Thus, this is a distance graph in $\mathbb{R}^n$: its vertices are points in $\mathbb{R}^n$, and edges are the pairs of points at distance 2.

Problem 24. Let $K_{l_1,\ldots,l_r}$ be the complete r-partite graphs, with parts of sizes $l_1, \ldots, l_r$. Prove that distance graphs in $\mathbb{R}^n$ do not contain subgraphs of the form $K_{\underbrace{3,\ldots,3}_{[n/2]+1}}$.

3 Problems after intermediate finish

Problem 25. Prove that if in the statement of problem 22 to impose additionally the condition $|W_n| = o(n^2)$, then the estimation of problem 22 (i. e. usual Turán's estimation) cannot be asymptotically improved. In other words, for every function g such that $g(n) = o(n^2)$, $h = o(g(n))$ there exists sequence W_n such that $|W_n| \sim g(n)$ and $r(W_n) \sim \frac{|W_n|^2}{2n}$.

Problem 26-27. We say that any k points in $\mathbb{R}^n$ are the *vertices of right simplex*, if all the pairwise distances between them are equal to 1. Prove such sets exist for all $k \leqslant n+1$ (problem 26) and do not exist for all $k \geqslant n+2$ (problem 27).

Problem 28. Let $G_n = (V_n, E_n)$, $n = 1, 2, \ldots$ be unit distance graphs in $\mathbb{R}^n$. Denote their independence numbers by α_n. Let W_n be an arbitrary subset of the set of vertices of graph G_n (as usual, for each n we consider its own set W_n). Denote by $r(W_n)$ the number of the edges, both ends of which belong to W_n. Let $n\alpha_n = o(|W_n|)$ as $n \to \infty$. With the help of problem 26-27 prove that $r(W_n) \geqslant f(n)$, where f is some function asymptotically equal to the value $\frac{|W_n|^2}{\alpha_n}$.

For sequence $G_n = G(n, 3, 1)$ problem 28 give the estimation that is approximately 2 times better than the estimation in problems 22 and 25 (twise better than Turán's estimation). There are no contradiction here, because these problems have different (in fact opposite) limitations for the number of vertices $|W_n|$: in problem 25 $|W_n| = o(n^2)$ and in problem 28 (check!) $n^2 = o(|W_n|)$. It turns out that for graphs $G(n, 3, 1)$ even stronger estimations of Turán's kind can be obtained, by temporary refuse of using the independence number. The idea is to consider the vertices containing an element of set $\{1, \ldots, n\}$, to estimate the corresponding numbers of edges and to apply some standard inequalities.

Problem 29. Let W_n be an arbitrary subset of the set vertices of graph $G(n, 3, 1)$. Let $n^2 = o(|W_n|)$ as $n \to \infty$. Prove that $r(W_n) \geqslant f(n)$, where f is some function asymptotically equal to the value $4.5 \cdot \frac{|W_n|^2}{n}$. By the other words, we have obtained the estimation, approximately 4.5 times better than in problem 28!

Problem 30. Prove that the estimation of problem 29 in the standard sense cannot be asymptotically improved.

The notation "$G(n, 3, 1)$" itself prompts that this graph has the generalization. It is graph $G(n, r, s)$. Its vertices are all r-element subsets of set $\{1, \ldots, n\}$, and two vertices are connected by edge, if and only if the intersection of the corresponding sets contains exactly s elements. In other words, the vertices are n-dimensional points with "coordinates" 0 or 1, where the number of 1's is exactly r. Edge is drawn if and only if the scalar product of the vertices equals s. Graphs $G(n, r, s)$ are called *Johnson graphs*, and the particular case of them, graphs $G(n, r, 0)$, are called *Kneser graph.*

Problem 31. Find the number of the edges of graph $G(n, r, s)$.

Problem 32. Find the number of the triangles of graph $G(n, r, s)$.

Problem 33*. Prove that the analogue for the results from problems 29 and 30 is the estimation of the form $\frac{|W_n|^2}{n^s} \cdot \frac{C_r^s \cdot r!}{2 \cdot (r-s)!}$ that asymptotically cannot be improved. Here we have to demand $n^{r-1} = o(|W_n|)$.

The following result you can apply without proof.

Erdős–Ko–Rado theorem. *Let $n \geqslant 2r$. Then $\alpha(G(n, r, 0)) = C_{n-1}^{r-1}$.*

Problem 34. Prove that if W_n is an arbitrary subset of the set of vertices of graph $G(n,r,0)$ and $l = |W_n| > \alpha(G(n,r,0))$, then

$$r(W_n) \geqslant \frac{l\left(l - \left(C_n^r - C_{n-r}^r\right)\right)}{2}.$$