Determinants in graph theory

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1 Preliminaries

Graphs. Informally speaking, a *graph* is a set of points (vertices, or nodes) connected with linear segments (edges). The formal definition is as follows: a graph G is determined iff we have chosen the set of its *vertices* $V(G)$ and the set of its *edges* $E(G)$ which consists of some pairs (v_1v_2) of elements from V. A graph is called *undirected* if the order of elements in a pair is ignored, that is, pairs (v_1v_2) and (v_2v_1) are considered as the same pair. Otherwise the graph is called *directed* or briefly *a digraph*. In the sequel we assume that the set $V(G) = \{v_1, v_2, \dots, v_n\}$ is *finite*, so that the set $E(G)$ is finite as well.

For $(v_1v_2) \in E(G)$ we say that *the edge v_1v_2 connects the vertices v_1 and v_2*. Also we say that this edge is *incident* to the vertices v_1 and v_2 and these vertices are *incident* to the edge v_1v_2. Then the vertices v_1 and v_2 are called *adjacent*, otherwise *non-adjacent*. If any two vertices are adjacent then the graph is called *complete*.

The valency, or degree, of a vertex of a graph is the number of edges incident to this vertex. The valency of a vertex v is denoted $\deg v$. *The exit valency* and *the entry valency*, respectively, is the number of edges which exit or enter the given vertex.

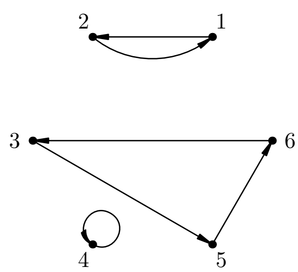
A square matrix A of order n is a table $n \times n$ whose cells contain reals: the number at the intersection of ith row and jth column is denoted a_{ij}. *The adjacency matrix* of a non-directed graph G is the $n \times n$ matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. This matrix is necessarily symmetric (relative to the *main diagonal* that connects the upper left and lower right corners).

The adjacency matrix of a directed graph G is the $n \times n$ matrix such that $a_{ij} = 1$ if G contains an edge from v_i to v_j, and $a_{ij} = 0$ otherwise. This matrix is not necessarily symmetric.

A path in a graph is a sequence $v_{i_1}v_{i_2}, \dots, v_{i_{k-1}}v_{i_k}$ of its edges ($k \geqslant 1$), where the first vertex of the next edge coincides with the second vertex of the preceding one. A set consisting of a single edge is also considered as a path. If $v_{i_k} = v_{i_1}$, so that the path is closed then it is called *a cycle*. A graph is called *connected* if there exists a path between any of its vertices or the graph has a single vertex. A graph is called *a tree* if it is connected and contains no cycles. It is easily seen that then and only then any two of its vertices are connected by a single path.

Permutations and their parity. *A permutation* of the set $\{1, \dots, n\}$ is a mapping of this set into itself such that distinct elements map to distinct ones and each element is the image of some element (maybe the same one). If π is a permutation then $\pi(i)$ denotes the element obtained from i under action of this permutation.

A directed graph such that the exit and the entry valency of each vertex equals 1 will be called *univalent*. It may contain *loops* (edges such that both their endpoints coincide). Each permutation π determines a univalent graph on the zet of vertices $\{1, 2, \dots, n\}$: this graph contains some directed edge ij iff $\pi(i) = j$. The figure illustrates the case $n = 6$: $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 5$, $\pi(4) = 4$, $\pi(5) = 6$, $\pi(6) = 3$.



The product of permutations σ and τ is the permutation realized by consecutive execution of σ and τ. *Transposition* is a permutation which interchanges two elements and fixes all the others.

It is not difficult to see that any permutation is a product of transpositions. No permutation can be represented as a product of both even and odd number of transpositions. If a permutation is a product of even number of transpositions then it is called *even*, otherwise *odd* permutation. *The sign of a transposition* is $+1$ for an even permutation and -1 for an odd one. The sign of a permutation π is denoted $\operatorname{sgn}(\pi)$. If a permutation corresponds to a univalent graph then its sign equals $(-1)^m$, where m is the number of even (including an even number of edges) cycles in the graph.

The determinant of a matrix. To each square matrix $A = (a_{ij})$ there corresponds a real number which is called its *determinant* and is equal to the sum of products

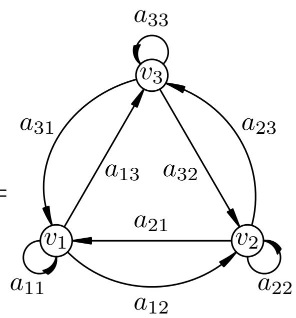
$$\det A = \sum_{\pi} \operatorname{sgn}(\pi) a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdot \ldots \cdot a_{n,\pi(n)}, \tag{1}$$

where π runs over all permutations of $\{1, \ldots, n\}$, so the sum contains $n!$ summands. If $\det A = 0$ then the matrix A is called *degenerate*, otherwise *non-degenerate.*

For every matrix A consider a complete digraph $\mathcal{A}$ on n vertices (with loops), such that to each edge $v_i v_j$ there corresponds real a_{ij} called *the weight* of this edge.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$\mathcal{A} =$



In the sequel, we consider only the univalent subgraphs of $\mathcal{A}$ that contain all its vertices.

Each permutation π in the sum (1) determines a univalent subgraph in $\mathcal{A}$ (and conversely, each univalent subgraph determines a permutation). For each univalent subgraph consider the product of weights of all its edges; if the subgraph contains an odd number of cycles with even number of vertices then multiply this product by -1. The result will be called *the weight* of the subgraph. Thus the weight of the univalent subgraph corresponding to a permutation π is equal to a summand of (1), and so we may accept the alternative definition.

The determinant of a matrix A is the sum of weights of all univalent subgraphs in $\mathcal{A}$.

For the graph $\mathcal{A}$ from our example, all univalent subgraphs and their weights are shown at fig. 1. Thus

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31}.$$

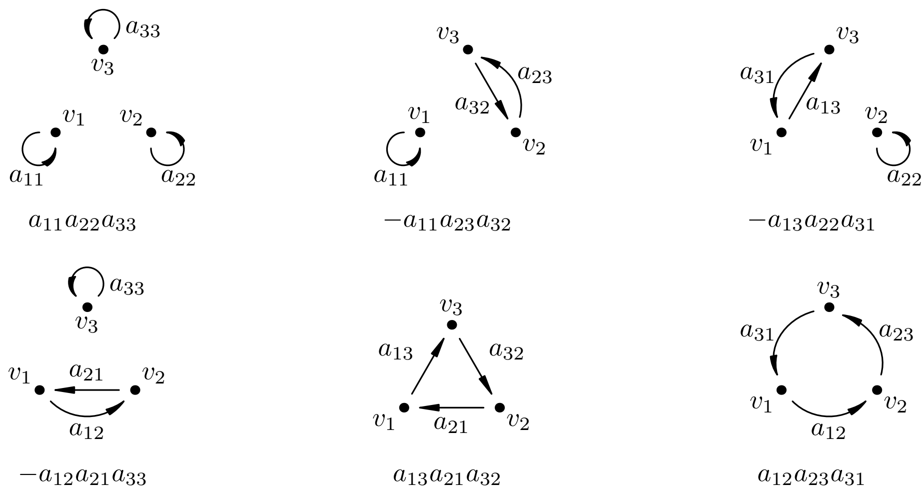


Рис. 1. Univalent subgraphs of the graph $\mathcal{A}$ and their weights

2 Properties of determinants

2.1. Using the combinatorial definition of the determinant, prove its following properties.

a) If each element of some row of a matrix A is multiplied by a real c then the determinant is multiplied by c as well.

b) If two rows of a matrix are interchanged then the determinant changes its sign.

c) Suppose in the kth column of a matrix A the diagonal element is 1 and the other elements are 0. Let $\widetilde{A}$ be the matrix obtained from A by deletion of the kth row and the kth column. Then $\det A = \det \widetilde{A}$.

d) Given a matrix $A = (a_{ij})$ and matrices $A^{(1)}$ and $A^{(2)}$ identical to A except the jth row, and for the elements of the jth row we have

$$a_{ij} = a_{ij}^{(1)} + a_{ij}^{(2)}.$$

Then $\det A = \det A^{(1)} + \det A^{(2)}$.

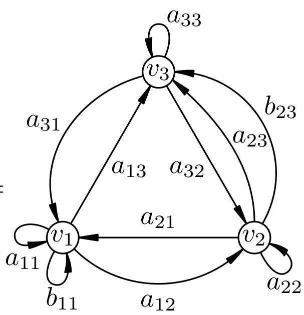
e) Let A_{ij} be the matrix obtained from the matrix A by deletion of the ith row and the jth column. Then we have the following formula for development along the ith row:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

Regarding Problem 2.1 d), let us mention the following lifehack.

2.2. Given a matrix $A = (a_{ij})$ such that some matrix elements are represented as sums, see the example below. Consider the digraph $\mathcal{A}$ containg a single edge for each summand. Unlike the standard case, this graph may contain multiple edges and multiple loops (with different weights). Then the determinant of A is still equal to the sum of weights of all univalent subgraphs in the constructed graph $\mathcal{A}$.

$$A = \begin{pmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathcal{A} =$$



2.3. Suppose that the sum of elements in each row of a matrix A equals 0: specifically, each diagonal cell of A contains the number equal to the minus sum of the remaining elements of the same row.

a) Prove that $\det A = 0$.

b) Prove that for each i and any j_1, j_2 we have $\det A_{ij_1} = (-1)^{j_2-j_1} \det A_{ij_2}$ (notation from 2.1 e).

2.4. Suppose that the graph G corresponding to the matrix A contains a subgraph H on 8 vertices, shown at fig. 2 at the left. Furthermore the graph may contain edges incident to white vertices and not shown at the figure, however the black vertices have valency 3, so all their edges are shown. Replace this subgraph with the subgraph H' shown at fig. 2 at the right, where the new weights are of the form

$$x' = \frac{y}{wz - xy}, \quad y' = \frac{x}{wz - xy}, \quad z' = \frac{w}{xy - wz}, \quad w' = \frac{z}{xy - wz}. \tag{2}$$

Let A' be the matrix corresponding to the new graph. Then

$$\det A = (xy - wz)^2 \det A'. \tag{3}$$

2.5. Suppose that G is the graph shown at fig. 3 (consisting of $m-1$ «concentric» squares), A is the adjacency matrix of this graph. Prove that

$$\det A = \begin{cases} m^2 & \text{for } m \text{ odd}, \\ 0 & \text{for } m \text{ even}. \end{cases}$$

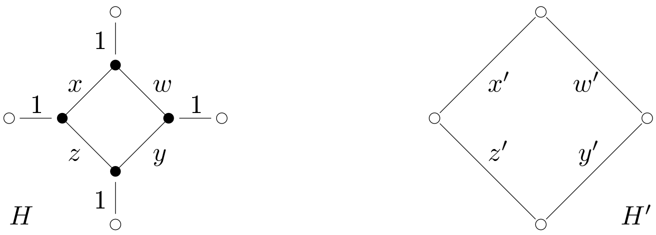


Рис. 2. Restructuring of a subgraph.

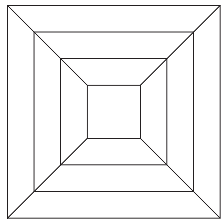


Рис. 3. Graph $C_4 \times P_{m-1}$.

3 Determinants of checkered figures

Suppose F is a bounded checkered figure on the checkered plane, G_F is the graph dual to F. This means that the vertices of G correspond to the cells of F, and the edges connect the cells adjacent by side. It is suitable to consider G_F and its subgraphs as directed, assuming that two adjacent cells are connected by two edges with opposite directions. Let A_F denote the adjacency matrix of G_F. *Tilings* mean dissections of F into dominoes. A vertical edge of a univalent subgraph in G_F will be called *ascending* if it is directed up, and *descending* if it is directed down. (Remind that we consider only univalent subgraphs that contain all vertices of the graph.) The figure F is called simple connected if it is «without holes», or more formally, if its dual graph is connected and the dual graph of its complement is connected too.

3.1. Prove that the number of univalent subgraphs of G_F equals the square of the number of tilings of F.

3.2. Let P be a simple connected checkered polygon such that its border contains a points with even y-coordinate and b points with odd y-coordinate, and there are d points with integer coordinates inside it. Then the sum of lengths of vertical sides of P is equivalent to $a-b+2d+2$ modulo 4.

3.3. Let F be a simple connected checkered figure with even area. Then either for each univalent subgraph of G_F the number of ascending edges and the number of cycles have the same parity or for each univalent subgraph of G_F these parities are opposite.

The sign of a figure F will mean the number $\operatorname{sgn} F = (-1)^h$, where h is the number of horizontal dominoes in some tiling of the figure. Let c_k denote the number of tilings of F, containing just k vertical dominoes. The polynomial $f_F(x) = \sum\limits_{k=0}^{+\infty} c_k \cdot x^k$ will be called *the vertical polynomial* of F. A pair of tilings will be called *good* if the numbers of vertical dominoes in these differ just by 2.

