On inverse images of the Feuerbach point,
poles of a triangle, and Kulanin's theorem
Solutions

0. Auxiliary facts

0.1 Since $\angle BHC = 180^\circ - \angle BAC$, points symmetric to H in BC and in the midpoint of BC, lie on the circumcircle. Homothety with center H and ratio $1/2$ maps the circumcircle to Euler circle.

0.2 Perpendicular bisectors of the original triangle are the altitudes for the medial triangle, hence O is its orthocenter. Homothety with center M and ratio $-1/2$ maps the original triangle to the medial one, hence it maps H to O. Hence M belongs to the segment HO and $HM : MO = 2$. By the previous problem, E belons to the segment HO and $HE : EO = 1 : 1$. The equality $HE : EM : MO = 3 : 1 : 2$ follows.

0.3 Let theee lines form a traingle ABC, while the fourth line intersects AB, AC, BC at D, E, F, respectively. Let the circles (ABC) and (CEF) intersect at $\neq C$. Further, we prove that P lies on the circle (BDF). It is sufficient to prove that $\angle(BP, PF) = \angle(BD, DF)$. It is clear that $\angle(BP, PF) = \angle(BP, PC) + \angle(PC, PF) = \angle(BA, AC) + \angle(EC, EF) = \angle(BD, AC) + \angle(AC, DF) = \angle(BD, DF)$. Similarly, P lies on the circle (ADE).

0.4 Let I_a, I_b, I_c be the centers of the excircles of ABC. Note that I_aA, I_bB, I_cC are the altitudes in $I_aI_bI_c$. The corresponding sides of $I_aI_bI_c$ and $G_aG_bG_c$ are parallel, i.e. these triangles are homothetic. Hence, their Euler lines are parallel. Note that I is the circumcenter of $G_aG_bG_c$ and the orthocenter of $I_aI_bI_c$. Hence both Euler lines pass through I, and therefore, coincide. O is the center of Euler circle in triangle $I_aI_bI_c$, hence O belongs to its Euler line.

0.5 By angle chasing, follows from the solution of the next problem.

0.6 Consider the homothety with center P and ratio 2. Let P'_a, P'_b, P'_c, H'_a, H'_b, H'_c be points symmetric to P and H in the corresponding sidelines. It suffices to prove that P'_b, H, P'_c are collinear (the proof of collinearity of P'_a, H, P'_c is analogous). By symmetry in AB, $\angle(P'_cH, HB) = \angle(BH'_c, H'_cP)$ $\angle(CH, HP'_b) = \angle(PH'_b, H'_bC)$. Since P, H'_b, H'_c are concyclic, we have $\angle(P'_cH, HP'_b) = \angle(P'_cH, HB) + \angle(HB, CH) + \angle(CH, HP'_b) = \angle(BH'_c, H'_cP) + \angle(HB, CH) + \angle(PH'_b, H'_bC) = \angle(BA, AP) + \angle(HB, CH) + \angle(PA, AC) = \angle(BA, AC) + \angle(HB, CH) = 0$, therefore, P'_b, H, P'_c are collinear.

0.7 Let P'_a, P'_b, P'_c be points symmetric to P in the corresponding sidelines. It is easy to check that the line symmetric to AP in AI is the perpendicular bisector of $P'_bP'_c$. Therefre, P' is the circumcenter of $P'_aP'_bP'_c$.

0.8 We prove the following properties of isogonal conjugacy:

- Follows from the previous problem and the fact that the reflections of H in the sidelines lie on the circumcircle.
- Using the previous and applying homothety with center P and ratio $1/2$, we obtain that the midpoint of PP' is the center if the pedal circle of P. The midpoint of PP' is equidistant from the projections of P and P' onto any sideline. It follows that the pedal circles of P' and P coincide.

- By the previous problem, the isogonal conjugate of P is the intersection of the perpendicular bisectors of $P'_aP'_b$, $P'_bP'_c$, $P'_cP'_a$. Note that each perpendicular bisector is perpendicular to Simson line of P.
- Clear.

0.9 By the previous problem, P_b, P_c, Q_b, Q_c lie on a circle centered at the midpoint of PQ. Let X and Y be reflections of P_b and P_c in the center, respectively. Hence X lies on QQ_b, while Y lies on Q_c. By Pascal theorem for $P_bQ_cYP_cQ_bX$, the midpoint of PQ, the point Q, and the point $P_bQ_c \cap P_cQ_b$ are collinear. This means that P_bQ_c, P_cQ_b, PQ are concurrent.

