Phase spaces

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Abstract

We offer participants a number of problems to solve, for which it is useful to consider phase spaces. In the section before the semifinal we show a variety of problems illustrating this principle. In the section after the semifinal we introduce a couple of challenging ideas. These ideas will require the participants to explore geometrical and topological properties of the phase spaces.

1 Problems before the semifinal

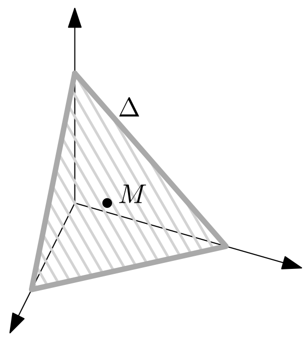
1.1 Introduction

It is convenient to consider a "set of all possible states" of a system for solving many mathematical problems. Such a set is usually called *a phase space* of the system. Information about the state of the system may include not only points' coordinates, but also, as is common in mechanics, points' velocities.

A phase space for some problem should be viewed not just as a set of points: usually it is useful to consider an additional structure. For example, geometric structure (as on a set of points in the plane); function of distance between points; notion of area/volume for subsets of a phase space; incidence relation (if a phase space is a set of graph vertices). The right choice of such a structure can simplify a statement of a problem, then the problem itself becomes trivial.

Let us consider the following example.

Problem 0. Three kids sit around a table, and each of them has a plate of porridge. Every minute they simultaneously do the following: each kid divides his or her porridge into 2 equal parts and puts these parts to each of the other kids' plates. Prove that after several minutes all porridge will be spread evenly between the kids with an accuracy of 1%.



Solution. Suppose that the total amount of the porridge is equal to 1. By (x_1, x_2, x_3) denote a set of numbers, where x_i corresponds to an amount of porridge in i-th kid's plate. All possible states of our system lie in the plane $x_1 + x_2 + x_3 = 1$, and $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$. Hence, *our phase space is a regular triangle* Δ *with vertices* $(1, 0, 0)$, $(0, 1, 0)$, *and* $(0, 0, 1)$.

Note that the transformation

$$(x_1, x_2, x_3) \mapsto (y_1, y_2, y_3) = \left(\tfrac{x_2+x_3}{2}, \tfrac{x_1+x_3}{2}, \tfrac{x_1+x_2}{2}\right)$$

is a homothetic transformation of the triangle Δ with ratio $-\frac{1}{2}$ and origin $M = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ — the intersection point of the medians. Try to prove that yourself.

Let us mark all the system's states, where the porridge is spread evenly between the kids with an accuracy of 1%. This subset of the triangle Δ contains a small disk of centre M. Then, after a sufficiently large number of iterations all the points of Δ will be inside the disk. This implies the required statement of the problem. $\square$

1.2 Geometric probability

In each of the following problems a notion of probability needs to be defined. Then, you should use your definition to find the answer.

Problem 1. Two friends agreed to meet near a huge oak between midday and 1 p.m., but they didn't decide on the exact time of the meeting. They arrive there randomly at that time interval. They are willing to wait for each another for 10 minutes, after which they go away. What is the probability of their meeting?

Problem 2. What is the probability that a stick randomly broken in two points can form a triangle? Compare the answers for the following ways *to break it randomly*:

(a) the first point we choose randomly; after breaking the stick we select one of two parts with equal probability; then we choose the second point randomly on the selected part;

(b) we choose the two points randomly and independently;

(c) we choose a random representation of the stick's length l as a sum of the three ordered summands $l = x_1 + x_2 + x_3$ (cf. problem 0).

Problem 3. A Mars rover travels around an even surface of an inhospitable planet (the surface can be considered flat). The rover chooses a random direction and moves one kilometer this way. What is the probability that after making three iterations and covering three kilometers the rover will cross its own trail?

Problem 4 (Buffon's needle)**.** A plane is ruled with parallel lines 1 cm apart. A needle of length 1 cm is dropped on the plane. What is the probability that the needle crosses a line?

1.3 Configuration spaces

Problem 5. A polygon of area > 1 is drawn on a coordinate plane. Prove that there exist two points A and B in its interior such that both coordinates of the vector AB are integer.

Problem 6. A polygon of area < 1 and 1000 points are drawn on a plane. Prove that the polygon can be moved in a vector of length $< \sqrt{\frac{1000}{\pi}}$ in such a way that the polygon will not cover any of the given points.

Problem 7. A unit sphere is given. *A great circle* is a circle with radius 1 laying on the sphere. We call a curve on the sphere *polygonal* if it consists of arcs of great circles.

(a) There is a polygonal curve γ of length $< \pi$ on the unit sphere. Prove that there exists a great circle that does not intersect γ.

(b) There is a non-self-intersecting polygonal curve γ of length $> \pi k$ on the unit sphere. Prove that there exists a great circle intersecting γ in more than k points such that it does not contain any edges of γ.

Problem 8. There are two tetrahedrons in space. For every plane the following statement holds: the projections of these tetrahedrons on the plane are both either triangles or quadrilaterals. Prove that the tetrahedrons are similar.

Problem 9. There are k nails hammered into a plane. Consider all lines in the plane such that none of the nails lies on these lines. We call two lines equivalent if one of them can be moved onto another without touching any nails. Then, the set of all considered lines is divided into disjoint equivalence classes, where each class consists of the lines equivalent to each other. Find the **(a)** minimal **(b)** maximal number of the equivalence classes for a given k.

Problem 10. Eight cars entered a cross-country race. A route is a straight road with several swampy sections. Every car moves at a constant speed on a dry section of the road (each one with its own), and at a different constant speed on a swampy section of the road (each one also with its own). All the cars start the race from the same point, but at a different time (start times may be not equally distributed). There are 500 judges on the route.

Each judge noticed the order of the cars passed by, and there was no overtaking right in front of the judges. Prove that there exist two judges such that the cars passed by them in the same order.

1.4 Descrete phase spaces

Problem 11. Alice picked a two-digit number. Bob suggests two-digit numbers and his goal is to find a number which differs from Alice's number no more than by 1 in every digit. What is the minimum number of attempts required?

Problem 12. Alice picked a two-digit number. Now, Bob's goal is to find a number which equals Alice's number in one digit and differs from it no more than by 1 in the other digit. Is it enough for him to do **(a)** 18 attempts; **(b)** 20 attempts; **(c)** 22 attempts?

Problem 13. Several boxes are arranged in a circle with some beads placed inside the boxes. For each move it is allowed to take all the beads from any box and place them one by one to the next boxes in a clockwise order.

(a) Prove that if for each move the beads are taken from a box to which the last bead was placed in the previous move, then after some number of moves the placement of beads from the start appears again.

(b) Prove that it is possible to get any desired bead's placement form any starting placement for several moves.

Problem 14. **(a)** An invisible ship is placed on some cell of a cellular strip infinite in both directions. For each turn a player can shoot at one square and after each shot the ship moves by some number of cells (the number of cells and the direction of the move is the same for every turn, but player does not know neither of them). Find a strategy to shoot the ship.

(b) The same problem can be stated for a cellular plane: the ship is placed on some cell and after each shot it moves in a vector with integer coordinates (this vector is the same for every move). A player can shoot at one square for each turn.

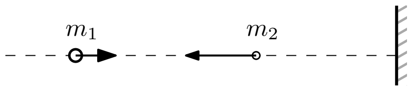
1.5 Selected problems

Problem 15. You have 6-liter, 7-liter, and 12-liter jugs. The two smaller jugs are filled with water. Is it possible to measure exactly 9 liters of water if you are allowed just to pour water from one jug to another?

Problem 16. There are three water tanks. Water is running out of the first tank with a constant speed, and running in the second and the third tank (also with a constant speed). In the beginning, the first tank was filled with the same amount of water as two other tanks together; after some period of time the second tank had the same amount of water as two other tanks together; in the end the third tank had the same amount of water as two other tanks together. Is it possible that neither in the beginning nor in the end none of the tanks was empty?

Problem 17 (Simplified "predator-pray" model)**.** There are x rabbits and y wolves living in a forest. Without wolves the rabbit population grows with constant rate a_0. But if there are $y > 0$ wolves living in a forest, then these wolves eat $a_1 y$ rabbits per a unit of time (i.e., $x'(t) = a_0 - a_1 y$). Also, without rabbits the wolves population decrease with constant rate b_0, but if there are $x > 0$ rabbits then the wolves population grows with rate b_1x (i.e., $y'(t) = -b_0 + b_1x$). Find the time dependence for an amount of rabbits and wolves. For what values a_0, a_1, b_0, and b_1 do these amounts change periodically?

Problem 18. Two balls of masses m_1 and m_2 are moving along a straight line as it is shown in the figure. Elastic collisions occur between the balls and between the balls and the wall.



(a) Prove that there will be a finite number of collisions between the balls.