3.4. Prove that the definition of the sign of a figure is correct.

3.5. For each simple connected checkered figure F

$$\det A_F = \operatorname{sgn} F \cdot \sum_{\pi} (-1)^{\text{the number of ascending edges in } \pi}.$$

$$\det A_F = \operatorname{sgn} F \cdot f_F^2(\mathbf{i}).$$

In the first formula summation is spread over all univalent subgraphs of G_F, in the second formula $\mathbf{i} = \sqrt{-1}$.

3.6. Let F be an arbitrary simple connected checkered figure with area $2s(F)$. If all tilings of F can be split into good pairs then $\det A_F = 0$. If all tilings except one can be dissected into good pairs then $\det A_F = (-1)^{s(F)}$.

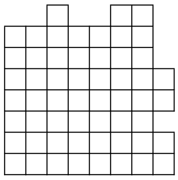


Рис. 4. A regular 8-stamp

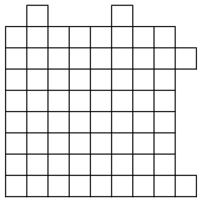


Рис. 5. An irregular 9-stamp

Let *an n-stamp* mean a square $n \times n$ maybe with removal of some cells adjoining to the upper of right side. Enumerate the row of an n-stamp by numbers from 1 to n upwards, and the columns from left to right. A cell will be denoted by the pair consisting of the numbers of its row and column. Call an n-stamp *regular* (fig. 4) if in any pair of cells (n, i) and (i, n) with $i < n$ just one was removed and the cell (n, n) was removed as well. The other stamps will be called *irregular* (fig. 5).

3.7. Let F be any regular n-stamp. Then $\det A_F = (-1)^{n(n-1)/2}$. And if F is an irregular n-stamp then $\det A_F = 0$.

3.8. For an arbitrary $n \times m$ rectangle

$$\det A_{n\times m} = \begin{cases} 0, & \text{if } (n+1, m+1) \neq 1; \\ (-1)^{\frac{n \cdot m}{2}}, & \text{if } (n+1, m+1) = 1; \end{cases}$$

where (n, m) is the greatest common divisor of n and m.

4 Spanning trees

Let G be an arbitrary (non-directed) connected graph. *A spanning tree* of G is a tree whose set of vertices is the same as for G, and the set of edges is included in the set of edges of G.

Our goal is to determine the number of spanning trees of a given connected graph if we know its structure. For this, we require the *Laplacian matrix* of G: this is the $n \times n$ matrix $L = (\ell_{ij})$ such that

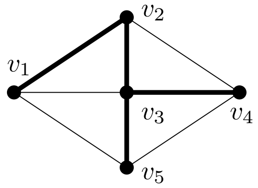
$$\ell_{ij} = \begin{cases} \deg v_i & \text{for } i = j; \\ -1 & \text{for } i \neq j \text{ and adjacent vertices } v_i \text{ and } v_j; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

By assertion of Problem 2.3 a we have $\det L = 0$. Let L^- denote the matrix obtained from the Laplacian matrix by removing the last row and the last column.

We have the Matrix-tree theorem: *the number of spanning trees of a connected graph equals* $\det L^-$.

A suitable starting point to prove this theorem is the lifehack from Problem 2.2 or the following construction.

Let us express the determinant of L^- as the sum of products of its elements, representing each diagonal element as a sum of units or minus units, and expanding brackets. The resulting expression will be called the *superexpansion* of the determinant. To each summand in the superexpansion we associate the following digraph having the vertices $1, 2, \ldots, n$ and signs «$+$», «$-$» at the edges (fig. 6). Circle the factors of this summand (these are units and minus units, one in each row and in each column). If a minus one is circled at the meet of row i and column j then we draw a *negative* edge (with the minus sign) from v_i to v_j. If in a diagonal element a_{ii} we circle the kth unit then we draw a *positive* edge (with the plus sign) from v_i to the kth smallest neighbor of V_i (it is the vertex v_j adjacent to v_i and such that there are just $k-1$ vertices adjacent to v_i and having numbers smaller than j). Clearly each of the resulting digraphs arises from a single summand of the superexpansion. Now we may define *the sign of the digraph* as a whole, meaning the sign of the corresponding summand in the superexpansion.



$$\begin{pmatrix} 1+1+1 & -1 & -1 & 0 \\ -1 & 1+1+1 & -1 & -1 \\ -1 & -1 & 1+1+1+1 & -1 \\ 0 & -1 & -1 & 1+1+1 \end{pmatrix}$$

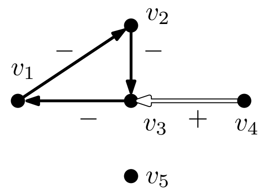


Рис. 6. Left part: the graph G and one of its spanning trees. Central part: the matrix L^- for G. The circled numbers form one of summands in the superexpansion. The sign of the summand is $(-1)^m\mathrm{sgn}(\pi)$, where m is the number of chosen minus units, $\mathrm{sgn}(\pi)$ is the sign of the associated permutation, in our case $\pi = (2314)$ and its sign is «+». Right part: the digraph associated with this summand.

4.1. Prove the Matrix-tree theorem.

4.2. Let $E_{i,j}$ denote the $n \times n$ matrix such that the cell in the ith row and jth column contains 1 and the others contain zeroes (such matrices are called matrix units). Prove that the number of spanning trees of G equals

a) $\det(L + E_{i,i})$ for each i, b) $\det(L + E_{i,j})$ for each i, j.

4.3. Let L again denote the Laplacian matrix of a graph G. Fix a set of vertices $v_1, v_2, \ldots, v_k$ of G ($k < n$) and remove the corresponding rows and columns from L. Denote the resulting $(n-k) \times (n-k)$ matrix by L_k^-. The subgraph of G that contains all vertices of G and consists of k trees without common vertices or edges, containing, respectively, vertices $v_1, v_2, \ldots, v_k$ will be called the *spanning forest* based on $v_1, v_2, \ldots, v_k$. Some trees of the spanning forest may consist of a single vertex.

Prove that the determinant of the matrix L_k^- is equal to the number of spanning forests based on $v_1, v_2, \ldots, v_k$.

4.4. Let G be a digraph without loops. Its Laplacian matrix L^- is defined by the formula (4) with the only modification: for $i = j$ take the exit valency of v_i, and for $i \neq j$ put $\ell_{ij} = -1$ only if there is an edge from v_i to v_j. *A rooted directed spanning tree* with the root v is the subgraph of a digraph G, such that it contains all vertices of G and for each vertex there is a unique path to v. Prove the following version of the Matrix-tree theorem for digraphs: $\det L^-$ equals the number of rooted directed spanning trees with the root v_n.

4.5. In a digraph without loops, the entry valency of any vertex equals its exit valency. Prove that the number of rooted directed spanning trees with a given root does not depend on the root.

4.6. Using the Matrix-tree theorem, prove the *Cayley formula*: the number of trees on n enumerated vertices equals n^{n-2}.

A graph G is called *bipartite* if its set of vertices consists of two disjoint *color classes* such that no two vertices of the same class are adjacent.

4.7. Prove that the number of spanning trees of a complete bipartite graph $K_{k,m}$ equals $k^{\ell-1}\ell^{k-1}$.

4.8. A graph G has an even number of vertices, and the valency of each vertex is even. Prove that the number of spanning trees of G is even.

5 Determinants and existence of perfect matchings

In this section we work with (non-directed) bipartite graphs. In this section we assume $n = 2m$ and we consider only bipartite graphs such that both classes contain the same number m of vertices. *The bipartite adjacency matrix* of a bipartite graph G is the $m \times m$ matrix $B = (b_{ij})$ such that $b_{ij} = 1$ if the ith vertex of the first class is adjacent to the jth vertex of the second class, and $b_{ij} = 0$ otherwise. If in this matrix each element b_{ij} equal to 1 is replaced by a variable x_{ij} then the resulting matrix $\tilde{B}$ is called *the variable bipartite adjacency matrix* of G. *A perfect matching* in a graph G is a subset $M \subseteq E(G)$ of the set of its edges, such that each vertex of G is incident to a single edge from M.

5.1. Prove that if the color classes of a bipartite graph G contain equal number of vertices then the existence of a perfect matching in G is equivalent to the fact that $\det \tilde{B}$ is not the zero constant (as a polynomial).

How can we check, in a sensible time, that the bipartite adjacency matrix A is non-degenerate? For this, the following theorem is useful.

5.2. The Schwartz–Zippel theorem. Suppose d is a positive integer, S is a set of s reals, $p(x_1, \dots, x_m)$ is a polynomial of degree d in m variables with real coefficients. Then the number of tuples $(r_1, \dots, r_m)$, where $r_1, \dots, r_m \in S$ and $p(r_1, \dots, r_m) = 0$, does not exceed ds^{m-1}.

5.3. Given a bipartite graph G whose color classes consist of m vertices each. Using the Schwartz–Zippel theorem, construct an algorithm which checks existence of a perfect matching and fails in at most half of cases.

Repeated application of this algorithm enables us to make the probability of an error arbitrarily small.

6 Counting perfect matchings in a planar bipartite graph

A graph is called *planar* if it can be drawn in the plane without self-intersections. This means that its vertices can be represented by points of the plane, and the edges incident to them by curves with ends in these points so that internal points of these curves don't belong to other such curves. We call *a domain* any part of the plane such that its border consists of images of edges, and the internal (remaining) points don't belong to images of edges. For any such picture of a connected planar graph we have *Euler formula*

$$v - e + f = 2,$$

where v is the number of vertices, e is the number of edges, f is the number of domains.

If in the expression (1) for the determinant of a matrix A all summands are taken with the plus sign then we obtain *the permanent* of A, denoted by $\mathrm{per}(A)$.

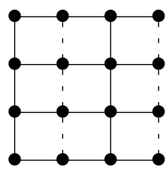
6.1. Prove that the number of perfect matchings in a bipartite graph equals the permanent of the bipartite adjacency matrix.

Calculation of the permanent requires much more work than that of the determinant. So the question arises whether it is possible to reduce the calculation of the permanent of a bipartite adjacency matrix to the calculation of its determinant changing signs of some elements of the matrix?

Let G be a planar bipartite graph. To its edges we attach signs «+» and «−». In the bipartite adjacency matrix B of this graph replace 1 by -1 if the corresponding edge has minus sign. We denote this signing by σ, and the resulting matrix by B^σ. *A Kasteleyn signing* is a signing σ such that

$$|\det B^\sigma| = \mathrm{per}(B).$$

6.2. The picture shows a graph on 16 vertices. The continuous lines represent edges with plus sign, and the dotted lines correspond to edges with minus sign. Prove that this signing is Kasteleyn one.



Suppose C is a cycle of length 2ℓ in a bipartite graph, σ is a signing of edges, n_C is the number of negative edges in C. The cycle C will be called *properly signed* with respect to σ if n_C and ℓ are of opposite parity. The cycle C is *evenly placed* if the removal of all its vertices and adjacent to them edges from G results in a graph having a perfect matching. A graph is called *2-connected* if each its edge belongs to a cycle.

6.3. Suppose each evenly placed cycle is properly signed relative to a signing σ. Then σ is a Kasteleyn signing. (No planarity assumed.)

6.4. Let G be a planar bipartite 2-connected graph. Let us fix some planar drawing of G. Let σ be a signing of G such that the boundary cycle of every bounded domain in the drawing is properly signed. Then σ is a Kasteleyn signing.

6.5. Prove that any planar 2-connected bipartite graph has a Kasteleyn signing.

6.6. a) Suppose in a digraph G we have marked n vertices having only outgoing edges (these vertices are «inlets» from which we may start to move along edges of the graph) and n vertices having only incoming edges («outlets»). Let $a_{i,j}$ be the number of ways to pass from the ith inlet to the jth outlet. Prove that the permanent of the matrix $(a_{i,j})$ equals to the number of sets such that each set consists of n paths from an inlet to an outlet, and the beginning vertex and the ending vertex of any two paths do not coincide.

b) In addition suppose that the graph be planar, all inlets are located to the left from all outlets, and all edges in all paths are passed from the left to the right. Prove that the determinant of $(a_{i,j})$ equals the number of sets such that each set consists of n non-intersecting paths where the ith path leads from the ith inlet to the ith outlet (with the same i).

6.7. a) In the left lower corner of an $n \times n$ board there is a lame king which may move only in three directions: to the right, up, and up to the right (along a diagonal). Denote by A_n the number of all its paths to the opposite corner of the board, and denote by B_n the number of these paths such that they are disjoint with the left column and the upper row (except the initial and final cells). Prove that $B_n = 2A_{n-1}$.

b) Consider the matrix $A^{(n)} = (A_{i,j})_{1 \leqslant i.j \leqslant n}$, where $A_{i,j}$ is the number of paths of the lame king on the $i \times j$ board, leading from the left lower to the right upper corner. Determine $\det A^{(n)}$.

6.8. Let H be the matrix $(n+1) \times (n+1)$, where $h_{i,j} = C_{i+j} = \frac{1}{n+1}\binom{2n}{n}$ are Catalan numbers. Then $\det H = 1$.

7 Problems on trees

7.1. Let G be a tree with n vertices. Prove that

$$\det A_G = \begin{cases} (-1)^{\frac{n}{2}}, & \text{if } G \text{ has a perfect matching;} \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of a graph G is the determinant of the matrix $\tilde{A}_G = (\tilde{a}_{ij})$, where

$$\tilde{a}_{ij} = \begin{cases} x & \text{for } i = j; \\ -1, & \text{ifthe vertices } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

Thus the characteristic polynomial is a polynomial in x and is denoted by $\chi_G(x)$.