0.10 Since $ABCH$ is non-convex, each its circumconic is a hyperbola. Let P, Q be its points at infinity. It suffices to prove that the directions defined by P and Q are perpendicular. Use pascal theorem for $ABCHPQ$. Let $X = AB \cap HP$, $Y = CH \cap QA$. XY passes through $BC \cap PQ$, i.e. $XY \parallel BC$, or, equivalently, $XY \perp AH$. Hence Y is the orthocenter of AHX, and it follows $AY \perp HX$. Thus $AQ \perp HP$, QED.

0.11 Choose points P, Q, R on a line ℓ. Take a conic c through its isogonal conjugates P', Q', R' and through B, C. Consider two maps $\ell \to c$ keeping cross-ratios. The first map to each $X \in \ell$ assings the intersection of c with the reflection of BX in the bisector of the angle B. The second map is the same with B replaced by C. These two maps coincide at P, Q, R, hence in fact they coincide. Therefore, the image of ℓ under isogonal conjugacy is a conicn through B and C. It is clear that this conic also passes through A.

1. Feuerbach theorem

1.1 Since S_{ab} is the intersection point of λ_a and the midline M_aM_b, triangle BM_aS_{ab} is isosceles. We have $\angle S_{ab}BM_a = \frac{1}{2}\angle S_{ab}M_aC = \frac{1}{2}\angle ABC = \angle IBC$. Hence S_{ab} lies on BI.

Since $\angle BS_{ab}C = 90^\circ$, I, G_b, S_{ab}, C are concyclic (lie on the circle with diameter CI). Hence $\angle S_{ab}G_bC = \angle S_{ab}IC = 90^\circ - \frac{1}{2}\angle BAC = \angle AG_bC$. Therefore, S_{ab} lies on the line G_bG_c (see fig. 1).

1.2 The angle between H_aM_a and ε is equal to the half of arc H_aM_a in ε, hence it is equal to $\angle H_aM_bM_a = |\angle CM_aM_b - \angle CH_aM_b| = |\angle CBA - \angle H_aCM_b| = |\angle B - \angle C|$.

1.3 Note that BI and BI_a are internal and external bisectors of the angle ABL_a. Hence A, I, L_a, I_a is a harmonic quadruple. The required statement follows from projection onto BC.

1.4 By the previous problem, the inversion in the circle with diameter $G_aG'_a$ takes the nine-point circle ε to the line ℓ passing through L_a. On the other hand, ω and ω_a are invariant under this inversion, since each of them is orthogonal to circle of inversion. Therefore, the angle between ℓ and BC equals $|\angle B - \angle C|$, that is the angle between ω BC. The line symmetric to BC in AI forms the same angle with BC. Therefore, ℓ is symmetric to BC in AI, hecne ℓ is tangent to ω and ω_a (see fig. 2).

1.5 The fact of tangency of Eulaer circle with the excircle ω_a at some point F_a follows from the solution of the previous problem. L_a is the center of homothety with a negative ratio. The required perspectivity follows from Monge theorem for the incircle, the excircle, and Euler circle.