(b) How does the number of collisions depend on a ratio of the balls' masses and initial velocities?

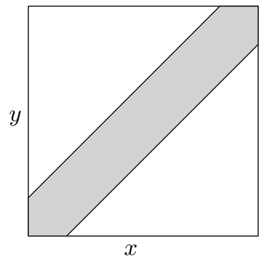
The law of conservation of momentum ($\sum m_i v_i = \text{const}$) and the law of conservation of energy ($\sum m_i v_i^2 = \text{const}$) apply to the balls' collisions. Collisions with the wall are absolutely elastic (with the conservation of velocity). Assume that a ball which is the closest to the wall moves away from it initially.

Hints and solutions for problems before the semifinal

Geometric probability

To solve these problems we need to define a notion of probability. Let X be a rectangle such that its points correspond to the set of outcomes in our problem. Let $A \subset X$ be some subset of X. We say that *the probability that a point is inside* A is equal to the relation of the area[1] of A to the area of X. In such a case we call X *a sample space* and we call its subsets *events*. More detailed introduction to probability theory see e. g in [Sh].

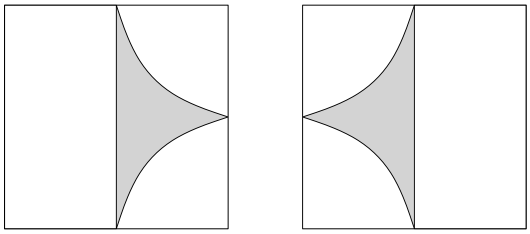
Solution for problem 1. Answer: $\frac{11}{36}$.



Let us identify the time interval for friend's arrival to the oak with the segment $[0;1]$. Then a possible time of two friends' arrival to the oak is identified with the square $[0;1]\times[0;1]$. Namely a point (x,y) is identified with a situation when the first friend comes at the time x, and the second friend, at the time y.

Let us colour the points corresponding to the situations when the friends meet. These points form a strip with area $\frac{11}{36}$. □

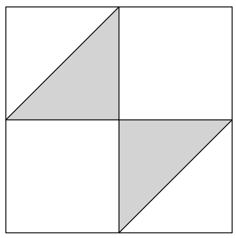
Solution for problem 2. (a) Answer: $\ln 2 - \frac{1}{2}$.



Suppose we choose the first point randomly from the line segment $[0;1]$. Then identify each of the resulting parts with its own copy of the line segment $[0;1]$. Hence our sample space consists of two squares (a coordinate of the first choice is marked in each of the squares on the horizontal axis and a coordinate of the second choice is marked on the vertical axis).

Let us colour the points of X corresponding to the situations when the broken stick can form a triangle. Suppose the first place where the stick is broken is in the point $x \in [0;1]$. Then suppose that we choose the first part and identify it with $[0;1]$, and then choose a point y in it. It is possible to form a triangle if $x > \frac{1}{2}$, $xy < \frac{1}{2}$, and $x(1-y) < \frac{1}{2}$. This subset has area

$$2\int\limits_{\frac{1}{2}}^{1}\left(\frac{1}{2x}-\frac{1}{2}\right)dx = \ln 2 - \frac{1}{2}.$$

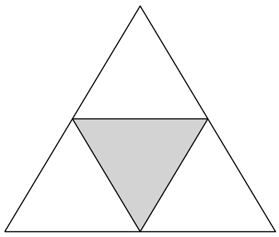


The subset of the second square corresponding to the situations where a triangle can be formed can be constructed similarly.

(b) Answer: $\frac{1}{4}$.

Identify the stick with the line segment $[0;1]$. A choice of two points $x, y \in [0;1]$ is identified with a point in the square. It is possible to form a triangle if $x < y$ and also $x < \frac{1}{2}$, $y > \frac{1}{2}$, and $y - x < \frac{1}{2}$, or if $y < x$ and also $x < \frac{1}{2}$, $y > \frac{1}{2}$, and $y - x < \frac{1}{2}$. This subset has area $\frac{1}{4}$.

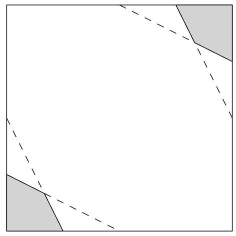
(c) Answer: $\frac{1}{4}$.



A sample space for this problem can be identified with the phase space in problem 0. It is possible to form a triangle if $x_1 < \frac{1}{2}$, $x_2 < \frac{1}{2}$, and $x_3 < \frac{1}{2}$. This subset has area of a quarter of the triangle's area. □

[1] Generally speaking, for some subsets $A \subset X$ area may not be defined; therefore we can consider only the probability that a point is inside a "suficciently good" subset such that its area is well-defined.

Solution for problem 3. Answer: $\frac{1}{12}$.

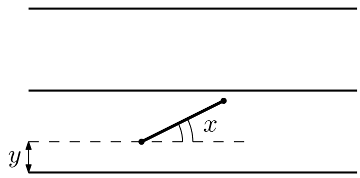


Let us identify the sample space with the square $[0; 2\pi) \times [0; 2\pi)$ where coordinates correspond to the values of angles of the polygonal curve for rover's trail. Besides the boundary $x = 0$ and $y = 0$ (that have area 0), the rover will cross its trail if either (x, y) is a pair of angles in some triangle such that both angles are strictly not the greatest ones, or $(2\pi - x, 2\pi - y)$ is a pair of angles in some triangle such that both angles are strictly not the greatest ones. These situations correspond to the following systems of inequations

$$\begin{cases} x \leqslant \pi - x - y, \\ y \leqslant \pi - x - y \end{cases} \quad \text{and} \quad \begin{cases} (2\pi - x) \leqslant \pi - (2\pi - x) - (2\pi - y), \\ (2\pi - y) \leqslant \pi - (2\pi - x) - (2\pi - y). \end{cases}$$

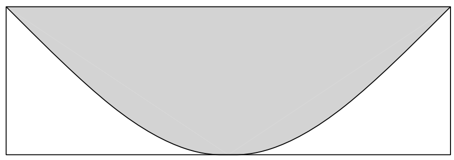
Each of the systems corresponds to a quadrilateral with area equal to the square's area divided by 24. □

Solution for problem 4. Answer: $\frac{2}{\pi}$.



For the sake of argument suppose that the lines are horizontal. A sample space for the problem can be identified with the rectangle $[0; \pi) \times (0; 1]$ where the first coordinate is for the orientated angle of the needle with respect to the lines and the second coordinate is for the distance between the bottom end of the needle and the closest line from below.

Then the set of the outcomes we are interested in is described by inequality $1 - y \leqslant \sin x$. The set has area $\int_0^\pi \sin x dx = 2$. □



Configuration spaces

Note about terminology. Suppose we have a subset X of Euclidean space. If we identify the set of *points* X with the set of *positions* of some objects (for example, with a set of lines in a plane or with a set of great circles on a sphere or with a set of n-tuplets of numbers with a fixed sum), then X is called *configuration space*. This term is considered commonly-accepted and more accurate than the term "phase space", although these terms have the equivalent meaning.

Solution for problem 5. Let us consider a "grid" consisting of lines $x = n$ and $y = n$ for all possible $n \in \mathbb{Z}$. It divides the plane into squares. Let us call the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ a *base square*. It is clear that each square is obtained from the base square using translation by some integer vector. Suppose that all the squares look like $[0; 1) \times [0; 1)$ that is the base square without its upper and right edges.

Let us consider the rectangle from the statement of the problem. The grid divides it into a finite number of parts. Let us translate each part by a corresponding integer vector such that all these parts are inside the base square. In fact, we constructed a *set of points of the polygon up to an integer translation.*

Since the sum of the areas of the parts is > 1, the parts will definitely overlap. As points A and B it is sufficient to take any pair of points such that these points become the same point after the move. □

Solution for problem 6. Let us draw disks of radius $\sqrt{\frac{1000}{\pi}}$ and centre in each of the points. Let us colour the points of the polygon inside each disk. Then let us stack all the disks using translations. Coloured points will cover some set P. Vectors starting at points of P and ending at the centre of the disk are *such vectors that the polygon cannot be moved in those directions.*

Each coloured subset has area < 1, therefore the area of P is < 1000. But the area of the disk is exactly 1000 thus there exists a point inside it that is not coloured. The vector starting at that point and ending at the centre of the disk is the required vector for our problem. □

Solution for problem 7. An idea used in this solution is very similar to projective duality (see e. g. [S]). This method allows us to work conveniently with *the set of great circles* on a sphere. Namely suppose we have a sphere of centre O. Identify each point A on the sphere with a great circle in the plane passing through O and perpendicular to OA.