Let m_k be the number of ways to choose k edges of the graph G so that no two edges are incident to the same vertex. *The matching polynomial* of G is

$$m_G(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k m_k x^{n-2k}.$$

7.2. Let G be a tree. Prove that $\chi_G(x) = m_G(x)$.

Solutions

2.1. a) This is obvious. Multiplication of the kth row of a matrix A by a number c corresponds to multiplication of weights of all edges from v_k by c. Each univalent subgraph contains just one of these edges. Hence its weight is also multiplied by c.

b) If a univalent graph contains an edge from a vertex v_k to a vertex v_i then let us call v_i a successor of v_k. Interchange of the jth and kth rows of a matrix A determines the following transformation of univalent subgraphs: the edge from v_k leads now not to the successor of v_k but to the successor of v_j, and the edge from v_j leads now to the successor of v_k. If v_k and v_j were in distinct cycles of length ℓ_k and ℓ_j respectively then now they occur to be in the same cycle of length $\ell_k + \ell_j$. And if the vertices were in the same cycle then it decomposes into two parts with the same total length. In both cases the number of even cycles changes by 1.

c) Every univalent subgraph of $\mathcal{A}$ must contain a loop at the vertex v_k, so the edges leading to v_k from the other vertices cannot belong to any univalent subgraph. Hence there exists a natural bijection between univalent subgraphs of $\mathcal{A}$ with nonzero weight and univalent subgraphs of $\widetilde{\mathcal{A}}$ with nonzero weight: it removes the loop at v_k. Since the weight of the loop is 1, this bijection saves the weight of the subgraph as well.

d) Each univalent subgraph contains one edge which corresponds to some element a_{ij} of the jth row. The weight of this subgraph equals to the product of a_{ij} and the product of weights of the remaining edges (and maybe also -1). Replacing the weight of this edge by $a_{ij}^{(1)}$ or $a_{ij}^{(2)}$ we obtain univalent subgraphs from the decompositions of $\det A^{(1)}$ and $\det A^{(2)}$, and the sum of weights of these subgraphs equals the weight of the first subgraph.

e) Represent the ith row of the matrix A as the sum of n rows:

$$(a_{i1}, 0, \ldots, 0) + (0, a_{i2}, 0, \ldots, 0) + \ldots + (0, 0, \ldots, a_{in}).$$

By the assertion of the preceding part of the problem, it suffices to check for each j that if the ith row of A equals $(0, \ldots, 0, a_{ij}, 0, \ldots, 0)$ then $\det A = (-1)^{i+j} a_{ij} \det A_{ij}$.

A particular case of this assertion for $j = i$ was already considered in part c). For certainty suppose $i < j$. Let us consecutively interchange the ith row of A with the $(i+1)$th one, then the $(i+1)$th row with the $(i+2)$th one and so on, the $(j-1)$th row with the jth one. As the result, the ith row of A gets the jth position and we obtain a matrix B such that its jth row is of the form $(0, \ldots, 0, a_{ij}, 0, \ldots, 0)$: the nonzero element a_{ij} is at the jth position, that is, on the diagonal. Moreover $B_{jj} = A_{ij}$, $\det B = (-1)^{j-i} \det A = (-1)^{j+i} \det A$, and then we have by c)

$$\det A = (-1)^{j+i} \det B = (-1)^{j+i} a_{ij} \det B_{jj} = (-1)^{j+i} a_{ij} \det A_{ij}$$

as required.

As we see, the argument is not "purely combinatorial". To justify it, let us consider the case $j = i+1$ in combinatorial language, for the particular case $n = 6$, $i = 3$, $j = 4$. We wish to check the equality

$$\det \left(\begin{array}{cc|cc|cc} a & b & c & d & e & f \\ g & h & k & \ell & m & n \\ \hline 0 & 0 & 0 & a_{i\,i+1} & 0 & 0 \\ o & p & q & r & s & t \\ \hline u & v & w & x & w & z \\ \alpha & \beta & \gamma & \delta & \epsilon & \phi \end{array} \right) = -a_{i\,i+1} \cdot \det \left(\begin{array}{cc|c|cc} a & b & c & e & f \\ g & h & k & m & n \\ \hline o & p & q & s & t \\ \hline u & v & w & w & z \\ \alpha & \beta & \gamma & \epsilon & \phi \end{array} \right).$$

Consider any digraph corresponding to the matrix on the left hand, remove the edges entering v_{i+1} and turn the edge $v_i v_{i+1}$ into a new vertex v. The edges from v_{i+1} now exit from v, and the edges to v_i enter v. Then we obtain the digraph corresponding to the right-hand matrix. Now observe that since any univalent subgraph of the original graph contains the edge $v_i v_{i+1}$ (the only edge from v_i), it contains no other edge entering v_{i+1}. Thus the transformation leads to removal of the single edge $v_i v_{i+1}$ which has weight $a_{i\,i+1}$ indicated on the right side. The sign of the subgraph will change since a single cycle will change its parity. On the other hand, each univalent subgraph for the right-hand matrix clearly is obtained from a uniquely determined univalent subgraph for the left-hand matrix.

2.2. Suppose an element of A, for instance at position $(1,1)$, is represented as $a_{11}+b_{11}$. In the digraph $\mathcal{A}$ replace two loops at v_1 with weights a_{11} and b_{11} by a single loop with weight $a_{11}+b_{11}$, obtaining a graph $\widetilde{\mathcal{A}}$. Each univalent subgraph H of the digraph $\widetilde{\mathcal{A}}$ with a loop at v_1 corresponds to two univalent subgraphs in $\mathcal{A}$ with a loop at v_1 with weight a_{11} or b_{11} respectively. The sum of the weights of these subgraphs equals the weight of H. And if a univalent subgraph of $\widetilde{\mathcal{A}}$ contains no loop at v_1 then $\mathcal{A}$ contains a subgraph isomorphic to it.

Summing up over all univalent subgraphs we see that joining of two loops or, equivalently, replacement of the formal sum $a_{11}+b_{11}$ by its value retains the determinant of the matrix unchanged as required.

2.3. a) Consider the matrix

$$A=\begin{pmatrix} a_{12}+a_{13}+a_{14} & -a_{12} & -a_{13} & -a_{14}\\ -a_{21} & a_{21}+a_{23}+a_{24} & -a_{23} & -a_{24}\\ -a_{31} & -a_{32} & a_{31}+a_{32}+a_{34} & -a_{34}\\ -a_{41} & -a_{42} & -a_{43} & a_{41}+a_{42}+a_{43} \end{pmatrix}.$$

This form of recording (non-diagonal elements are with minus signs, their sums on the diagonal are with pluses) will be considered as standard. The following argument for this example is easily generalized.

According to the statement of the preceding problem, for calculating $\det A$ we have to list univalent subgraphs in graph $\mathcal{A}$ shown at fig. 7 left. However we will describe the construction, which enables us to restrict this listing by the subgraphs of graph $\mathcal{A}_1$ obtained from $\mathcal{A}$ by removal of all loops and signs at the weights of edges (fig. 7 right).

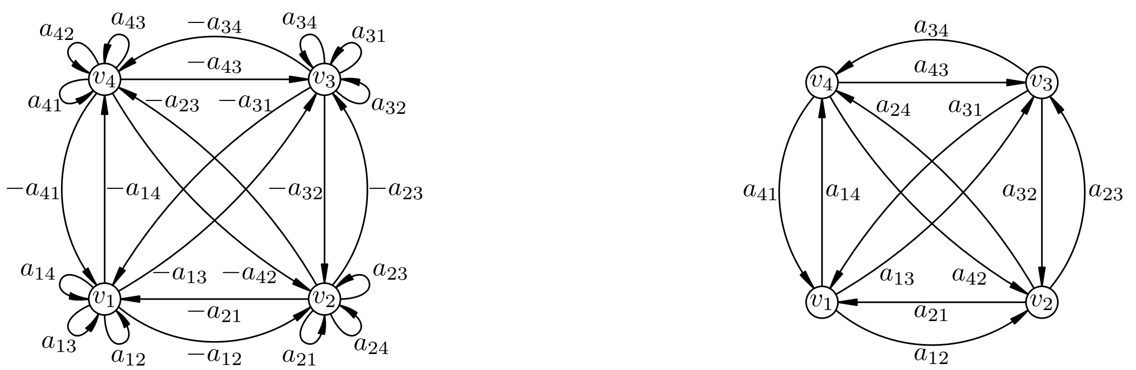
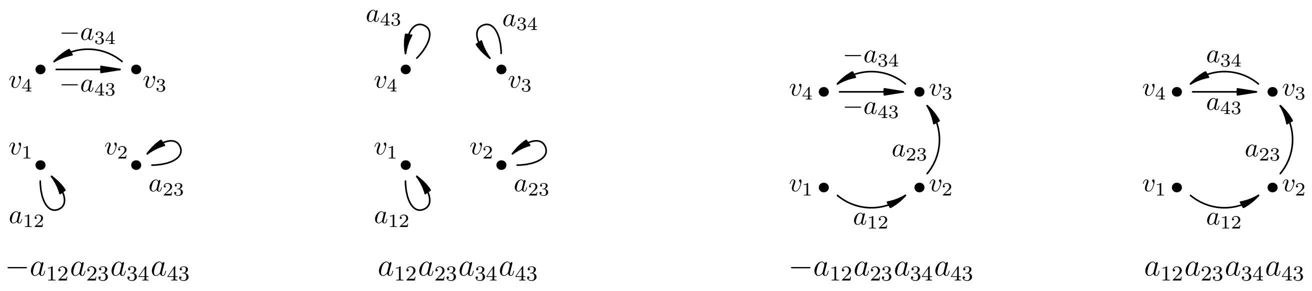


Рис. 7. Graphs $\mathcal{A}$ and $\mathcal{A}_1$

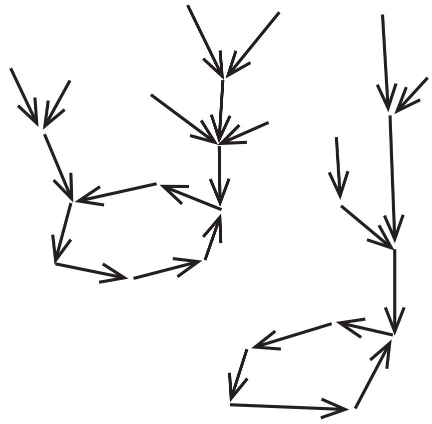
Since the labels in graph $\mathcal{A}$ are repeated, some subgraphs may have equal weight (see fig. 8,a; at the moment, we ignore signs of weights). The labels on the edges of graph $\mathcal{A}$ are the numbers $\pm a_{ij}$, and indexes i and j for each label are different. It is evident that the edge outgoing from vertex v_i is marked by a label with the first index equal to i. This implies that labels of all edges of any univalent subgraph of graph $\mathcal{A}$ are pairwise different. Then introduce *new rule for the imaging of univalent subgraphs* of graph $\mathcal{A}$. Write all the labels of edges of a univalent subgraph and redraw the subgraph (on the same set of vertices), namely: for each label $\pm a_{ij}$ we draw the edge leading from the vertex v_i to v_j marked by the label itself $\pm a_{ij}$ (see fig. 8,b). It is evident that the graph drawn by the new rule does not have loops. It is in fact a subgraph of graph $\mathcal{A}_1$, the only difference is that the weights of edges of our subgraph may have a superfluous sign minus. A little later we will remove this difference.



a) Subgraphs with almost equal weights

b) The corresponding sesquivalent subgraphs

Рис. 8. New way for drawing graphs



A sesquivalent graph

The obtained subgraphs themselves are not univalent. Only the property that each vertex has one outgoing edge is remained from univalent graphs. A vertex may have either no incoming edge or several incoming edges. Call such graphs *sesquivalent*.

It is evident that any sesquivalent graph contains at least one cycle (move along arrows, then sometimes you will come to a vertex, where you have already been earlier), and if it contains more than one cycle then these cycles have no common vertices. Besides the cycles it may contain several trees planted on vertices of the cycles.

As it can be seen from fig. 8,*b*, identical sesquivalent graphs may have different edge labeling. The label of edge v_iv_j of a sesquivalent graph may be equal to $+a_{ij}$ or $-a_{ij}$. However if this graph is obtained by the described rule from a univalent oone then the assigning of the signs is not quite chaotic. Namely: a sesquivalent graph may be obtained from a univalent one if and only if:

1) for each of the cycles of this graph the property holds: either all edge labels are of the form $+a_{ij}$ (such a cycle we call *positive*, in the univalent graph some set of loops corresponded to it), or all of them are of the form $-a_{ij}$ (such a cycle we will call *negative*, in the univalent graph the same cycle corresponded to it);

2) the edges not belonging to any cycle, have labels of the form $+a_{ij}$.

The graphs possessing this property of edge labeling are called *admissible*.

The rule for calculation of the weight of a sesquivalent graph is the same as for a univalent graph: the weight is equal to the product of edge weights, taken with the minus sign if the graph contains odd number of negative even cycles. We take into account only negative even cycles, since positive cycles of a sesquivalent graph correspond to sets of loops in the univalent graph, and loops do not affect the weight sign.