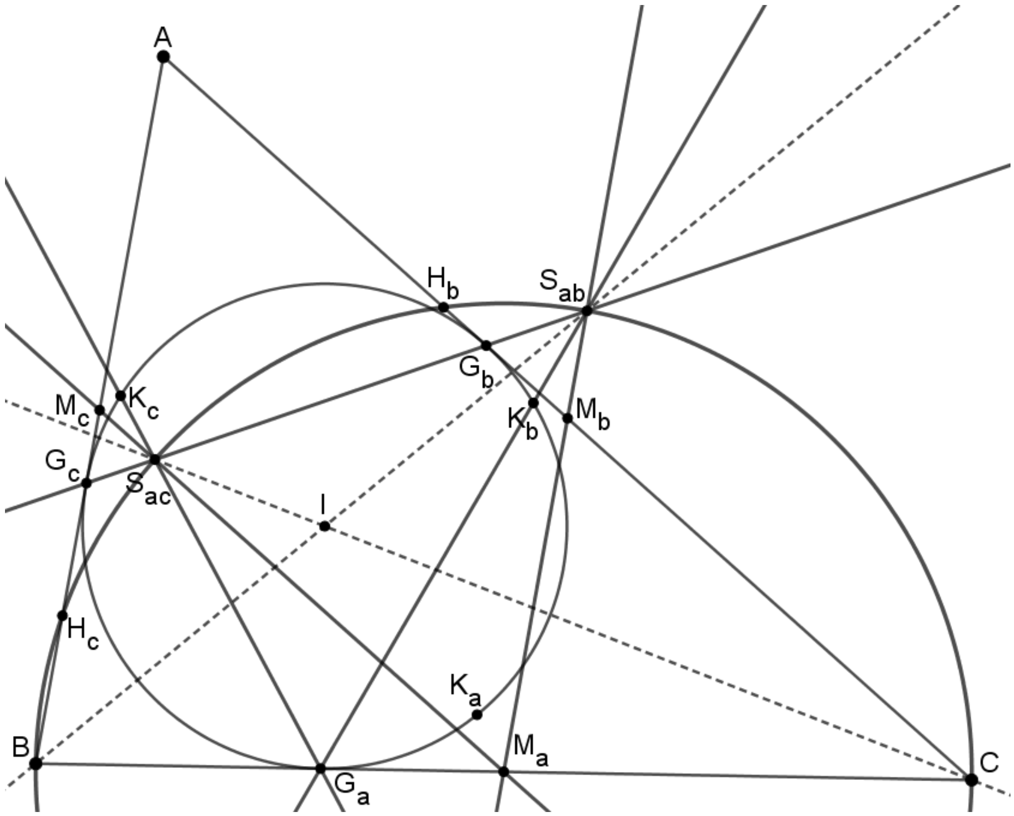

Figure 1.

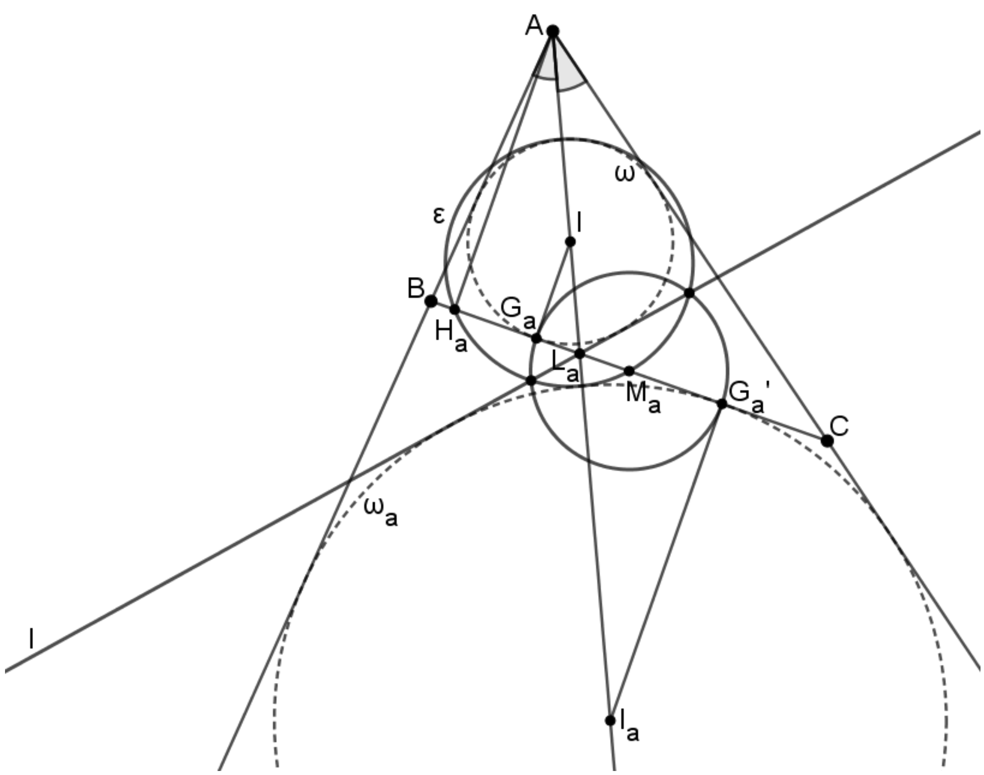

Figure 2.