This determines a one-to-one correspondence between pairs of diametrically opposite points on a sphere and great circles. We call such correspondence *duality* and denote it by δ. Note that $\delta(\delta(A)) = \pm A$ for any point A on the sphere and $\delta(\delta(\omega)) = \omega$ for any great circle ω. Moreover, if the point A is on a great circle ω, then the great circle $\delta(A)$ passes through the pair of points $\delta(\omega)$.

(a) Suppose we have an ark α of a great circle. Suppose α is of length πt for $t \in (0; 1)$. Let us consider the great circles $\delta(A)$ for all $A \in \alpha$. Note that area of the union of $\delta(A)$ is a fraction t of the area of the sphere.

The polygonal curve γ from the statement of the problem is of length $< \pi$. Hence, the great circles dual to the points of γ cannot cover the whole of sphere. Let us consider a point B such that it is not covered. Then, the great circle $\delta(B)$ does not intersect γ.

(b) The proof is analogous to the previous one. For each of the edges γ take the union of the great circles that are dual to the points of the edge. We will get a pair of "spherical segments". Since the length of γ is greater than πk, the union of such segments covers some part of the sphere with non-zero area more than k times.

In particular there exist infinitely many points on the sphere covered by more than k segments. Let us mark points $B_1, \ldots, B_m$ which are dual to the extensions of the edges of γ. We can choose a point C such that it is covered by more than k segments and is not equal to $B_1, \ldots, B_m$. Then a big circle $\delta(C)$ intersects more than k edges of γ and does not contain any edges. □

Hint for problem 8. Let us inscribe each tetrahedron into a sphere. Let us apply translation and homothetic transformation so that the spheres take the same place.

Then consider lines l passing through the centre of the sphere such that projections of the tetrahedrons on the plane perpendicular to l are triangles. □

Solution for problem 9. Let us consider a disk D such that all the nails are inside it. If a line from some equivalence class intersects D, then we call such equivalence class *restricted*. It is clear that there exists exactly one class which is not restricted.

Let us draw all possible line segments between the nails such that every segment does not pass through any nails except its ends. One can construct a correspondence between *the set of the segments* and *the set of restricted equivalence classes of the lines*. To do this we should rotate each line counterclockwise until it touches two nails, this pair of nails is the ends of the segment corresponding to the equivalence class of this line

(a) Answer: k.

Indeed, we can always draw $k-1$ line segments. This number of segments is possible when all the nails lie on the same line.

(b) Answer: $C_k^2 + 1$.

The number of the segments is not greater than the number of edges of complete graph. This estimate is possible when any three nails do not lie on the same line. □

Another idea for problem 9. Using projective duality one can identify the configuration space of all the lines in $\mathbb{R}^2$ with $\mathbb{RP}^2$ minus one point. The removed point corresponds to the "line at infinity". See e. g. [S].

The set of lines passing through a point $A \in \mathbb{R}^2$ corresponds to a line in $\mathbb{RP}^2$. As a result, our problem reduces to finding the number of parts into which $\mathbb{RP}^2$ is divided by k lines (the removed point does not matter for the number of parts). □

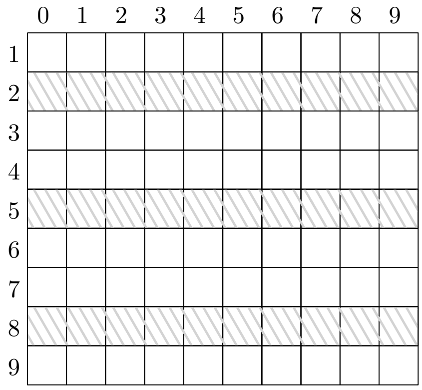
Solution for problem 10. Suppose S is the starting point. For each point A on the route denote the sum of lengths of dry sections from S to A by x, and the sum of lengths of swampy sections from S to A by y. Then we will assign the point (x, y) in the plane to the point A on the route.

Let us draw a plot in space for every car's movement with coordinates x, y, t. Each of the plots is a polygonal curve. Let us note that each of these polygonal curves lies in some plane $t = t_0 + \frac{x}{u} + \frac{y}{v}$, where t_0 is the passing time of point S, u and v are velocities of the car on dry and swampy road sections respectively. Denote these planes by $\alpha_1, \ldots, \alpha_8$.

Let us draw vertical lines $l_1, \ldots, l_{500}$ for the judges. We need to show that the points of intersection of some two lines with the planes are in the same order.

Let us consider pairwise intersections of the planes $\alpha_i \cap \alpha_j$ and project them onto a horizontal plane β. We get no more than $C_8^2 = 28$ different lines. They divide β into no more than $1 + C_{29}^2 = 407$ parts. Using pigeonhole principle we get that some two lines of $l_1, \ldots, l_{500}$ pass through the same part. $\square$

Descrete phase spaces

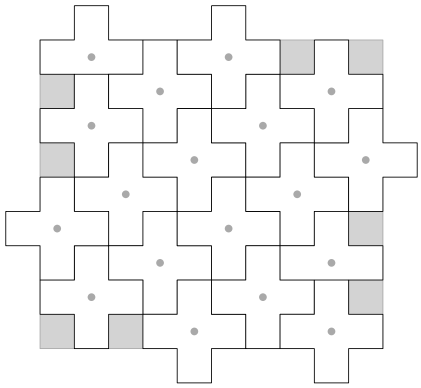


Solution for problem 11. Answer: 12 attempts.

Let us identify every two-digit number with a cell in a table 9×10 where rows correspond to the ten's digit and columns, to the one's digit. If Bob suggests a number, then it covers a square 3×3 in the table. But to cover by such squares all marked rows (all numbers starting with 2, 5, and 8) we need to place 12 such squares, that is to suggest minimum 12 numbers.

An example is easy to construct by looking at the table: Bob should suggest numbers 21, 24, 27, 29, 51, 54, 57, 59, 81, 84, 87, and 89. $\square$

Solution for problem 12. An idea is the same, but this time we need to cover all the cells of the rectangle 9×10 by "crosses" constructed from five cells.



(a) Answer: No.

Genuinely, if 18 crosses were enough then they would cover the rectangle without any intersections and stay within the rectangle. But it is impossible because, for example, a corner cell cannot be covered this way.

(c) Answer: Yes.

Let us put 18 crosses as shown in the figure (for more clarity a centre of each cross is marked) and we can cover eight not covered cells with the rest 4 crosses. $\square$

Solution for problem 13. **(a)** A current state of the system is determined by a number of beads in every box and a box from which we will start to place the beads next time. Hence the total amount of the system's states is finite.

From every state it is possible to get to some other well-defined state by placing beads.

On the other hand, we can uniquely determine the systems' state before the last beads placement knowing a current state of the system. Genuinely, the last placement ended at the marked box. Hence we need to take one bead from that box and then going in a counterclockwise order take one bead from every box while we can to restore the previous state. When we meet an empty box, we place all the beads we took into it and mark this box.

Let us construct *an oriented graph of the states* of the system. For this purpose we will denote the systems' states by points and the possibility of transfer from one state to another, by an arrow connecting the corresponding points. There will be only one arrow going out from a point and only one arrow going in.

Let us start to move along the arrows starting with the given state A_1. We get a sequence of states $A_2, A_3, \ldots$ We get that at some moment we will have a repetition in our sequence $\{A_i\}$ because the number of the states is finite. Suppose, for example, $A_k = A_l$ where $k < l$. As long as there is only one arrow going in the point A_k we obtain from the equality $A_k = A_l$ that $A_{k-1} = A_{l-1}, \ldots, A_1 = A_{l-k+1}$. Thereby we return to the state A_1 after $l - k$ moves.

(b) As opposed to (a) now the system's state is defined only by the way the beads are placed into the boxes. Note that if a move goes from state A to state B, then according to (a) we can (in several moves) get back from B to A. If we can get from A to C in several moves then we can get back from C to A by "rolling back" our moves one by one.

Hence if we can get from any state to some fixed state M, then we can get from any state to any other state going through M. Denote a state where all the beads lie in some fixed box m by M. Let us take the beads from the closest to m (in a clockwise order) non-empty box for each move. Then either the number of beads in m will grow or the closest to m non-empty box will get closer. Sooner or later all the beads will be in m. □

Hint for problem 14. A phase space for this problem is a set of all possible pairs (x, v) where x is an initial position of the ship and v is a vector in which the ship moves at each turn. It is easy to show that the phase space is countable and this condition is sufficient for constructing an algorithm for shooting. □

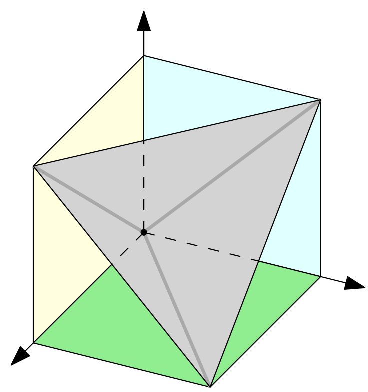
Selected problems

Hint for problem 15. Construct a configuration space of the water in jugs as in problem 0 and examine how a state of the system changes under the operation of pouring water from one jug to another. See [CG, Chapter 4, §6] □

Solution for problem 16. Answer: it is impossible.