Dealing with sesquivalent graphs, it is convenient to write edge labels without signs at all but remember in addition about the sign of each cycle. In this case the weight of a sequivalent graph equals the product of weights of all its edges multiplied by $(-1)^k$, where k is the number of negative even cycles.

S t a t e m e n t. The determinant of matrix A equals the sum of weights of all admissible sesquivalent subgraphs in graph $\mathcal{A}_1$.

Properly, there is nothing to be proven. Admissible sesquivalent subgraphs of graph $\mathcal{A}_1$ are in one-to-one correspondence with univalent subgraphs of graph $\mathcal{A}$, and this correspondence saves weights of subgraphs.

Turn to the problem solution. Now it is easy to check that $\det A = 0$. Indeed, consider whatever admissible sesquivalent subgraph and change the sign of the cycle passing through the vertex with the least number. We obtain another admissible subgraph. The second applying of this operation leads to the initial subgraph. Thus, with the help of this operation the set of all admissible sesquivalent subgraphs is split into pairs. But the sum of the weights of graphs in one pair equals zero: if the pair is constructed by changing the sign of an even cycle then the numbers of negative cycles in graphs of this pair have different parity, and the sets of edge weights are identical; and if the pair is constructed by changing the sign of an odd cycle then the sign of the product of labels of this cycle does change.

b) Not to get involved in linear algebra, prove the problem statement generalizing the Matrix-tree theorem for the case of weighted oriented graphs. Let a complete digraph $\mathcal{A}_1$ on n vertices without loops be given, in which between any two vertices v_i and v_j ($i \neq j$) there exists as edge v_iv_j with weight a_{ij} as well as edge v_jv_i with weight a_{ji} (fig. 7, to the right). Matrix A from the problem condition is the Laplacian matrix of this graph.

We call a *rooted oriented spanning tree with root* v_i the acyclic graph on the set of vertices $\{v_1, v_2, \ldots, v_n\}$, in which the outgoing degree for vertex v_i equals 0, and for any other vertex equals 1. In other words, for any vertex there exists the only path from it to v_i. The weight of the rooted oriented spanning tree equals the product of weights of the edges belonging to it.

The problem statement immediately follows from t h e o r e m: the number $(-1)^{i+j}\det A_{ij}$ equals the sum of weights of all rooted oriented trees with root v_i.

Let us prove this theorem. It is sufficient to consider the case $i = n$, the other cases are similar. Let $\widetilde{A}$ be an $n \times n$ matrix such that its n-th row equals $(0, \ldots, 0, 1, 0, \ldots, 0)$ (the unit is on j-th place), and the other matrix elements are the same as in matrix A. Then from the expansion of $\det \widetilde{A}$ in terms of n-th row we obtain

$$\det \widetilde{A} = (-1)^{n+j} \cdot 1 \cdot \det A_{nj} \tag{5}$$

We will calculate the determinant $\det A_{nj}$ with help of technique of sesquivalent graphs. As an example, we restrict the discussion to the case $n = 5$, let $j = 2$:

$$A_{25} = \begin{pmatrix} a_{12}+a_{13}+a_{14}+a_{15} & -a_{13} & -a_{14} & -a_{15} \\ -a_{21} & -a_{23} & -a_{24} & -a_{25} \\ -a_{31} & a_{31}+a_{32}+a_{34}+a_{35} & -a_{34} & -a_{35} \\ -a_{41} & -a_{43} & a_{41}+a_{42}+a_{43}+a_{45} & -a_{45} \end{pmatrix}.$$

It is convenient to assume that the rows of matrix A_{nj} are numbered from 1 to $n-1$, and the columns by numbers from 1 to n, but with omission of number j.

As in the previous part of the problem, one should choose one number in each row and each column (if the corresponding matrix element is written as a sum, one should take only one of the summands), after that construct edges of graph $\mathcal{A}_1$ corresponding to the chosen numbers, and if an edge has weight with «superfluous» minus sign (for example, element $-a_{13}$ has been chosen in the matrix) then we call such edge negative. We have obtained a graph on vertices $\{v_1, \ldots, v_n\}$ possessing the following properties (fig. 9).

1) Outgoing degree of vertices v_1, …, v_{n-1} equals 1, outgoing degree of vertex v_n equals 0.

2) One «negative» edge goes out from vertex v_j, and only «positive» edges may come in.

3) Negative edges form several cycles and one more «incomplete cycle», it is a path from v_j to v_n (this path becomes cycle, if we would supplement it by edge v_nv_j corresponding to element a_{nj}=1 of matrix $\widetilde{A}$).

4) Positive edges may also form several cycles and also several oriented rooted trees planted in some vertices of the cycles (including the incomplete cycle).

The sign of such subgraph equals $(-1)^{c+\ell}$, where c is the number of negative even cycles, ℓ is the number of edges in the incomplete cycle. Indeed, let us supplement the set of the chosen matrix elements generating the considering subgraph, by element $a_{nj} = 1$, and add edge v_nv_j to the subgraph. We have obtained the sesquivalent subgraph used in the calculation of $\det \widetilde{A}$. The incomplete cycle will turn into a usual negative cycle, its contribution to the calculation of the sign of the subgraph will be just equal to $(-1)^{\ell}$. Let π be a permutation of the set $\{1, 2, \ldots, j-1, j+1, \ldots, n\}$ which enumerates the columns of matrix A_{nj}. The operation of adding a new factor a_{nj} to the matrix elements chosen at the positions $(1, \pi_1)$, $(2, \pi_2)$, …, $(n-1, \pi_{n-1})$ creates a permutation of the set $\{1, 2, \ldots, n\}$, which has by $n - j$ transpositions

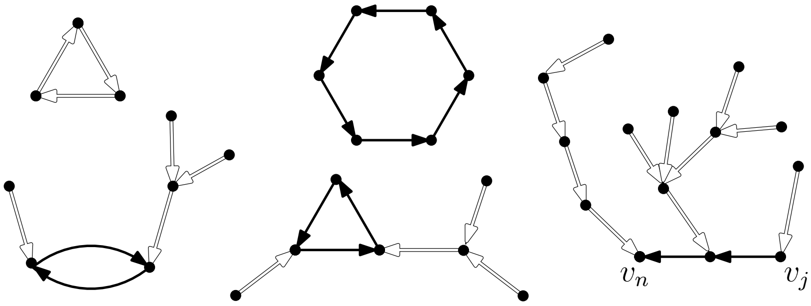


Рис. 9. «Negative» edges are indicated by black color, «positive» by white

more than π. Corrective sign $(-1)^{n-j} = (-1)^{n+j}$ is assigned to all sesquivalent subgraphs in consideration and we see just it in formula (5).

Thus, the calculation of the determinant $\det A_{ij}$ is reduced to the summation with appropriate signs of weights of all sesquivalent subgraphs with properties 1)–4). Applying the involution from the previous part of the problem (the changing of the sign of the cycle passing through the vertex with the least number), we cancel all subgraphs containing at least one cycle. Consider any of the remaining subgraphs, it is an oriented tree with root v_n. Let it contain ℓ negative edges (these are the edges of the incomplete cycle), the sign of this subgraph equals $(-1)^\ell$, and the same is the product of the signs of "negative" edges. Therefore, the total weight of this subgraph is the product of the weights of edges of the tree.

Thus, the calculated determinant is the sum of the weights of all oriented trees with root v_n.

2.4. See [2, lemma 2.1]. In the following solution *1-factors* are just the *univalent* graphs. Split all the 1-factors of the graphs G and G' into groups, such that in every group the intersection of 1-factors with the subgraphs H and H' is the same. We will construct a bijection between the groups (and sometimes between individual 1-factors) that preserves the total weight.

1) If a 1-factor of the graph G has cycles that pass through edges x and y of the subgraph H, we map this 1-factor to the 1-factor of the graph G' in which the corresponding parts of these cycles are replaced by new edges x', y' (as in fig. 10, left). The parts of the initial cycles in the subgraph H contribute xy to the weight of the 1-factor. After the replacement, the contribution is $x'y' = \frac{xy}{(wz-xy)^2}$ but we also have the multiplier $(wz - xy)^2$ on the right hand side of (3). The total weight remains unchanged.

The case where the cycles contain edges w and z is treated analogously.

Рис. 10. Rebuilding of 1-factors

2) If a 1-factor of the graph G contains a long cycle passing through edges x, w in the subgraph H, and a cycle of the length 2 on the vertical edge (see fig. 10, right), then we map it to a 1-factor of the graph G' obtained by removing the 2-cycle and replacing the part of the long cycle by the new pair of edges z', y'. The preservation of weights can be checked as in the previous case.

We treat similar configurations in an analogous way.

3) Collect together all 1-factors of the graph G that coincide outside the subgraph H and contain a cycle passing through edges z, y, w (the contribution of this configuration to the weight of the 1-factor equals yzw) or which contain a long cycle that passes through x and a 2-cycle along the edge y (this configuration contributes xy^2 to the total weight). Observe that these two configurations have opposite signs, because the numbers of even cycles in them differ by 1. We map this set of 1-factors to the set of 1-factors in G that have the same structure outside the

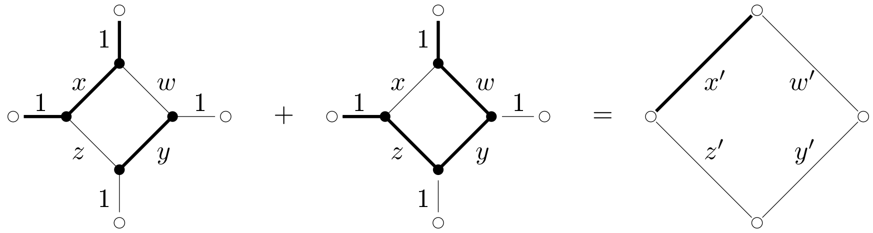


Рис. 11. Rebilding of groups of 1-factors

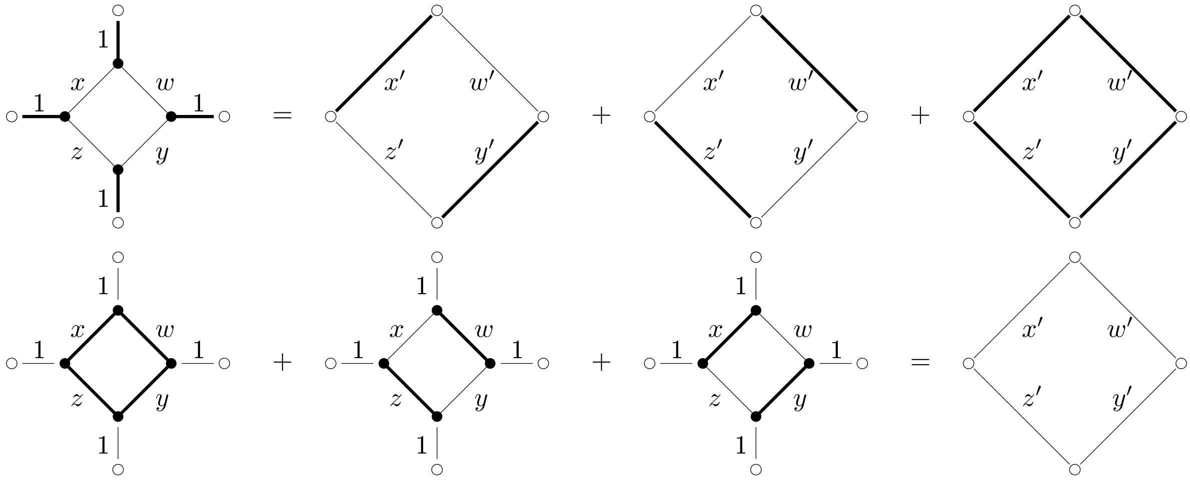


Рис. 12. The remaining identities

subgraph H (and so the contribution of the outer part is the same for both sets of 1-factors) and contain the edge x' (fig. 11, left). Thus again we see that the weights are equal because $yzw - xy^2 = (wz - xy)^2 x'$. The "dual" case is treated similarly (fig. 11, right).

4) We consider the remaining cases analogously (fig. 12). Note that the right cycle on the top of fig. 12, and the left cycle on the bottom of fig. 12 should be taken into account with two different orientations, which doubles their contribution. The equality of the weights at fig. 12 is due to the identities

$$1 = (wz - xy)^2 (x'^2 y'^2 + w'^2 z'^2 - 2x'y'w'z') \quad \text{and} \quad x^2y^2 + w^2z^2 - 2xywz = (wz - xy)^2.$$

2.5. We have taken this problem from [2, example 2.2].

Applying the previous problem, we remove 4-cycles on the boundary of the cylinder step by step (fig. 13). To avoid zeros in denominators, we assume that the edges of the first cycle H_0 on the boundary have weights $x_0 = a$, $z_0 = 1$, $y_0 = a$, $w_0 = 1$, and all other edges in the graph have weight 1. After a single operation, the cycle H_0 disappears, and we obtain a new cycle H_1 on the boundary. The weights of its edges are the sums of the initial weights (equal to 1) and the new weights $\frac{a}{1-a^2}$, $\frac{1}{a^2-1}$, $\frac{a}{1-a^2}$, $\frac{1}{a^2-1}$ obtained by the formula (2). Thus

$$x_1 = 1 + \frac{a}{1-a^2}, \quad z_1 = 1 + \frac{1}{a^2-1}, \quad y_1 = 1 + \frac{a}{1-a^2}, \quad w_1 = 1 + \frac{1}{a^2-1}.$$

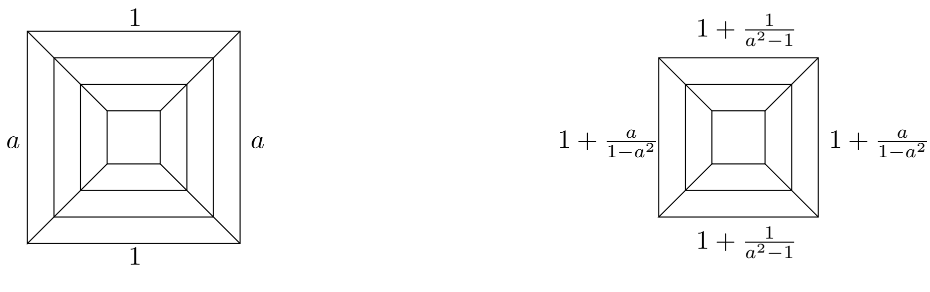


Рис. 13. Removing a cycle on the boundary of the cylinder

Then one can check by induction that after $2n$ applications of these operations (it is more convenient to use induction step 2, because the formulae are slightly different for even and odd number of iterations) the weights of the edges of the boundary cycle are equal to

$$x_{2n} = y_{2n} = \frac{na^2 + a}{2na + 1}, \qquad z_{2n} = w_{2n} = \frac{1 + 2na - na^2}{2na + 1},$$

and the product of the determinants of all removed cycles equals $\det A(H_0) \det A(H_1) \ldots \det A(H_{2n-1}) = (2na + 1)^2$.

If m is even, say $m - 1 = 2n + 1$, then after $2n$ operations the remaining graph consists of the unique 4-cycle with the weights given by the above formulas. The determinant of the matrix of this cycle equals

$$(x_{2n}y_{2n} - z_{2n}w_{2n})^2 = \Big(\frac{(2n + 1)a + 1}{2na + 1}\Big)^2 (a - b)^2,$$

and the total determinant equals

$$\det A(H_0) \det A(H_1) \ldots \det A(H_{2n})(x_{2n}y_{2n} - z_{2n}w_{2n})^2 = \big((2n + 1)a + 1\big)^2 (a - b)^2.$$

For $a = b = 1$ this expression vanishes. If m is odd, we obtain by the similar reasoning that the determinant is equal to $\big((m - 1)a + 1\big)^2$, and for $a = b = 1$ this is equal to m^2.

3.1. [1, theorem 2.1]. Consider a chess coloring of the figure, and split the edges of each univalent subgraph onto two groups: edges which start from black vertices and edges which start from white vertices. Edges of each group determine a matching, which can be interpreted as a tiling. This map is bijective.

3.2. [1, lemma 2.3]. Induction on the area. If the dual graph contains terminal vertex then cut the corresponding cell. Otherwise cut a suitable corner cell.

3.3. [1, theorem 2.4]. Consider an arbitrary univalent subgraph in G_F. Obviously the number of rising edges in it equals to the number of falling edges, denote this number by v. Let the univalent subgraph consist of k cycles. Each cycle is a polygon. Since all of the cycles have even length, they contain all the vertices in total, and the figure is simply connected, there is an even number of cells inside each cycle. Therefore applying the statement of the previous problem to each cycle, we can omit the term $2d$ in the left hand side of the congruence. Now if we sum up over the set of all cycles then we obtain

$$A - B + 2 \cdot k \underset{\text{mod } 4}{\equiv} \text{the total length of all vertical sides } = 2 \cdot v,$$

where A is equal to the number of integer points with even ordinates and B is the number of integer points with odd ordinates on the boundary of cycles. Since the univalent subgraph covers all integer points of the figure, the difference $A - B$ is even and does not depend on the subgraph. Put $A - B = 2 \cdot t$. Then $2 \cdot v \equiv 2 \cdot t + 2 \cdot k \pmod 4$, and so $v \equiv t + k \pmod 2$. Since t does not depend on the subgraph, the theorem is proven.

3.4. [1, lemma 2.5]. The statement is trivial if we use "horizontal zebra" coloring, but we will demonstrate another reasoning.

Let us interpret a tiling as a univalent subgraph (each domino is interpreted as 1-cycle), then the number of cycles equals the number of dominoes, the number of rising edges is equal to the number of vertical dominoes. Then

$$\begin{aligned} &\text{number of cycles} + \text{number of rising edges} = \\ &\quad = \text{number of dominoes} + \text{number of vert. dominoes} \underset{\text{mod } 2}{\equiv} \text{number of horiz. dominoes.} \end{aligned} \tag{6}$$

We know from the previous problem that the parity of the sum at the l.h.s. does not depend on the subgraph. Therefore the parity of the number of horizontal dominoes in all tilings is the same. Hence the definition of sign of does not depend on tiling.

Let us formulate one generalization of the above observations. If $\operatorname{sgn} F = 1$, i. e. an even number is in the right-hand side of the formula (6) then in any tiling (and therefore in any univalent subgraph by the statement of problem 3.3) the parity of the number of rising edges coincides with the parity of the number of the cycles, and if $\operatorname{sgn} F = -1$ then these parities are opposite. By the other words, for any univalent subgraph π in graph G_F

$$(-1)^{\text{number of rising edges in } \pi} = \operatorname{sgn} F \cdot (-1)^{\text{number of cycles in } \pi}. \tag{7}$$

Remind that graph G_F is bipartite, all its cycles are even, therefore we obtain the formula

$$\det A_F = \operatorname{sgn} F \cdot \sum_{\pi} (-1)^{\text{number of rising edges in } \pi}. \tag{8}$$

3.5. [1, theorem 2.7]. Denote by γ_k the number of univalent subgraphs of G_F with k rising edges. Then

$$\det A_F = \operatorname{sgn} F \cdot \sum_{\pi} (-1)^{\text{number of ascending edges in } \pi} = \operatorname{sgn} F \cdot \sum_{k=0}^{+\infty} \gamma_k \cdot (-1)^k. \tag{9}$$

It is clear due to the bijection from the problem 3.1 that the coefficient of x^k in the expression $f_F(x)^2$ is equal to the number of univalent subgraphs with exactly k vertical edges. The number of ascending edges in a univalent subgraph equals one half of the number of vertical edges, therefore $f_F(x)^2 = \sum\limits_{k=0}^{+\infty} \gamma_k x^{2k}$. Substitute $x = \mathbf{i}$, and the formula (9) gives us the first equality. In particular, we have proven the equality

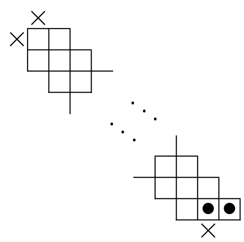
$$\det A_F = \operatorname{sgn} F \cdot f_F(\mathbf{i})^2. \tag{10}$$

3.6. [1, theorem 2.8]. Let us calculate $\det A_F$ by the formula (10). If a good pair consists of the tiling with k vertical dominoes and the tiling with $k+2$ vertical dominoes then its contribution to $f_F(\mathbf{i})$ is equal to $\mathbf{i}^k + \mathbf{i}^{k+2} = 0$. Therefore all good pairs contribute zero to $f_F(\mathbf{i})$ and the first claim of the problem follows.

If the set of all tilings, except one, can be split into good pairs then we denote the number of vertical and horizontal dominoes in the remaining tiling by v and h, $h + v = s(F)$. Then $f_F(\mathbf{i}) = \mathbf{i}^v$ by the previous reasoning, $\operatorname{sgn} F = (-1)^h$ and therefore

$$\det A_F = \operatorname{sgn} F \cdot {f_F}^2(\mathbf{i}) = (-1)^{h+v} = (-1)^{s(F)}.$$

3.7. [1, theorem 2.11]. The expression $n(n-1)$ in the formula equals the area of any regular stamp. By problem 3.6 it is sufficient to check that the set of all tilings of each regular stamp, except one, can be split into good pairs and that the set of all tilings of each irregular stamp can be split into good pairs. We will check both statements by the induction on n. We need the following lemma [5, lemma 2.1].



L e m m a (about a “halfdiagonal”). Suppose figure F contains three diagonal sequences of cells, like on the figure to the right, and cells which are marked by crosses do not belong to the figure. Then the set of all tilings of figure F which do not contain the domino marked with bold circles can be split into good pairs.

P r o o f o f t h e l e m m a. Consider the middle of one of these diagonal sequences. Prove that there exists a 2×2 square consisting of two dominoes of the tiling and containing two squares of this diagonal sequence (rearranging the dominoes in such square we easily split the set of the tilings into pairs). Indeed, if there is no such a square then looking through this sequences of squares starting from the left corner, we see that each next domino covering the diagonal square has also to cover either the square to the right or the square downward. Coming to the lower right corner we obtain the contradiction.

Turn to the problem solution.

1) We will check by induction on n (the size of stamp) that the set of all tilings of each regular stamp, except one, can be split into good pairs. The base is trivial.

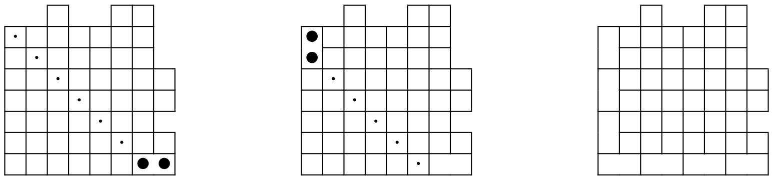


Рис. 14. Construction of an «unpaired» tiling of $(n+1)$-stamp

Step of induction, $n \to n+1$. Consider a regular $(n+1)$-stamp. We will split the set of its tilings into good pairs. For this we take a look at the bottom-right and upper-left corner cells of the $(n+1)$-stamp. One of these cells lies inside the $n \times n$ square, let it be the upper-left cell. Apply the halfdiagonal lemma in the bottom-right direction starting from this cell. Then the set of tilings which do not contain the marked domino (figure 14, left) can be split into good pairs. Let's look at tilings, which contain this domino. Apply the halfdiagonal lemma again in the upper-left direction starting from the cell to the left of the marked domino (figure 14, middle). By this lemma the set of tilings which does not contain the marked domino in the upper-left corner, can be split into good pairs. The remaining tilings contain this domino. Apply the halfdiagonal lemma once again in the bottom-right direction from the cell below the domino, and so on. As a result of repeated application of the halfdiagonal lemma, we split the set of tilings into pairs except the tilings containing all the dominoes on the left and the bottom sides of our $(n+1)$-stamp (fig. 14, right). There is a bijection between the remaining tilings and the tilings of the remaining n-stamp. Therefore all tilings except one can be split into good pairs.

2) Check by induction on n that the set of all tilings of each irregular stamp can be split into good pairs. The base is trivial.

Step of induction, $n-1 \to n$. Consider an arbitrary n-stamp. We mark some cells of its $n \times n$ square as at figure 15.

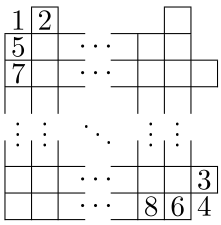


Рис. 15. Layout of the cells for the irregular stamp

Consider the following cases.

1) The figure does not contain cells 1 and 4. Consider the diagonal from 5 to 6. By the halfdiagonal lemma the set of all tilings can be split into pairs (because the marked domino does not belong to the figure). The case when four cells 1, 2, 3, 4 don't belong to the stamp is considered similarly.

2) The stamp contains the cell 1 but not the cell 4 (or vice versa). Consider the first case, the second one is similar. Apply the halfdiagonal lemma in the direction from 6 to 1. As in the proof of the previous item, we split the set of all tilings into pairs, except those tilings for which the position of dominoes on the leftmost column and bottom row is fixed as at figure 14, right. The set of exceptional tilings can be split into good pairs by induction hypothesis.

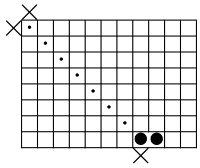
3) Cells 1 and 4 belong to the stamp but 2 and 3 do not belong. Then each tiling contains dominoes $1-5$ and $4-6$. Cut them. By the halfdiagonal lemma which we apply in the direction from 7 to 8, the set of all tilings can be split into good pairs.

4) Cells 1, 2, 4 belong to the stamp but 3 does not belong (or similarly 1, 3, 4 belong to the stamp but 2 does not belong). Obviously each tiling contains the dominoes $4-6$. Cut it. Apply the halfdiagonal lemma in the direction from 8 to 7. Observe that each tiling contains the

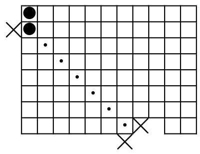
dominoes $5-7$ and therefore each tiling contains the dominoes $1-2$. We cut these dominoes and finish the proof by induction, like in item 2.

3.8. [1, theorem 2.12]. Check that the number of tilings of $m \times n$ rectangle is odd iff the numbers $m+1$ and $n+1$ are coprime, in the case when the number of tilings is even all these tilings can be split into good pairs, and if it is odd then all tilings except one can be split into good pairs. Then the statement of the problem follows from Problem 3.6.

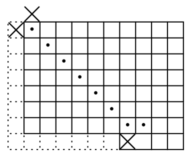
Induction realizes Euclidean algorithm. Consider the diagonal sequence of squares outgoing from the corner. By the halfdiagonal lemma, it is sufficient to investigate the parity of the numbers of tilings containing the marked domino.