A configuration space for an amount of water in all three tanks is a subset $X \subset \mathbb{R}^3$ consisting of points with non-negative coordinates. A point corresponding to a current state of the system moves in a line segment I connecting starting and ending states.

An intersection of the planes $x + y = z$, $y + z = x$, and $z + x = y$ with X is a boundary of an infinite triangular cone. Since the cone is convex the line segment I intersects its boundary in not more than two points. But according to the problem's statement segment I passes trough at least one point from every face of the cone. Hence I passes through at least one edge of the cone. Finally note that the edges of the cone lie in the planes $x = 0$, $y = 0$, and $z = 0$. □



Solution for problem 17. A configuration space of amounts of rabbits and wolves is the first quadrant of the plane $\mathbb{R}^2$. Denote the quadrant by X. The position of the system changes over time: a point with positive coordinates (x, y) moves in a velocity vector $(a_0 - a_1 y,\ -b_0 + b_1 x)$.

Note that the velocity vector is equal to the null vector at the point $(\frac{b_0}{b_1}, \frac{a_0}{a_1})$. Let us move the origin of the plane to this point. Then the matching each point to its velocity vector is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$. Note that this linear transformation maps the x axis to the $-y$ axis, and the y axis to the x axis. So this transformation can become a rotation by scaling coordinate axis.

Namely, consider the linear substitution

$$\tilde{x} = \frac{-b_0 + b_1 x}{\sqrt{b_1}} \quad \text{and} \quad \tilde{y} = \frac{-a_0 + a_1 y}{\sqrt{a_1}}.$$

In this new coordinate system a velocity vector of a point $(\tilde{x}, \tilde{y})$ not from the boundary of X is written down as $\sqrt{a_1 b_1} \cdot (-\tilde{y}, \tilde{x})$. This means that all such points move in circles of centre in the origin of the plane. As a result the movement of all the points such that their circles do not intersect the boundary of X is periodical. If a point gets to the boundary of X then it keeps moving along the boundary with a constant speed. That is movement of such points that

$$\tilde{x}^2 + \tilde{y}^2 \leqslant \min\left(\frac{b_0^2}{b_1}, \frac{a_0^2}{a_1}\right),$$

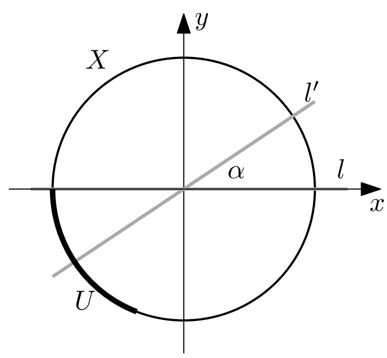
is not periodic, that is the distance to the origin (in the new coordinates) is not greater than the distance to at least one of the lines $x = 0$ and $y = 0$. □

Solution for problem 18. (a) Let us trace just the *velocities* of the balls. In a space with coordinates v_1, v_2 the law of conservation of energy $m_1 v_1^2 + m_2 v_2^2 = \text{const}$ plots an ellipse. For the sake of convenience we can multiply coordinates by positive numbers in such a way that the ellipse becomes a circle.

Namely suppose $x = \sqrt{m_1} \cdot v_1$ and $y = \sqrt{m_2} \cdot v_2$. (The first ball is the furthest from the wall and we suppose that velocity is positive if it is directed towards the wall.) From the law of conservation of energy it follows that the points corresponding to the system's positions are in a circle $x^2 + y^2 = \text{const}$. This circle is our phase space, let us denote it by X.

A collision between a ball and the wall is represented in the phase space as a map $(x, y) \mapsto (x, -y)$, that is a reflection in a line l where l is the line $y = 0$. Note that collision can happen if and only if $v_2 > 0$ that is if $y > 0$.

With every collision between the balls there is a reflection in a line, passing through the origin and a point $(\sqrt{m_1}, \sqrt{m_2})$, in the phase space. It follows from the law of conservation of momentum. Denote this line by l', it is written as $\frac{x}{\sqrt{m_1}} = \frac{y}{\sqrt{m_2}}$. Note that a collision can happen if and only if $v_1 > v_2$, that is $\frac{x}{\sqrt{m_1}} > \frac{y}{\sqrt{m_2}}$.



Let us denote an angle between l and l' by α. We know that $\alpha = \operatorname{arctg} \frac{\sqrt{m_2}}{\sqrt{m_1}}$. Obviously a composition of the two described transformations of X is a rotation on an angle -2α.

Let us mark an ark $[\pi; \pi + 2\alpha]$ on the circle and denote it by U. Sooner or later every point will get to the set U. In such position of the system there will be no more collisions with the wall.

(b) Suppose (x_0, y_0) is an initial state of the system. It follows from the statement of the problem that $y_0 < 0$. Define β as an angle of the vector (x_0, y_0). Formally $\beta = \operatorname{arcctg} \frac{x_0}{y_0} - \pi$. Then the number of collisions between the balls is equal to $\lceil \frac{\pi + \beta - \alpha}{2\alpha} \rceil$ and the number of collisions between the second ball and the wall is equal to $\lceil \frac{\pi + \beta - 2\alpha}{2\alpha} \rceil$.

We can advise you to take a look on some of the additional materials for this problem on YouTube: summarizing video [1] where the connection between the problem and the decimals for π is shown, video [2] with a solution in the space of velocities, and also video [3] with the solution using billiards in the space of coordinates. □

Problems after the semifinal

Problems here are divided into two independent topics. In section 2 we study different topological properties of phase spaces that appear in problems with quite simple statements. In section 3 we deal with geometry of phase spaces, namely with transformations preserving volume. *Because of the expanse of these topics we suggest the participants to focus their effort on just one of them.*

The main results of §2 are problems 20^m and 22^m. The main results of §3 are problems 42^m and 51^m. Problems marked with asterisks require knowledge of some techniques (such as continuous mappings), if you do not know the formal definitions needed, then you do not have to solve these problems.

Sometimes we formulate some lemmas as hints for difficult problems. These lemmas tell us much about the nature of considered objects and therefore might be interesting on their own. Proofs of these lemmas are divided into smaller problems placed after the lemmas' statements.

Also we give statements of some theorems for reference, you do not need to send us their proofs.

2 Topology of phase spaces

2.1 Problem about wagons

Problem 19. At 9 a. m. a tourist left her home. She was walking the whole day and arrived to a camp by 9 p.m. She stayed over the night and at the next day she was walking home by the same road from 9 a.m. till 9 p.m. Her velocity is not constant and she can stop or sometimes even go back. Prove that she was in some point of the road at the same time both days.

Problem 20^m (N. N. Konstantinov)**.** Two nonintersecting roads lead from city A to city B. We know that two cars connected by a 10-meter rope manage to go from A to B along different roads without breaking the rope. Can two circular wagons of radius 6 meters, which centres move along the different roads in the opposite directions, pass each other without colliding?

Further we map a path to the *formal* solution for problem 20^m for *piecewise-linear* trajectory of the cars and wagons. It means that the time interval is divided into finite number of parts, and velocities of the cars and wagons are constant on each part. Then we can prove the following preliminary conjecture for solving problem 20^m.

Lemma 1. *There are two polygonal curves α and β in a square $ABCD$. The polygonal curve α connects vertices A and C and the polygonal curve β, vertices B and D. Then $\alpha \cap \beta \neq \varnothing$.*

The polygonal curves can be self-intersecting unless we explicitly specify otherwise.

Why don't we consider *continuous curves* instead of polygonal curves? The following problem illustrate the difficulties that may arise due to switching from piecewise-linear to continuous setting (answers for (a) and (b) are different).

Problem 21. There is a mountain in the middle of an even steppe. There are two paths up to the top of the mountain (these paths do not go below the steppe level). Two mountain climbers started their way to the top using different paths and staying on the same height. Is it possible to reach the top of the mountain moving continuously if **(a)** paths are polygonal curves; **(b*)** paths are arbitrary plots of continuous functions?

(c) Solve an analogue for (a) for an arbitrary number of climbers.

Lemma 1 seems obvious, but it is difficult to give a rigorous proof for it. Hints for such a proof see in §2.3.1.

2.2 Rectangle inscribed in a curve

Problem 22^m. We have a closed non self-intersecting curve in the plane. Prove that there exists a rectangle such that its vertices lie on this curve.

Consider points $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, and $D = (0, 1)$ in the plane. We can define the *Möbius strip* as the square $ABCD$ (with its interior) with sides AD and CB identified by the map $(0, t) \mapsto (1, 1 - t)$. Let us denote the Möbius strip by M.