Now we can take the diagonal sequence of squares below the row just considered, and looking through it from bottom to top apply the halfdiagonal lemma again.



Continuing to move down the diagonal in question and not changing the parity of the number of tilings, we can remove all the squares of the first column and also all the squares of the lower row to the left from the marked domino. Now we can apply the halfdiagonal lemma again.



Continuing further we will remove the squares of the second column and the second row (lying to the left from the domino that we had found). Acting in such a way, we will remove the rectangular part of size $(n+1) \times n$ from our $m \times n$ rectangle finally. We have obtained $(m-n-1) \times n$ part that satisfies the induction hypothesis.

4.1. Let $\mathcal{D}$ denote the set of all signed digraphs D obtained as described before the formulation of the problem from the terms of the superexpansion. We write π_D for the associated permutation. We divide $\mathcal{D}$ into three parts as follows:

- $\mathcal{T}$, the $D \in \mathcal{D}$ with no directed cycle;
- $\mathcal{D}^+$, the $D \in \mathcal{D}$ with $\operatorname{sgn}(D) = +1$ and at least one directed cycle;
- $\mathcal{D}^-$, the $D \in \mathcal{D}$ with $\operatorname{sgn}(D) = -1$ and at least one directed cycle.

Here is a plan for the rest of the proof. We will show that all $D \in \mathcal{T}$, the "acyclic objects", have positive signs, and they are in one-to-one correspondence with the spanning trees of G; thus they count what we want. Then, by constructing a suitable bijection, we will prove that $|\mathcal{D}^+| = |\mathcal{D}^-|$— so the "cyclic objects" cancel out. We then have $\det(L^-) = \sum_{D\in\mathcal{D}} \operatorname{sgn}(D) = |\mathcal{T}| + |\mathcal{D}^+| - |\mathcal{D}^-| = |\mathcal{T}|$ and the theorem follows.

To realize this plan, we first collect several easy properties of the signed digraphs in $\mathcal{D}$.

(i) *If $i \to j$ is a directed edge, then $\{i, j\}$ is an edge of G.* (Clear.)

(ii) *Every vertex, with the exception of n, has exactly one outgoing edge, while n has no outgoing edge.* (Obvious.)

(iii) *All incoming edges of n are positive.* (Since L^- has only $n-1$ rows and columns.)

(iv) *No vertex has more than one negative incoming edge.* (This is because two negative incoming edges $j \to i$ and $k \to i$ would mean two circled entries ℓ_{ji} and ℓ_{ki} in the ith column.)

(v) *If a vertex i has a negative incoming edge, then the outgoing edge is also negative.* (Indeed, a negative incoming edge $j \to i$ means that the off-diagonal entry ℓ_{ji} is circled, and hence none of the 1s in the diagonal entry ℓ_{ii} may be circled—which would be the only way of getting a positive outgoing edge from i.)

Claim A. These properties characterize $\mathcal{D}$. That is, if D is a signed digraph satisfying (i)–(v), *then $D \in \mathcal{D}$.*

Доказательство. Given D, we determine the circled entry in each row i, $1 \le i \le n-1$, of L^-. We look at the single outgoing edge $i \to j$. If it is positive, we circle the appropriate 1 in ℓ_{ii}, and if it is negative, we circle ℓ_{ij}. We cannot have two circled entries in a single column, since they would correspond to the situations excluded in (iv) or (v). □

Next, we use (i)–(v) to describe the structure of D.

Claim B. Each $D \in \mathcal{D}$ has the following structure (illustrated below).

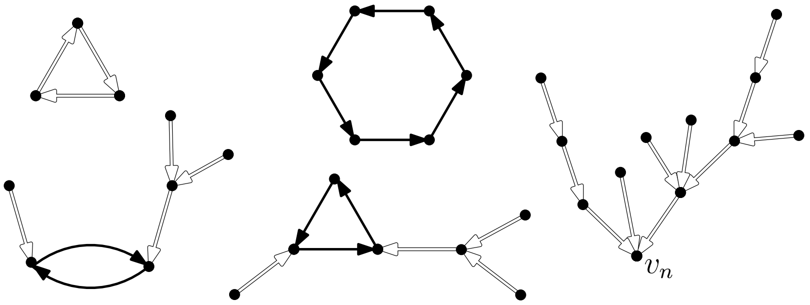


Рис. 16.

(a) The vertex set is partitioned into one or more subsets $V_1, V_2, \ldots, V_k$ corresponding to the components of D, with no edges connecting different V_i. If V_1 is the subset containing the vertex n, then the subgraph on V_1 is a tree with all edges directed toward n. The subgraph on every other V_i contains a single directed cycle of length at least 2, and a tree (possibly empty) attached to each vertex of the cycle, with edges directed toward the cycle.

(b) The edges not belonging to the directed cycles are all positive, and in each directed cycle either all edges are positive or all edges are negative.

(c) Conversely, each possible D with this structure and satisfying (i) above belongs to $\mathcal{D}$.

Sketch of a proof. Part (a), describing the structure of the digraph, is a straightforward consequence of (ii) (a single outgoing edge for every vertex except for n), and we leave it as an exercise. (If we added a directed loop to n, then every vertex has exactly one outgoing edge, and we get a so-called *functional digraph*, for which the structure as in (a) is well known.)

Concerning (b), if we start at a negative edge and walk on, condition (v) implies that we are going to encounter only negative edges. Thus, we cannot reach n, since its incoming edges are positive, and so at some point we start walking around a negative cycle. Finally, a negative edge cannot enter such a negative cycle from outside by (iv).

As for (c), if D has the structure as described in (a) and (b), the conditions are obviously satisfied and Claim A applies. This proves Claim B.

The first item in our plan of the proof is now very easy to complete.

All $D \in \mathcal{T}$ have a positive sign and they are in one-to-one correspondence with the spanning trees of G. Indeed, if $D \in \mathcal{D}$ has no directed cycles then D is a tree with positive edges directed toward the vertex n. Moreover, π_D is the identity permutation since all the circled elements in the term corresponding to D lie on the diagonal of L^-. Thus $\mathrm{sgn}(D) = +1$, and if we forget the orientations of the edges, we arrive at a spanning tree of G. Conversely, given a spanning tree of G, we can orient its edges toward n, and we obtain a $D \in \mathcal{T}$.

It remains to deal with the "cyclic objects". For $D \in \mathcal{D}^+ \cup \mathcal{D}^-$, let the *smallest cycle* be the directed cycle that contains the vertex with the smallest number (among all vertices in cycles). Let $\overline{D}$ be obtained from D by changing the signs of all edges in the smallest cycle.

Obviously $\overline{\overline{D}} = D$, and for $D \in \mathcal{D}$ we have $\overline{D} \in \mathcal{D}$ as well, as can be seen using Claim B. The following claim then shows that the mapping sending D to $\overline{D}$ is a bijection between $\mathcal{D}^+$ and $\mathcal{D}^-$, which is all that we need to finish the proof of the theorem.

Claim C. $\mathrm{sgn}(\overline{D}) = -\mathrm{sgn}(D)$.

Proof. We have $\mathrm{sgn}(D) = \mathrm{sgn}(\pi_D)(-1)^m$, where m is the number of negative edges of D and π_D is the associated permutation.

Let $i_1, i_2, \ldots, i_s$ be the vertices of the smallest cycle of D, numbered so that the directed edges of the cycle are $i_1 \to i_2$, $i_2 \to i_3, \ldots$, $i_{s-1} \to i_s$, $i_s \to i_1$.

In one of D and $\overline{D}$, the smallest cycle is positive; say in D (if it is positive in $\overline{D}$, the argument is similar). Positive edges correspond to entries on the diagonal of L^-, and thus the i_j are fixed points of the permutation π_D, i.e., $\pi_D(i_j) = i_j$, $j = 1, 2, \ldots, s$. In $\overline{D}$, the smallest cycle is negative, and so for $\pi_{\overline{D}}$ we have $\pi_{\overline{D}}(i_1) = i_2, \ldots$, $\pi_{\overline{D}}(i_{s-1}) = i_s$, $\pi_{\overline{D}}(i_s) = i_1$, which means that $i_1, i_2, \ldots, i_s$ form a *cycle* of the permutation $\pi_{\overline{D}}$. Otherwise, π_D and $\pi_{\overline{D}}$ coincide.

Now it is easy to check that $\pi_{\overline{D}}$ can be converted to π_D by $s-1$ transpositions (which "cancel" the cycle $(i_1, i_2, \ldots, i_s)$). Since each transposition changes the sign of a permutation, we have $\mathrm{sgn}(\pi_{\overline{D}}) = (-1)^{s-1}\mathrm{sgn}(\pi_D)$, and so

$$\mathrm{sgn}(\overline{D}) = \mathrm{sgn}(\pi_{\overline{D}})(-1)^{m+s} = (-1)^{s-1}\mathrm{sgn}(\pi_D)(-1)^{m+s} = -\mathrm{sgn}(D).$$

Claim C, and thus also the theorem, are proved.

4.2. a) Consider $\bar{G}$, in which $V(\bar{G}) = V(G) \cup \{w\}$, $E(\bar{G}) = E(G) \cup \{v_i w\}$. Denote the Laplacian matrix of graph $\bar{G}$ by $\bar{L}$. It is not difficult to understand that $\bar{L}^- = L + E_{i,i}$. The numbers of spanning trees of graphs G and $\bar{G}$ coincide too. Consequently, by the Matrix-tree theorem, the number of spanning trees of graphs G and $\bar{G}$ equals $\det \bar{L}^- = \det(L + E_{i,i})$, Q. E. D.

b) $\det(L + E_{i,j}) - \det(L + E_{i,i}) = \det M$; $M = (m_{kn})$, where

$$m_{kn} = \begin{cases} \ell_{kn} & \text{for } k \neq i; \\ -1, & \text{if } k = i \text{ and } n = j; \\ 1, & \text{if } k = i \text{ and } n = i; \\ 0 & \text{in the other cases.} \end{cases} \tag{11}$$

Note that the sum of elements in each row of matrix M equals 0, and by problem 2.3 $\det M = 0$. Therefore, $\det(L + E_{i,j}) = \det(L + E_{i,i})$, and the required statement follows from the previous part.

4.3. With light overwithundertwist this statement is immediately obtained similarly to the solution of problem 2.3.a).

4.4. The version with weights of this statement is proved in solution of problem 2.3.b).

4.5. This is the problem on linear algebra. One can read an elegant solution, for example, in D. Karpov "Graph theory" https://ru.overleaf.com/project/5fd1061de5a509b3447a4f55 .

4.6. Apply the Matrix-tree theorem and calculate the determinant of matrix L^-. For a complete graph it is an $(n-1) \times (n-1)$ matrix of the form

$$\det\begin{pmatrix} n-1 & -1 & \ldots & -1 \\ -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n-1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n-1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & n & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & n \end{pmatrix} = n^{n-2}.$$

Here in the first equality we have added the rows with numbers from 2 up to $n-1$ to the first row, the determinant has not changed from this action. In the second equality we conversely have added the first row to all the others. Finally, only the graph consisting of $n-1$ loops has nonzero weight among univalent graphs, and this implies the third equality.

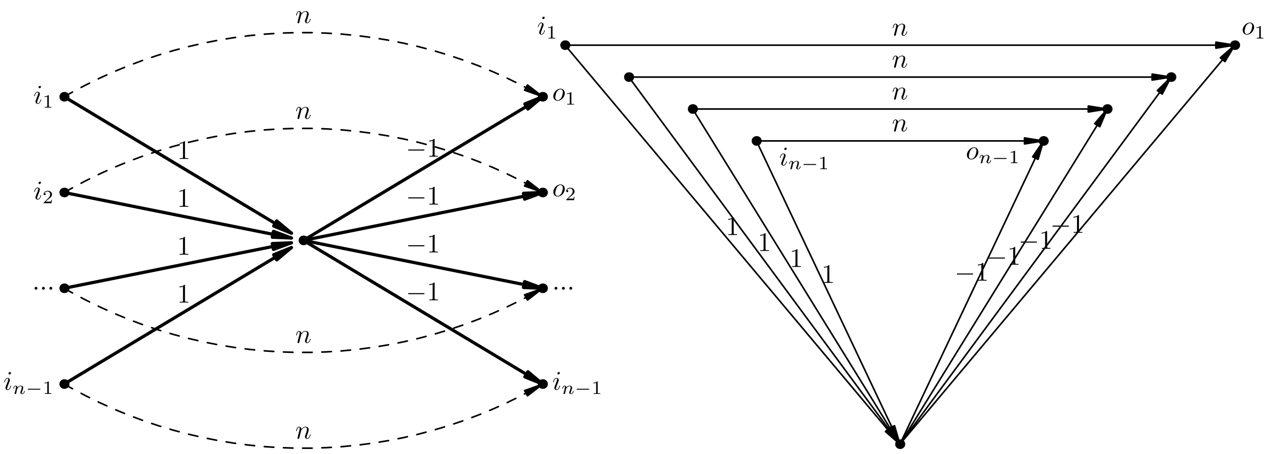


Рис. 17.

We give, however, a more combinatorial argument applying the generalization of the statement of problem 6.6 b).