The following two conjectures are useful for solving problem 22^m.

Problem 23. Identify M with the configuration space of pairs of points in a circle.

Lemma 2. *Suppose we have a plane α in space $\mathbb{R}^3$. Then there is no continuous embedding $M \to \mathbb{R}^3$ such that the boundary of M is in α and all the interior points are in one of the semispaces.*

(By "embedding" we mean a map such that different points of M are mapped to different points of $\mathbb{R}^3$.)

Generally speaking, notion of continuity is used in Lemma 2 and careful treatment for this notion is far beyond the scope of our project. Nevertheless we can prove rigorously the *piecewise-linear* variation of Lemma 2. Comments and hints for the proof see in §2.3.2.

2.3 Topological lemmas

2.3.1 Closed curves in the plane

The aim of this section is to prove Lemma 1.

Definition. A set of points in the plane is called *set in general position* if
(1) any three points do not lie on a line and
(2) any 6 of them cannot be covered by three lines passing through the same point (this point may not be in the set).

Problem 24. Is it true that if a set of points consists of at least 6 points then condition (1) of the definition above follows from condition (2)?

Problem 25. Show that any finite set of points in the plane can be transformed into a set in general position by arbitrary small movement (i. e. each point is moved by not more than some given distance).

Problem 26. We have a square $ABCD$ and polygonal curves $\alpha = A_0A_1 \dots A_m$, and $\beta = B_0B_1 \dots B_n$ where $A_0 = A$, $B_0 = B$, $A_m = C$, $B_n = D$ and the rest of the vertices are inside the square. Suppose the set of all the vertices is in general position. Then the number of intersections of α and β is odd. (Hint: use induction on m and n.)

Problem 27. Is the statement of problem 26 true if for the set of vertices of α and β the following does not hold: **(a)** condition (1) or **(b)** condition (2) from the definition of set in general position?

Problem 28. **(a)** Prove Lemma 1. **(b*)** Prove the variation of Lemma 1 for continuous curves.

Likewise the following conjecture can be deduced from problem 26.

Jordan curve theorem. We say that a subset A of the plane is *connected* if there exists a polygonal curve between any two of its points such that the curve lies inside A. Then **(a)** any closed non self-intersecting polygonal curve divides the plane into parts (that is, complement of the curve is not connected); **(b)** moreover, there are exactly two of these parts.

2.3.2 Linked closed curves in space

The aim of this section is to prove Lemma 2.

Definition. A set of points in space is called *set in general position* if
(1) any three points do not lie on the same line,
(2) any four points do not lie in the same plane and
(3) any eight points cannot be covered by two planes and a line passing through the same point (this point may not be in the set).

Polygonal curves in space are called polygonal curves in general position if the set of their vertices is a set in general position.

Problem 29. Show that any finite set of points in space can be transformed into a set in general position by arbitrary small movement.

Problem 30. Show that if a union of a set of the vertices of a closed polygonal curve in space and a set of vertices of any tetrahedron is a set in general position, then the number of intersections of the polygonal curve with the surface of the tetrahedron is even.

Definition. Suppose $\alpha, \beta \subset \mathbb{R}^3$ are two closed polygonal curves in general position. Take point $O \in \mathbb{R}^3$ in general position with their vertices. Suppose $A_0, A_1, \dots, A_n = A_0$ are the vertices of α. Let us count for $i = 0, \dots, n-1$ a remainder modulo 2 for the number of points of intersection of the triangle OA_iA_{i+1} and polygonal curve β. The sum n of these remainders is called linking number of α and β modulo 2. It is denoted by $\mathrm{lk}(\alpha, \beta)$.

Problem 31. Show that $\mathrm{lk}(\alpha, \beta)$ does not depend on the choice of point O. (Hint: take point O' and consider the intersection of tetrahedrons $OO'A_iA_{i+1}$ and polygonal curve β.)

(b) Show that $\mathrm{lk}(\alpha, \beta) \equiv \mathrm{lk}(\beta, \alpha)$. (By $\equiv$ we denote congruentness modulo 2.)

Problem 32. Suppose $\alpha, \beta \subset \mathbb{R}^3$ are two non self-intersecting and non intersecting polygonal curves.

(a) Prove that there exists $\varepsilon > 0$ such that for any movement of each vertex of α and β on a distance less than ε the polygonal curves remain non self-intersecting and non intersecting.

(b) We can obtain two polygonal curves in general position α', β' by moving the vertices of α, β on a distance less than ε. Prove that $\mathrm{lk}(\alpha', \beta')$ is well-defined, i. e. it does not depend on this move.

Problem 32 allows us to define linking number modulo 2 for *arbitrary* pair of non intersecting and non self-intersecting closed polygonal curves $\alpha, \beta \subset \mathbb{R}^2$. We say that α and β are *linked modulo* 2 if $\mathrm{lk}(\alpha, \beta) \equiv 1$.

It is convenient to use the following property of the graph K_6 (complete graph with 6 vertices) for an easier proof of Lemma 2. We call a map $K_6 \to \mathbb{R}^3$ *piecewise-linear* if images of all the edges under this map are polygonal curves.

Lemma 3. *For any piecewise-linear embedding $K_6 \to \mathbb{R}^3$ there exists a pair of non intersecting cycles linked modulo* 2.

Problem 33. Let us denote a sum of linking numbers modulo 2 for all pairs of non intersecting cycles for a piecewise-linear embedding $\gamma : K_6 \to \mathbb{R}^3$ by $\mathrm{slk}(\gamma)$. Construct some embedding $\gamma_0 : K_6 \to \mathbb{R}^3$ and check that $\mathrm{slk}(\gamma_0) \equiv 1$.

Problem 34. Suppose piecewise-linear embeddings $\gamma_1, \gamma_2 : K_6 \to \mathbb{R}^3$ are equal for all edges except one. Prove that $\mathrm{slk}(\gamma_1) \equiv \mathrm{slk}(\gamma_2)$.

Problem 35. Prove Lemma 3.

2.3.3 Embeddings of the Möbius strip into space

We call a graph $\Gamma \subset M$ such that all its faces are triangles a *triangulation* of the Möbius strip M. Note that in such a case the boundary of M consists entirely of the edges. A map $f : M \to \mathbb{R}^3$ is called *piecewise-linear* if for some triangulation of M map f transforms its faces into triangles in $\mathbb{R}^3$.

Problem 36. **(a)** Construct an embedding of K_6 into M such that one of the cycles goes to the boundary of M and the other cycle, to centre line of M. Let us call these cycles α and β.

(b) Show that if a map $f : M \to \mathbb{R}^3$ is given then no three cycles in $f(K_6)$ cannot be linked except $f(\alpha)$ and $f(\beta)$.

Problem 37. Suppose we have a piecewise-linear embedding $f : M \to \mathbb{R}^3$. Denote the boundary of M by α and denote the centre line of M by β. Prove that $f(\alpha)$ and $f(\beta)$ are linked polygonal curves in $\mathbb{R}^3$.

Problem 38. Prove that Lemma 2 for piecewise-linear embedding $M \to \mathbb{R}^3$ follows from problem 37

Problem 39*. Prove Lemma 2 for continuous embedding $M \to \mathbb{R}^3$.

3 Mirrors and billiards

3.1 Invisible systems of mirrors

In this section we consider *systems of mirrors*, which are formally just collections of curves in the plane. The law "the angle of reflection of a ray equals the angle of incidence" holds for light rays. Mirrors can be curved, then this law applies to a tangent line to the mirror at a point of collision. Rays hitting edges of the mirrors can be neglected. A system of mirrors have to be bounded. All the mirrors cannot be parallel to each other.

Problem 40. Construct a system of mirrors such that it is **(a)** invisible in some direction; **(b)** invisible in some two directions.

Invisibility in a direction of a line l means that every ray parallel to l and starting far enough after some number of reflections continues to follow the same line as there was no reflections at all.

Problem 41. **(a)** Suppose that all the mirrors in a system are line segments and their angles are $0°$ and $90°$. Prove that such system of mirrors cannot be invisible in directions $45°$ and $135°$. Angles are measured with respect to the horizontal axis.

(b) Suppose that all the mirrors in a system are line segments and their angles are $0, \frac{\pi}{n}, \frac{2\pi}{n}, \dots, \frac{(n-1)\pi}{n}$ for some natural n. We call these directions *obtainable*.

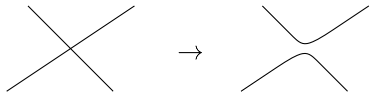
Suppose we have a finite set of lines $l_1, \dots, l_k$, such that any ray of light parallel to some line l_i after reflection in the mirror aligned in obtainable direction will continue its way in a direction parallel to some line l_j. Prove that such system of mirrors cannot be invisible in directions $l_1, \dots, l_k$.

Problem 42^m. Prove that there is no system of mirrors invisible in all directions.

We can consider a transformation of a phase space preserving volume for solving problem 42^m. Such transformations are considered further via less complex examples in problems 46 and 50. Another technique that is needed for solving problem 42^m is shown in the following descrete example.

Problem 43. An island has a shape of a disk D. There are $2n$ transportation hubs of n different companies on the boundary of D. Each company owns 2 hubs, that are connected by a straight railway on which trains travel from end to end. There are bridges at railway crossings so that each train moves in a straight line and cannot turn off at the crossing point.

Minister of Railway Transport decided to cut down on a number of bridges by replacing them by railway switches as shown in the figure.



Also it is required that each train continues to travel between transportation hubs of the same company. Prove that Minister cannot implement his idea.

3.2 Outer billiards

Suppose T is a convex bounded region in the plane $\mathbb{R}^2$. Consider a point x_0 outside T. There exist two supporting lines (tangents) to T from the point x_0. Choose the right one of these lines looking from x_0 and denote it by l. If l intersects T in one point y_1, then denote an image of x_0 under the reflection in y_1 by x_1.

Likewise one can construct the next point x_2 using x_1 (as long as the right supporting line from x_1 intersects T in a unique point), etc. The map R carrying a point in the plane into its image is called the *outer billiard map* and the sequence $x_0, x_1, x_2, \ldots$ is called the *orbit of point* x_0.

If at some step the supporting line intersects the boundary of T in more than one point, but by a line segment, then we say that the orbit of x_0 is not defined.

Problem 44. Suppose T is a square in the plane. **(a)** Draw a set of points for which orbits are defined. **(b)** How to find the length of the orbit of some arbitrary point in $\mathbb{R}^2$?

Problem 45. Solve the similar problem where T is an arbitrary triangle in the plane.

Problem 46. Prove that the map R preserves area (that is, for any region A non intersecting with T areas of A and $R(A)$ are equal), **(a)** if T is a convex polygon; **(b)** if T is an arbitrary strictly convex bounded region.

Let us call an orbit $x_0, x_1, x_2, \ldots$ *almost periodic* if for any $\varepsilon > 0$ and for some $n \in \mathbb{N}$ the distance between x_0 and x_n is less than ε.

Problem 47*. Is it true that any orbit of an outer billiard T is almost periodic if
(a) T is a convex polygon with rational vertices;
(b) T is a convex polygon;
(c) T is a strictly convex bounded region;
(d) T is a convex bounded region?

We do not know a simple solution for that problem. Although the answer for similar question about interior billiards is far more easier as we will see in the next section.

3.3 Interior billiards

Let T be a convex bounded figure on the plane. Let us take a point x inside T and choose some unit vector of velocity for that point. This defines billiard trajectory of the point as follows: inside T the point is moving with constant velocity and on the boundary it change the velocity vector according to the law "The angle of incidence is equal to the angle of reflection" (the angles are measured between current velocity and tangent line to the boundary of T at the point of collision).

If at some moment the point x hits boundary at the point where the tangent line does not exist we say that the trajectory of x is not defined. A trajectory of the point x is said to be *periodic* if after some time since the beginning of the path, x appears in the same place with the same speed.

Problem 48. Let T be a square. **(a)** Which pairs of starting points and velocities yield periodic trajectories? **(b)** Show that any periodic trajectory in T has even number of reflections from the border.

Further, T can be assumed to be a convex polygon or an arbitrary convex figure with smooth boundary.

Problem 49. Fix some vector of velocity v. Let $f_t(x)$ be a position of a point x after time t if it starts movement with velocity v. Show that the area of the image $f_t(A)$ of a region $A \subset T$ not necessarily equals the area of A (here $f_t(A)$ consists of images for all points in A which trajectories are defined).

Problem 50. For any starting point x and initial velocity v consider the position and the velocity of the point after time t. This gives us a mapping $F_t : T \times [0; 2\pi) \to T \times [0; 2\pi)$.

(a) Show that if T is a polygon, then F_t is defined everywhere on $T \times [0; 2\pi)$ except a set of zero volume.

(b) Show that for any t the mapping F_t preserves volume.

Consider point x with some initial velocity vector. The trajectory of x is said to be *almost periodic* if for any $\varepsilon_1, \varepsilon_2 > 0$ there is $t > 0$ such that the distance between x and $f_t(x)$ is less than ε_1 and the angle between velocities at point x and at point $f_t(x)$ is less than ε_2.

Problem 51^m. Show that almost all initial conditions $(x, \varphi) \in T \times [0; 2\pi)$ give us an almost periodic orbit. Here "almost all" means "all except some set of zero volume in $T \times [0; 2\pi)$".

Informally, if we shine a flashlight from a random place in a mirrored room in a random direction, the light will hit the back of our head with good accuracy.

The property of volume-preserving transformations, which is proposed to prove for solving the problem 51^m, can be formulated in the general case as follows:

Poincaré recurrence theorem. Let $U \subset \mathbb{R}^3$ be a bounded domain, $f : U \to U$ — a mapping which preserves volume. Consider lesser domain $A \subset U$. For a point $x \in A$ consider a sequence $f(x)$, $f(f(x))$, $f(f(f(x))), \ldots$. Define $Z \subset A$ as a set of points for which this sequence lies entirely outside of A. Then Z has zero volume.

In order to give mathematical rigor to this claim some specifications should be made about which 'domains' we consider and for which subsets of U the mapping f 'preserves volume'. (Famous paradoxes [4], [5] show that *there is no way* to define a notion of volume for all subsets. For theory see e. g. [Ox]).

Hints and solutions for problems after the semifinal

Topology. Problem about wagons

Closed curves in the plane

Solution for problem 24. The answer is “no”. Indeed (1) does not follow from (2).

Consider two parallel lines l, m. Choose three points on each of the lines: $A_1, A_2, A_3 \in l$ and $B_1, B_2, B_3 \in m$. If three lines A_1B_i, A_2B_j, A_3B_k do not intersect in one point for every $\{i, j, k\} = \{1, 2, 3\}$, then (2) holds and (1) does not hold by construction.

Let us prove that such six points exist. Choose arbitrary points $A_1, A_2, A_3 \in l$ and $B_1, B_2 \in m$. Then there are finitely many prohibited positions for the point B_3. Namely, this number is not greater than three. (For example, denote the intersection $A_1B_1 \cap A_2B_2$ by C. Then B_3 cannot lie on the intersection of A_3C and m.) □

Solution for problem 25. Suppose we have a set of arbitrary points $A_1, \ldots, A_n \in \mathbb{R}^2$. We can also suppose that all the points are different because otherwise we can fix that using arbitrary small movement. Then let us move these points one by one every time making the next point to be in general position with all previous points.

Suppose the points $A_1, \ldots, A_{k-1}$ are in general position. Consider a set of positions prohibited for A_k. Firstly, this set includes all the lines A_iA_j for all possible $i, j < k$. Secondly, this set includes all the lines A_jC where C is the intersection point $A_{i_1}A_{i_2} \cap A_{i_3}A_{i_4}$ for all possible $i_1, i_2, i_3, i_4, j < k$.

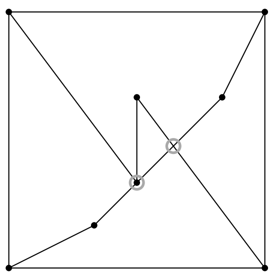
It is clear that the set of prohibited for A_k positions is a finite union of lines. Therefore we can get A_k out of this set by an arbitrary small movement. □

Solution for problem 26. The proof is by induction on m, n. The base of induction is trivial. If $m = n = 1$ then the claim of the problem is that square's diagonals intersect in an odd number of points.

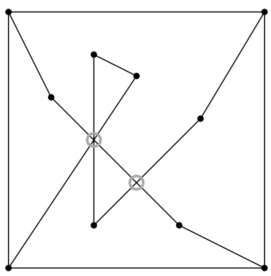
Now suppose $m > 1$. Draw an edge $A_{m-2}A_m$ instead of two edges $A_{m-2}A_{m-1}$ and $A_{m-1}A_m$ for polygonal curve α. Let us prove that β and the triangle's $A_{m-2}A_{m-1}A_m$ boundary have an even number of intersection points, to show that the parity of the number of intersection points between α and β stays the same.

Indeed, if vertices B_k, B_{k+1} are on *different sides* of the boundary $A_{m-2}A_{m-1}A_m$ (i. e. one of the vertices lies inside the triangle and the other one lies outside), then the line segment B_kB_{k+1} intersects the boundary $A_{m-2}A_{m-1}A_m$ in an odd number of points. But if these vertices are on the *same side*, then the intersection consist of an even number of points. Since A_0 and A_n are both on the same side (i.e. they are both outside the triangle), the number of intersection points is even. □

Solution of problem 27. Answer: **(a)** no; **(b)** no. You can see examples in the figures below.