The weighted version of problem 6.6 b). Let a graph satisfying to the condition of problem 6.6 a) be given. On each edge of the graph, write a real number which will be called the weight of the edge. The product of weights of the edges forming a path is called the weight of this path. The weight of a set of n paths is the product of weights of all paths from this set, multiplied by $(-1)^{|\pi|}$, where π is the permutation determined by this set of paths. Denote by a_{ij} the sum of weights of all paths from ith inlet to jth outlet. Then the determinant of matrix $(a_{i,j})$ is equal to the sum of weights of all sets consisting of n non-intersecting paths.

The proof is similar to the proof 6.6 b.

So, we want to find the determinant of matrix L^-. Look at the graph on the picture (fig. 17). Left and right parts contain $n-1$ vertices each. Note that the number of paths from ith inlet to jth outlet is just equal to ℓ_{ij}. Therefore, by the weighted version of problem 6.6 b), $\det L^-$ is equal to the sum of weights of sets consisting of n non-intersecting paths. And there exists one set of n paths of weight n^{n-1}, where ith inlet is immediately connected with ith outlet, and $n-1$ sets of weight $-n^{n-2}$. Hence, the sum of weights of sets consisting of n non-intersecting paths is equal to

$$n^{n-1} - (n-1)\cdot n^{n-2} = n^{n-2}.$$

According to the statement of the preceding problem, for calculating $\det A$ we have to list univalent subgraphs in graph $\mathcal{A}$ shown at fig. 7 left. However we will describe the construction, which enables us to restrict this listing by the subgraphs of graph $\mathcal{A}_1$ obtained from $\mathcal{A}$ by removal of all loops and signs at the weights of edges (fig. 7 right).

4.7. Apply the Matrix-tree theorem. The Laplacian matrix here looks as

$$L = \begin{pmatrix} k & \dots & 0 & -1 & \dots & -1 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & k & -1 & \dots & -1 \\ -1 & \dots & -1 & \ell & \dots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ -1 & \dots & -1 & 0 & \dots & \ell \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \ell \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} k \text{ rows} \end{matrix}$$

We will find the determinant of matrix $M = L_{\ell,\ell+1}$ (the notation from problem 2.1 e). By problem 2.3.b) $\det L^- = -\det M$.

$$M = \begin{pmatrix} k & \dots & 0 & 0 & -1 & \dots & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & k & 0 & -1 & \dots & -1 \\ -1 & \dots & -1 & -1 & 0 & \dots & 0 \\ -1 & \dots & -1 & -1 & \ell & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & \dots & -1 & -1 & 0 & \dots & \ell \end{pmatrix} \quad \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \ell - 1 \text{ rows} \\ \vphantom{a} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k - 1 \text{ rows} \end{matrix}$$

Give two combinatorial calculations of the determinant of matrix M. The first way of determinant calculation is similar to determinant calculation in problem 4.6. Namely, look at fig. 18 left. It is not difficult to check that the sum of weights of paths from i-th inlet to j-th outlet equals m_{ij}. Therefore, by the weighted version of problem 6.6 b), $\det M$ equals the sum of weights of the sets consisting of n non-intersecting paths. It is easy to understand that there exists exactly one such path, and its weight equals $-k^{\ell-1}\ell^{k-1}$, which finishes the calculation.

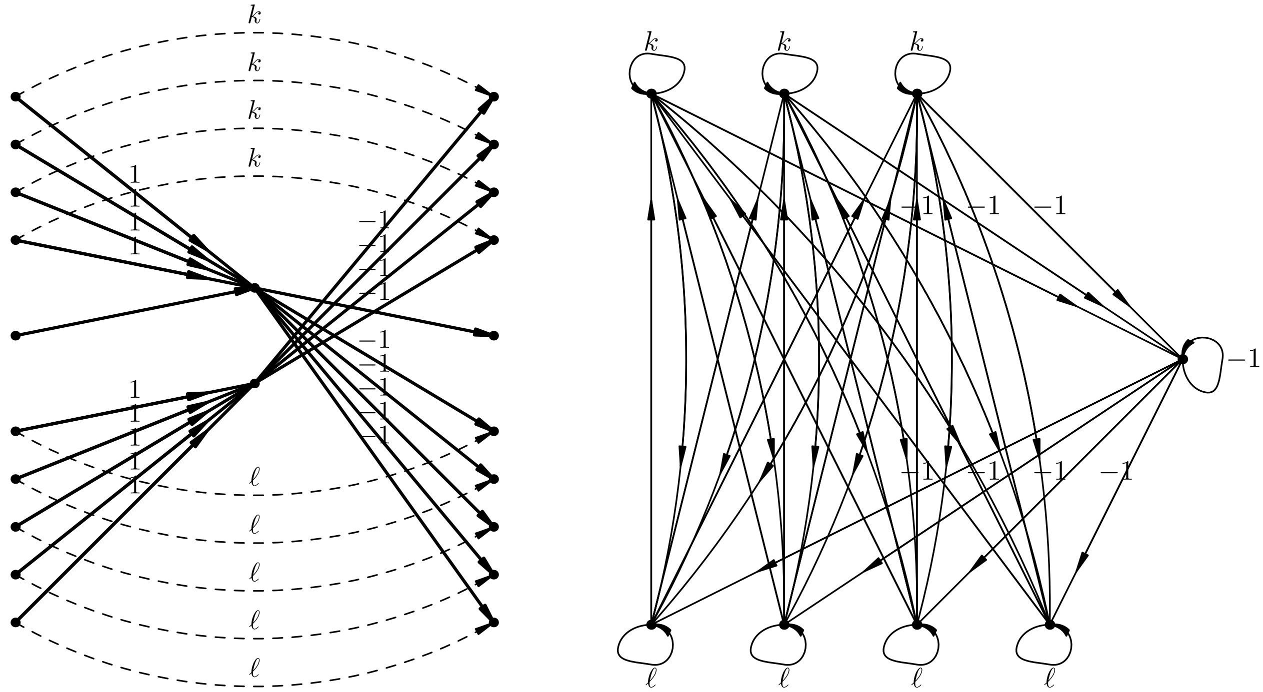
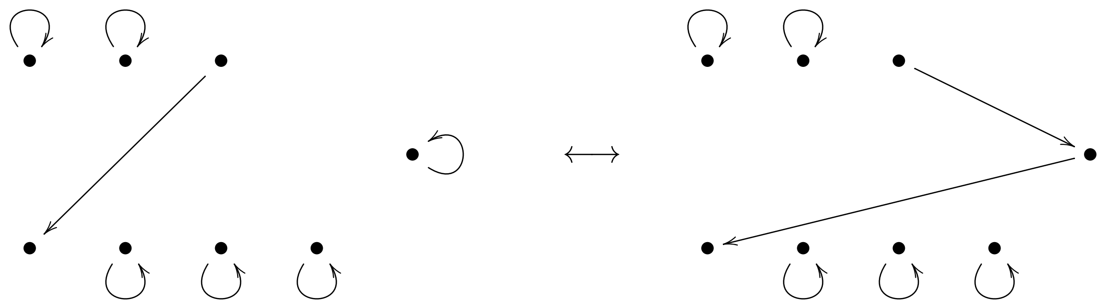


Рис. 18.

The second way of calculation uses the definition of determinant via univalent graphs. Fig. 18 right shows the graph such that $\det M$ counts its univalent subgraphs. In this graph $\ell - 1$ vertices of the first part are located on the top, $k - 1$ vertices of the second part are located on the bottom, and a single special vertex is located at the right.

First, split univalent graphs into pairs which contain at least 2 edges going from the second part to the first. Namely, among all the edges going from the second part to the first, choose two edges the ends of which have the least number and interchange the ends of these edges. Then the permutation will change its parity. It is not difficult to understand that this correspondence is bijective on the given set, therefore all corresponding graphs will be cancelled.

Second, split univalent graphs into pairs containing exactly one edge going from the second part to the first. As there exists exactly one edge outgoing from the second part, there exists either exactly one edge incomimg to the second part or a path consisting of two edges passing through the special vertex. Then rebuild the graph depending on from where this edge goes:



So we retain only univalent graphs such that no edge leads to the second part. It is clear that the only graph possessing this property is the graph consisting of all loops. The weight of this graph, and consequently also $\det M$, is equal to $-k^{l-1}l^{k-1}$.

4.8. Apply the Matrix-tree theorem. Expand $\det L^{-}$ into univalent graphs by the combinatorial definition. If there exists at least one loop in a univalent graph then the univalent graph has even weight. Otherwise since the number of vertices in a univalent graph is odd, there exists a cycle of length at least 3. Therefore reversing of all arrows leads to another univalent graph of the same weight. Thus we have split some univalent graphs into pairs of equal weight, and for the others we have understood that they give even contribution. This immediately implies that the corresponding determinant is even.

5.1. The formula for the determinant implies that if G has no perfect matching, then $\det(A)$ is the zero polynomial.

To show the converse, we fix a permutation π that defines a perfect matching, and we replace the variables in $\det(A)$ as follows: $x_{i,\pi(i)} := 1$ for every $i = 1, 2, \dots, n$, and all the remaining x_{ij} are 0. We have $\operatorname{sgn}(\pi) \cdot x_{1,\pi(1)} x_{2,\pi(2)} \cdots x_{n,\pi(n)} = \pm 1$ for this π.

For every other permutation $\sigma \neq \pi$ there is an i with $\sigma(i) \neq \pi(i)$, thus $x_{i,\sigma(i)} = 0$, and therefore, all other terms in the expansion of $\det(A)$ are 0. For this choice of the x_{ij} we thus have $\det(A) = \pm 1$.

5.2. We proceed by induction on m. The univariate case is clear, since there are at most d roots of $p(x_1)$ by a well-known theorem of algebra. (That theorem is proved by induction on d: if $p(\alpha) = 0$, then we can divide $p(x)$ by $x - \alpha$ and reduce the degree.)

Let $m > 1$. Let us suppose that x_1 occurs in at least one term of $p(x_1, \dots, x_n)$ with a nonzero coefficient (if not, we rename the variables). Let us write $p(x_1, \dots, x_m)$ as a polynomial in x_1 with coefficients being polynomials in $x_2, \dots, x_n$:

$$p(x_1, x_2, \dots, x_m) = \sum_{i=0}^{k} x_1^i p_i(x_2, \dots, x_m),$$

where k is the maximum exponent of x_1 in $p(x_1, \dots, x_n)$.

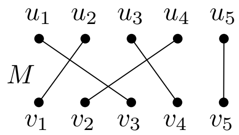
We divide the m-tuples $(r_1, \dots, r_m)$ with $p(r_1, \dots, r_m) = 0$ into two classes. The first class, called R_1, consists of the m-tuples with $p_k(r_2, \dots, r_m) = 0$. Since the polynomial $p_k(x_2, \dots, x_m)$ is not identically zero and has degree at most $d - k$, the number of choices for $(r_2, \dots, r_m)$ is at most $(d-k)|S|^{m-2}$ by the induction hypothesis, and so $|R_1| \leq (d-k)|S|^{m-1}$.

The second class R_2 are the remaining m-tuples, that is, those with $p(r_1, r_2, \dots, r_m) = 0$ but $p_k(r_2, \dots, r_m) \neq 0$. Here we count as follows: $r_2, \dots, r_m$ can be chosen in at most $|S|^{m-1}$ ways, and if $r_2, \dots, r_m$ are fixed with $p_k(r_2, \dots, r_m) \neq 0$, then r_1 must be a root of the univariate polynomial $q(x_1) = p(x_1, r_2, \dots, r_m)$. This polynomial has degree (exactly) k, and hence it has at most k roots. Thus the number of m-tuples in the second class is at most $k|S|^{m-1}$, which gives $d|S|^{m-1}$ altogether, finishing the induction step and the proof of the Schwartz–Zippel theorem.

5.3. Let us assume that G has a perfect matching and thus $\det(A)$ is a nonzero polynomial of degree n. The Schwartz–Zippel theorem shows that if we calculate $\det(A)$ for values of the variables x_{ij} chosen independently at random from $S := \{1, 2, \dots, 2n\}$, then the probability of getting 0 is at most $\frac{1}{2}$.

As usual, the probability of the failure can be reduced to 2^{-k} by repeating the algorithm k times.

6.1. Let S_n denote the set of all permutations of the set $\{1, 2, \ldots, n\}$. Every perfect matching M in G corresponds to a unique permutation $\pi \in S_n$, where $\pi(i)$ is defined as the index j such that the edge $\{u_i, v_j\}$ lies in M. Here is an example:



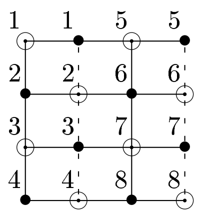
$\pi(1) = 3, \pi(2) = 1, \pi(3) = 4, \pi(4) = 2, \pi(5) = 5$

In the other direction, when does G have a perfect matching corresponding to a given permutation $\pi \in S_n$? Exactly if $b_{1,\pi(1)} = b_{2,\pi(2)} = \cdots = b_{n,\pi(n)} = 1$. Therefore, the number of perfect matchings in G equals

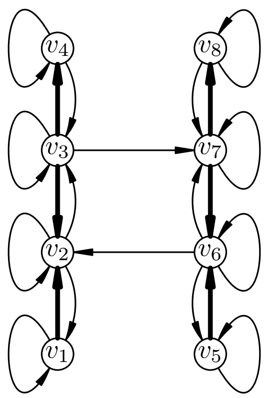
$$\sum_{\pi \in S_n} b_{1,\pi(1)} b_{2,\pi(2)} \cdots b_{n,\pi(n)},$$

and this is just the permanent of B.

6.2. Of course, the problem may be solved by calculation of the determinant, but not in our project! Number the vertices of the parts, write the bipartite adjacency matrix (black vertices correspond to the columns, white to the rows) and draw the graph, for which this matrix is the usual adjacency matrix:



$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$



The edges with weight -1 are drawn by bold arrows. It remains to check that any even cycle of any univalent subgraph of this graph contains an odd number of negative edges (and here are no odd cycles except loops). For this graph, it is evident.

6.3. Let the signing σ as in the condition be fixed, and let M be a perfect matching in G, corresponding to a permutation π. We define the *sign* of M as the sign of the corresponding term in $\det(B^\sigma)$; explicitly,

$$\operatorname{sgn}(M) := \operatorname{sgn}(\pi) b^\sigma_{1,\pi(1)} b^\sigma_{2,\pi(2)} \cdots b^\sigma_{n,\pi(n)} = \operatorname{sgn}(\pi) \prod_{e \in M} \sigma(e).$$

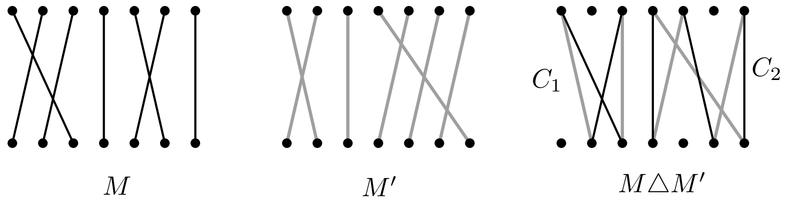
It is easy to see that σ is a Kasteleyn signing if (and only if) all perfect matchings in G have the same sign.

Let M and M' be two perfect matchings in G, with the corresponding permutations π and π'. Then

$$\begin{aligned} \operatorname{sgn}(M)\operatorname{sgn}(M') &= \operatorname{sgn}(\pi)\operatorname{sgn}(\pi') \Big(\prod_{e \in M} \sigma(e) \Big) \Big(\prod_{e \in M'} \sigma(e) \Big) \\ &= \operatorname{sgn}(\pi)\operatorname{sgn}(\pi') \prod_{e \in M \triangle M'} \sigma(e), \end{aligned}$$

where $\triangle$ denotes the symmetric difference.

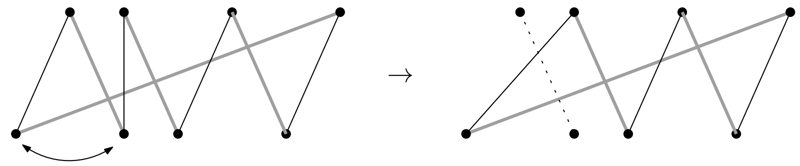
The symmetric difference $M \triangle M'$ is a disjoint union of evenly placed cycles, as the picture below illustrates.



Let these cycles be $C_1, C_2, \ldots, C_k$, and let the length of C_i be $2\ell_i$. Since C_i is evenly placed, it must be properly signed by the assumption in the lemma, and so we have $\prod_{e \in C_i} \sigma(e) = (-1)^{\ell_i - 1}$. Thus $\prod_{e \in M \triangle M'} \sigma(e) = (-1)^t$ with $t := \ell_1 - 1 + \ell_2 - 1 + \cdots + \ell_k - 1$.

It remains to check that π can be converted to π' by t transpositions (then, by the properties of the sign of a permutation, we have $\mathrm{sgn}(\pi) = (-1)^t \mathrm{sgn}(\pi')$, and thus $\mathrm{sgn}(M) = \mathrm{sgn}(M')$ as needed).

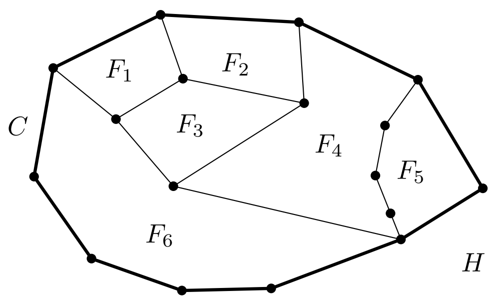
This can be done for one cycle C_i at a time. As the next picture illustrates for a cycle of length $2\ell_i = 8$, by modifying π with a suitable transposition we can "cancel" two edges of the cycle and pass to a cycle of length $2\ell_i - 2$ (black edges belong to M, gray edges to M', and the dotted edge in the right drawing now belongs to both M and M').



Continuing in this way for $\ell_i - 1$ steps, we cancel C_i, and we can proceed with the next cycle.

6.4. Let C be an evenly placed cycle in G; we need to prove that it is properly signed.

Let the length of C be 2ℓ. Let $F_1, \ldots, F_k$ be the inner faces enclosed in C in the drawing, and let C_i be the boundary cycle of F_i, of length $2\ell_i$. Let H be the subgraph of G obtained by deleting all vertices and edges drawn outside C; in other words, H is the union of the C_i.



We want to see how the parity of ℓ is related to the parities of the ℓ_i. The number of vertices of H is $r + 2\ell$, where r is the number of vertices lying in the interior of C. Every edge of H belongs to exactly two cycles among $C, C_1, \ldots, C_k$, and so the number of edges of H equals $\ell + \ell_1 + \cdots + \ell_k$. Finally, the drawing of H has $k + 1$ faces: $F_1, \ldots, F_k$ and the outer one.

Now we apply *Euler's formula*, which tells us that for every drawing of a connected planar graph, the number of vertices plus the number of faces equals the number of edges plus 2. Thus

$$r + 2\ell + k + 1 = \ell + \ell_1 + \cdots + \ell_k + 2. \tag{12}$$

Next, we use the assumption that C is evenly placed. Since the graph obtained by deleting C from G has a perfect matching, the number r of vertices inside C must be even. Therefore, from (12) we get

$$\ell - 1 \equiv \ell_1 + \cdots + \ell_k - k \pmod 2. \tag{13}$$

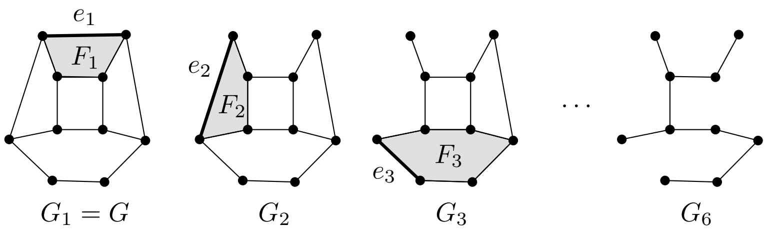
Let n_C be the number of negative edges in C, and similarly for n_{C_i}. The sum $n_C + n_{C_1} + \cdots + n_{C_k}$ is even because it counts every negative edge twice, and so

$$n_C \equiv n_{C_1} + \cdots + n_{C_k} \pmod 2. \tag{14}$$

Finally, we have $n_{C_i} \equiv \ell_i - 1 \pmod 2$ since the C_i are properly signed. Combining this with (13) and (14) gives $n_C \equiv \ell - 1 \pmod 2$. Hence C is properly signed. Now the result follows from the result of 6.3.

6.5. Given a connected, 2-connected, planar, bipartite G, we fix some planar drawing, and we want to construct a signing as in Lemma B, with the boundary of every inner face properly signed.

First we start deleting edges from G, as the following picture illustrates.



We set $G_1 := G$, and G_{i+1} is obtained from G_i by deleting an edge e_i that separates an inner face F_i from the outer (unbounded) face (in the current drawing). The procedure finishes with some G_k that has no such edge. Then the drawing of G_k has only the outer face.

Now we choose the signs of the edges of G_k arbitrarily, and we extend this to a signing of G by going backward, choosing the signs for $e_{k-1}, e_{k-2}, \ldots, e_1$ in this order. When we consider e_i, it is contained in the boundary of the single inner face F_i in the drawing of G_i, so we can set $\sigma(e_i)$ so that the boundary of F_i is properly signed. The assertion is proved.

6.6. We took this problem from [7].

a) The statement is evident. Choosing a permutation π, we define for each inlet i the outlet $\pi(i)$ where the route should go, and the number of ways to choose a route equals $a_{i\pi_i}$. Then the product $a_{1\pi_1}a_{2\pi_2}\ldots a_{n\pi_n}$ is equal to the number of ways to choose a set of routes. Summing over all π, we obtain the total number of sets.

b) As in the reasoning from part a), the product $a_{1\pi_1}a_{2\pi_2}\ldots a_{n\pi_n}$ is equal to the number of ways to choose a set of routes. The sets of non-intersecting routes are taken into account only in the product defined by identity permutation. In the sum defining the determinant, this product has sign plus. It remains to check that the sets of intersecting routes one can split into pairs counted with different signs, then all of them will be cancelled in this sum.

Enumerate each path by the number of the inlet where this path begins. For each set of n non-intersecting paths defined by permutation π, define by i the least of the numbers of paths which intersect with other paths. Denote by O the first point of intersection of ith path with some other path, let it be jth path.

Interchange the fragments of ith and jth paths after point O, the obtained set of paths will form a pair with the considered set. The permutation that describes the obtained set,differs from the initial by the transposition (π_i, π_j) and therefore has the opposite sign.

In this reasoning we did not use that the graph is planar. It was essential for us only that this graph is nonpermutable, i. e. in the set of n non-intersecting paths the number of the outlet is equal to the number of the inlet for each path.

6.7. a) We have taken this problem from [3]. Let A_{mn} be the number of paths of lame king from a corner of $m \times n$ rectangle to the opposite corner. Then

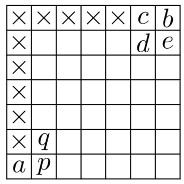
$$A_{n-1,n-1} = 2A_{n-2,n-1} + A_{n-2,n-2}$$

(to ensure this, consider three possibilities of the first king's move). Similarly,

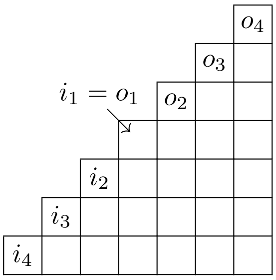
$$B_n = A_{n-2,n-2} + A_{n-1,n-1} + 2A_{n-1,n-2}$$

(consider four possible combinations of the first and the last moves). Substituting the first equality into the second one, we obtain the required.

We will present an argument which gives the bijective proof of the required equality, related to the formal proof written above.



Value B_n is the number of paths leading from square a to square b of an $n \times n$ board and not passing through the squares marked with asterisk and through square c. Value $2A_{n-1}$ is the number of paths leading from square p to square e plus the number of paths leading from square q to square b. One-to-one correspondence between the sets of such paths is constructed as follows. Consider an arbitrary path of lame king from p to e. Supplementing this path with steps a–p and e–b we obtain a path from a to b not passing through forbidden squares. The first step in the obtained path is horizontal, the last is vertical. Now consider an arbitrary path of lame king from q to b. If it does not pass through square c, then supplement it with step a–q. We obtain a path from a to b not passing through forbidden squares, in which the first step is diagonal, the last is diagonal or vertical. And in the case when this path passes through square c, remove step c–b from it, shift the obtained path q–c one square downward (we will obtain a path from p to d), and complete this path by steps a–p, d–b to obtain a path from a to b. In the resulting path the first step is horizontal, and the last is diagonal. It is evident that the constructed correspondence is one-to-one.



b) We have taken this statement in [4, lemma 5.1]. Consider matrix $B^{(n)} = (B_{i,j})_{1 \leqslant i,j \leqslant n}$, where $B_{i,j}$ is the number of routes of lame king on $i \times j$ board leading from the lower left corner to the upper right corner and not containing any cells of the left vertical and upper horizontal lines (except the initial and final positions). Similarly to the statement of the previous item it is proven that $B^{(n)} = 2A^{(n)}$. By the statement of problem 6.6, determinants $\det A^{(n)}$ and $\det B^{(n)}$ count the same number of the sets of routes. Thus, $\det A^{(n)} = 2^n \det A^{(n-1)}$, whence $\det A^(n) = 2^{n(n+1)/2}$.

6.8. Answer: 1. We have taken this problem in [7]. Consider inlets and outlets shown in the figure. The numbers of paths from inlets to outlets are just Catalan numbers. Then $\det H$ counts the number of the sets of non-intersecting routes from inlets to outlets. It is evident that such set of routes is unique.

7.1. Express $\det A_G$ through univalent graphs by the combinatorial definition. Since G does not contain cycles, a univalent graph occurs with nonzero weight only if all the cycles in it have length 2, and this corresponds to perfect matching of G. The required result follows from the fact that a tree has not more than one perfect matching.

7.2. Express $\det A_G$ through univalent graphs by the combinatorial definition. Since G does not contain cycles, a univalent graph occurs with nonzero weight only if there exist no cycles of length 3 or more. Then univalent graphs, consisting of k multiple edges and $n-2k$ loops, occur with weight $(-1)^k x^{n-2k}$. Now one can see that polynomials $\chi G(x)$ and $m_G(x)$ count the same combinatorial objects with the corresponding weights, so these polynomials coincide.

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