(a)



(b)

Remark. For the set of the polygonal curve's vertices (2) does not hold in our example for (a). There is no example for (a) such that (2) holds.

It is possible to prove that a number of intersection points is always odd for (b) by adding the restriction for polygonal lines not to be self-intersecting. To do so one should take notice of the intersection points while moving vertices of the polygonal curves to a general position (e. g. see proof of Lemma 1). □

Solution for problem 28. (a) Suppose $\alpha \cap \beta = \varnothing$. Take $\varepsilon > 0$ such that any distance between a point from α and a point from β is less than ε. Then, we cannot get any more points of intersection moving vertices of α and β by not more than $\frac{\varepsilon}{2}$ since any point on an edge also moves by not more than $\frac{\varepsilon}{2}$. But according to the problem 25 we can get α and β into general position moving each vertex by not more than $\frac{\varepsilon}{2}$. This concludes our proof.

Note that ε can be less than the minimal distance between the vertices of α and β. One way to compute ε is the following. Take a vertex from one polygonal curve and an edge from another curve, measure the distance between them. Do this for every possible combination of a vertex and an edge from different polygonal curves, and take ε less than any of these numbers.

(b) *Sketch of solution.* There is a way to approximate any continuous curve by a polygonal curve with accuracy of any $\varepsilon > 0$. This means that we draw points on a curve one by one in such a way that the distance between a curve's ark between two neighbouring points and the line segment connecting these points is less than ε. We need to construct such approximations for $\varepsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. Then for each ε we need to find an intersection point P_i of two polygonal curves. For each point P_i there are points A_i and B_i on continuous curves such that distance between A_i and B_i is not greater than $\frac{1}{2^{i-1}}$. Finally, we need to choose convergent subsequences with equal indices from A_i and B_i using the compactness of the line segments. These subsequences have equal limits, therefore this limit is the required intersection point of our curves. □

Problem about wagons

Solution for problem 19. Let us identify the road with the line segment $[0;1]$, and time with the line segment $[9;21]$. Time dependences for a current position of the tourist on the road are continuous functions $[9;21] \to [0;1]$. Denote such functions for her way to the camp and back by f and g respectively. We have $f(9) = g(21) = 0$ and $f(21) = g(9) = 1$.

Consider function $h = f - g$. Function h is negative at point 9 and positive at point 1. Using intermediate value theorem, we obtain the existence of some point $t \in [9;21]$ such that $h(t) = 0$. Therefore $f(t) = g(t)$ for some time t, i.e. the tourist was in some point of the road at time t both days. □

Solution for problem 20^m. Identify each road with the line segment $[0;1]$. Then the phase space for the two vehicles on these roads is the square $[0;1] \times [0;1]$.

piecewise-linear movement of the cars gives us a polygonal line in the square going from the vertex $(0,0)$ to the vertex $(1,1)$. piecewise-linear movement of the wagons gives us a polygonal line in the square going from the vertex $(0,1)$ to the vertex $(1,0)$. Using Lemma 1, we obtain the existence of the common point for these polygonal curves. Therefore, at some time the position of the cars will be similar with the position of the wagons. Hence, it is impossible for wagons to pass each other without colliding. □

Solution for problem 21. (a) Answer: yes.

Suppose that there is no horizontal sections of the path. Identify each of the paths with the line segment $[0;1]$. Then the set of all possible positions of the mountain climbers (on the same height) is a subset M of the square $[0;1] \times [0;1]$ containing the square's vertices $(0,0)$ and $(1,1)$.

Note that M consists of line segments. Let us consider it as a graph: states in which at least one of the climbers is in the vertex of the polygonal path are the vertices of the graph. It is easy to show that the degree of each vertex except $(0,0)$ and $(1,1)$ is even. Therefore there exists a path from $(0,0)$ to $(1,1)$.

If some sections of the mountain are horizontal, then "collapse" these sections into points. We call these points *special*. For the new mountain the algorithm exists as was proven above. Suppose a climber stays at this point for a minute when the climber passes through it. This adds finite amount of minutes to overall time. It is easy to obtain an algorithm for the initial mountain from this algorithm.

Also note that if no two vertices of the polygonal path are on the same height then there exists a unique way to get to the top such that the go through every position not more than one time.

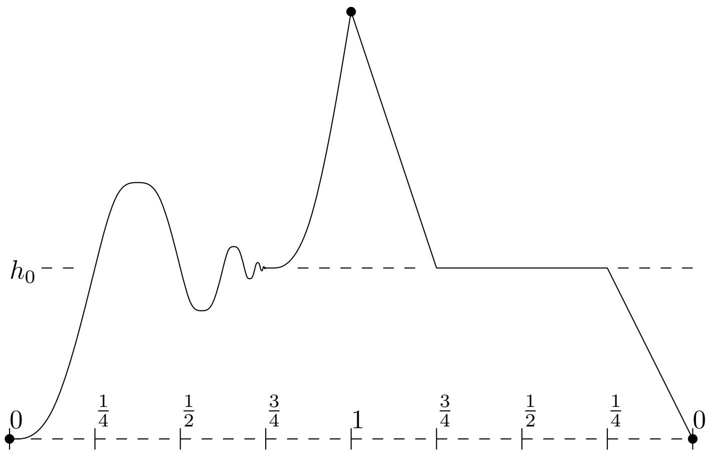
(c) Answer: yes. The proof is by induction on the number of climbers. The base of induction for one climber is trivial.

Consider the first two climbers and their algorithm for reaching the top from (a). Time dependence for their height is piecewise-linear. Draw the plot for this dependence and say that this is a new path to the top. Then we can swap the first two climbers for one climber moving along the new path. Thus, we can swap k climbers for $k-1$ climbers.

(b) Answer: no.

The example of such paths is shown in the figure. One of them is constructed the following way. In each of the neighbourhoods of the point $\frac{3}{4}$ our path has both monotonically increasing section and monotonically decreasing section. Each of these sections goes through the height h_0.

The second climber needs to travel from the point $\frac{1}{4}$ to the point $\frac{3}{4}$ back and forth infinitely many times to get to the top. Therefore time dependence for the second climber cannot be continuous under such condition. The proof of this conjecture is a great exercise on continuous functions.



We recommend participants to draw the set $M \subset [0;1] \times [0;1]$ for this mountain defined exactly as in (a), and then to show that the continuous path from the point $(0,0)$ to the point $(1,1)$ does not exist. □

Topology. Rectangle inscribed in a curve

Linkings of closed curves in space

Solution for problem 29. This solution is analogous to the solution for problem 25. Let us move the points one by one to get them into general position. The first three points we will move in such a way that they do not lie on the same line. The set of the prohibited positions of A_k after the k-th move ($k \geqslant 4$) consists of the finite amount of planes. Therefore we can get A_k out of this set by an arbitrary small move. □

Solution for problem 30. Since the set of all vertices is a set in general position, neither of the polygonal line's vertices lies on the tetrahedron's surface and any of its edges can intersect the tetrahedron's surface only in interior points of the faces.

Denote vertices of the polygonal line by $A_0, A_1, \ldots, A_n = A_0$. Colour point A_i red if it is inside the tetrahedron, otherwise colour it blue. Then the edge A_iA_{i+1} of the polygonal curve intersects the surface of the tetrahedron in an odd number of points if and only if A_i and A_{i+1} are coloured in different colours. We have an even number of such edges. $\square$

Solution for problem 31. **(a)** Denote the vertices of the polygonal curves α and β by $A_1, \dots, A_n$ and $B_1, \dots, B_m$ respectively.

Suppose, $\{A_1, \dots, A_n, B_1, \dots, B_m, O, O'\}$ is in general position. By problem 30 for each tetrahedron $OO'A_iA_{i+1}$ the number of intersections of its surface with β is even. Then take the sum for all such tetrahedrons and we are done.

But in general $\{A_1, \dots, A_n, B_1, \dots, B_m, O, O'\}$ may not be in general position. However, by the problem statement the sets $\{A_1, \dots, A_n, B_1, \dots, B_m, O\}$ and $\{A_1, \dots, A_n, B_1, \dots, B_m, O'\}$ are in general position. Choose a point $O'' \in \mathbb{R}^3$ such that when we add O'' to each of the sets it remains to be in general position. Then replace the point O by O'' and then O'' by O' as it was described the previous paragraph.

(b) *Hint.* Take two parallel planes in $\mathbb{R}^3$ such that α and β lie between them. Choose the point O on the first plane. We can compute $\mathrm{lk}(\alpha, \beta)$ and $\mathrm{lk}(\beta, \alpha)$ using the point O.

Project α and β onto the second plane from the point O. Denote the obtained polygonal curves by $\tilde{\alpha}$ and $\tilde{\beta}$. To show that $\mathrm{lk}(\alpha, \beta) \equiv \mathrm{lk}(\beta, \alpha)$ it is enough to check that $\tilde{\alpha}$ and $\tilde{\beta}$ intersect at an even number of points. This can be proved similarly to problem 26. $\square$

Solution for problem 32. **(a)** The proof is similar to problem 28. Take $\varepsilon > 0$ such that the distance between any point of α and any point of β is less than ε and also the distance between any two points on nonadjacent edges of α or nonadjacent edges of β is less than ε. Then if we move the vertices of α and β by the distance $< \frac{\varepsilon}{2}$, then intersection points do not appear since any point of any edge also moves by the distance $< \frac{\varepsilon}{2}$. By problem 29, we can transform α and β to general position moving each of the vertices to the distance $< \frac{\varepsilon}{2}$.

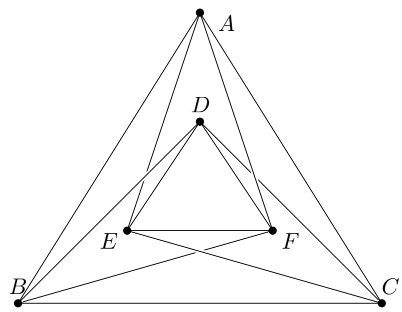
(b) Suppose the polygonal curves α' and β' are in general position. Move one vertex A'_i of α' so that when it moves, the edges $A'_{i-1}A'_i$ and $A'_iA'_{i+1}$ are disjoint with the other edges of α' and β'. Denote the obtained vertex by A''_i and the obtained polygonal curve by α''. Next we will show that $\mathrm{lk}(\alpha', \beta') \equiv \mathrm{lk}(\alpha'', \beta')$.

Take a point O in general position with all the vertices of α' and β' including A''_i. Consider the intersection of the tetrahedrons $OA'_{i-1}A'_iA''_i$ and $OA'_iA''_iA'_{i+1}$ with β'. By the construction, the faces $A'_{i-1}A'_iA''_i$ and $A'_iA''_iA'_{i+1}$ are disjoint with β'. Then $\mathrm{lk}(\alpha', \beta') \equiv \mathrm{lk}(\alpha'', \beta')$ by problem 30. $\square$

Solution for problem 33. Take an embedding $\gamma_0 : K_6 \to \mathbb{R}^3$ as in the figure.

For this embedding the cycle AEF is linked modulo 2 with the cycle DBC, the cycle BFD is linked modulo 2 with the cycle ECA, and the cycle CDE is linked modulo 2 with the cycle FAB.

There are no other pairs of cycles which are linked modulo 2 for this embedding $K_6 \to \mathbb{R}^3$. Therefore $\mathrm{slk}(\gamma_0) \equiv 1$. $\square$



Solution for problem 34. Suppose the embeddings $\gamma_1, \gamma_2 : K_6 \to \mathbb{R}^3$ differ only on the edge AB. Consider the pair of cycles ABC and DEF. Denote the polygonal curve $\gamma_1(ABC)$ by α, the polygonal curve $\gamma_2(ABC)$ by α', the polygonal curve $\gamma_1(DEF)$ by β, and the union $\gamma_1(AB) \cup \gamma_2(AB)$, which is also a closed polygonal curve, by α''. Then one can show that

$$\mathrm{lk}(\alpha, \beta) - \mathrm{lk}(\alpha', \beta) \equiv \mathrm{lk}(\alpha'', \beta).$$

There are 4 pairs of cycles in K_6 such that one of the cycles contains the edge AB. If we sum such the equalities for each of the pairs of the cycles, then we have

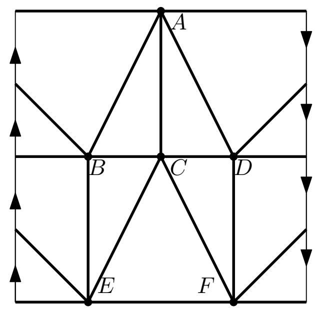
$$\mathrm{slk}(\gamma_1) - \mathrm{slk}(\gamma_2) \equiv \mathrm{lk}(\alpha'', \gamma_1(DEF)) + \mathrm{lk}(\alpha'', \gamma_1(CEF)) + \mathrm{lk}(\alpha'', \gamma_1(CDF)) + \mathrm{lk}(\alpha'', \gamma_1(CDE)).$$

It remains to show that the sum on the right is zero. This assertion should remind problem 30. The difference is that here instead of a tetrahedron in $\mathbb{R}^3$ we have a piecewise-linear embedding of the graph $K_4 = CDEF$, and instead of the number of intersection points of α'' with the surface of the tetrahedron we compute the sum of linking numbers modulo 2.

Take a point O in general position with all other vertices of the polygonal curves. Then the equality of the sum on the right to zero follows by the definition of linking number modulo 2. □

Solution for problem 35. By problem 34 the sum $\mathrm{slk}(\gamma)$ modulo 2 does not depend on the embedding $\gamma : K_6 \to \mathbb{R}^3$. Then by problem 33 for all embeddings $K_6 \to \mathbb{R}^3$ we have $\mathrm{slk} \equiv 1$. Therefore, for each embedding $K_6 \to \mathbb{R}^3$ there is a pair of cycles which images are linked. □

Embeddings of the Möbius strip into space



Solution for problem 36. (a) The embedding is shown in the figure.

(b) If a cycle in K_6 bounds a polygon in M, then the image of this cycle cannot be linked with the image of any other cycle.

Let us list all the cycles of length 3 that do not bound a polygon: ABD, AEF, BCD, BEF. There is only one pair of disjoint cycles: AEF and BCD. The first one of them is the curve α and the second one is the curve β. Since $\mathrm{slk}(f(K_6)) \equiv 1$, we have $\mathrm{lk}(\alpha, \beta) \equiv 1$. □

Solution for problem 37. If K_6 is embedded into M as in problem 36 (a), then f sets a piecewise-linear embedding $K_6 \to \mathbb{R}^3$. It remains to use problem 36 (b). □

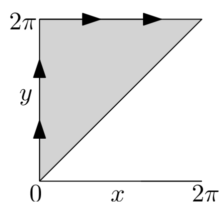
Solution for problem 38. Suppose such an embedding exists. As we have shown in problem 37, the image of the boundary of M must be linked with the image of the centre line of M. But then the image of the boundary lies in a plane and the image of the centre line lies in one of the semispaces. Therefore these curves cannot be linked, so we have a contradiction. □

Solution for problem 39*. One can try to reduce this problem to the piecewise-linear one, similarly to the proof of problem 28 (b). The difficulty is that the piecewise-linear maps $M \to \mathbb{R}^3$ which approximate the given continuous embedding also must be *embeddings*. It is hard to control this condition.

Another way that is much easier is to define the linking number modulo 2 for a pair of closed continuous curves in $\mathbb{R}^3$. To this end we approximate the curves by polygonal lines. One can show that if approximations are sufficiently close to the original curves then their linking number modulo 2 does not depend on the choice of the approximations. Further, we have to show that if one of the curves bounds a disk (continuously embedded into $\mathbb{R}^3$) which is disjoint from the other curve, then the linking number equals zero.

For more conceptual approaches to the definition of the linking number for closed continuous curves in $\mathbb{R}^3$ see [H, §1.2] and [FF, 17.6]. □

Rectangle inscribed in a curve



Solution for problem 23. We assign to each unordered pair of points in the circle an ordered pair of points $x \leqslant y$ in the segment $[0; 2\pi]$.

Then pairs $(0, a)$ and $(a, 2\pi)$ of points in the segment correspond to the same pair of points in the circle. So the configuration space of pairs of points in the circle can be identified with the triangle glued as in the figure to the right. We suggest the participations to prove it.

Note that the boundary of M corresponds to the set of pairs of coinciding points. □

Solution for problem 22^m. We denote the circle by S^1. Suppose we are given a continuous map $\gamma : S^1 \to \mathbb{R}^2$. We assume $\mathbb{R}^2$ to be embedded into $\mathbb{R}^3$ as the plane $z = 0$.

Next we define the map of the configuration space of pairs of points in the circle to $\mathbb{R}^3$ as follows. We set the image of an unordered pair (a, b) to be the point $\left(\frac{\gamma(a)+\gamma(b)}{2}, |\gamma(a) - \gamma(b)|\right)$. Here $\frac{\gamma(a)+\gamma(b)}{2}$ is the middle point of the segment in $\mathbb{R}^2$ connecting $\gamma(a)$ with $\gamma(b)$.

We obtain a map $M \to \mathbb{R}^3$. Note that the image of the boundary of M coincides with the given curve γ. According to Lemma 2 this map cannot be an embedding. So there are two different pairs of points in γ such that the segments between them have the same middle point and the same length. These four points are the vertices of a rectangle.

This proof is shown in the video [6] on YouTube. □

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