1.7 Consider the homothety with center F taking the incircle to the Euler circle. The line H_aM_a maps to the line parallel to H_aM_a and touching the Euler circle at X. By symmetry in the perpendicular bisector of H_aM_a, X is the midpoint of the arc H_aM_a of Euler circle. Hence FX is the bisector of $\angle H_aFM_a$. Now the statement follows, since the homothety maps G_a to X.

1.8 By M_{XY} denote the midpoint of XY Let A, B, C, P be points, and let $X \neq M_{BP}$ be the common point of the Euler circles of ABP and BCP. We have $\angle(M_{AP}X, XM_{CP}) = \angle(M_{AP}X, XM_{BP}) + \angle(M_{BP}X, XM_{CP}) = \angle(M_{AP}M_{AB}, M_{AB}M_{BP}) + \angle(M_{BP}M_{BC}, M_{BC}M_{CP}) = \angle(BP, AP) + \angle(CP, BP) = \angle(CP, AP) = \angle(M_{AP}M_{AC}, M_{AC}M_{CP})$.
This means that X lies on the Euler circle of ACP. Similarly, X lies on the Euler circle of ABC (see fig. 3).

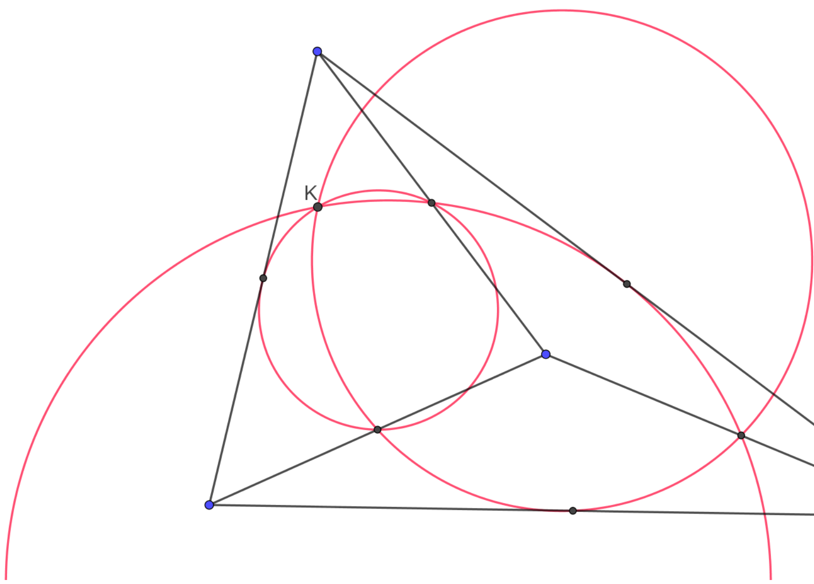

Figure 3.

1.9 Let P_a, P_b, P_c be P projections of P to the sides of the triangle ABC. We use notation from the solution to the previous problem.
$\angle(P_cX, XP_a) = \angle(P_cX, XM_{BP}) + \angle(M_{BP}X, XP_a) = \angle(P_cM_{AB}, M_{AB}M_{BP}) + \angle(M_{BP}M_{BC}, M_{BC}P_a) = \angle(P_cA, AP) + \angle(PC, CP_a) = \angle(P_cP_b, P_bP) + \angle(PP_b, P_bP_a) = \angle(P_cP_b, P_bP_a)$.

1.10 First, let us prive that the corresponding sides of $K_aK_bK_c$ and $M_aM_bM_c$ are parallel. Chords G_aK_b and G_bG_c are symmetric in BI, hence they are equal. The similar is true for chords G_bG_c and G_aK_c. Therefore, $K_cG_aK_b$ is isosceles, which implies $\angle G_aK_cK_b = \angle G_aK_bK_c = \angle CG_aK_b$. Hence $BC \parallel K_bK_c$. For the other pairs the proof is similar. Now by Desargues, $K_aK_bK_c$ are $M_aM_bM_c$ homothetic. The common point of lines M_aK_a, M_bK_b, and M_cK_c is the center of this homothety, and moreover, the center of homothety taking the incircle to the Euler circle, that is the Feuerbach point F (see fig. 4).

1.11 Follows from the Feuerbach theorem and problems **1.8**, **1.9** for quadrilateral $ABCI$ (see fig. 5).

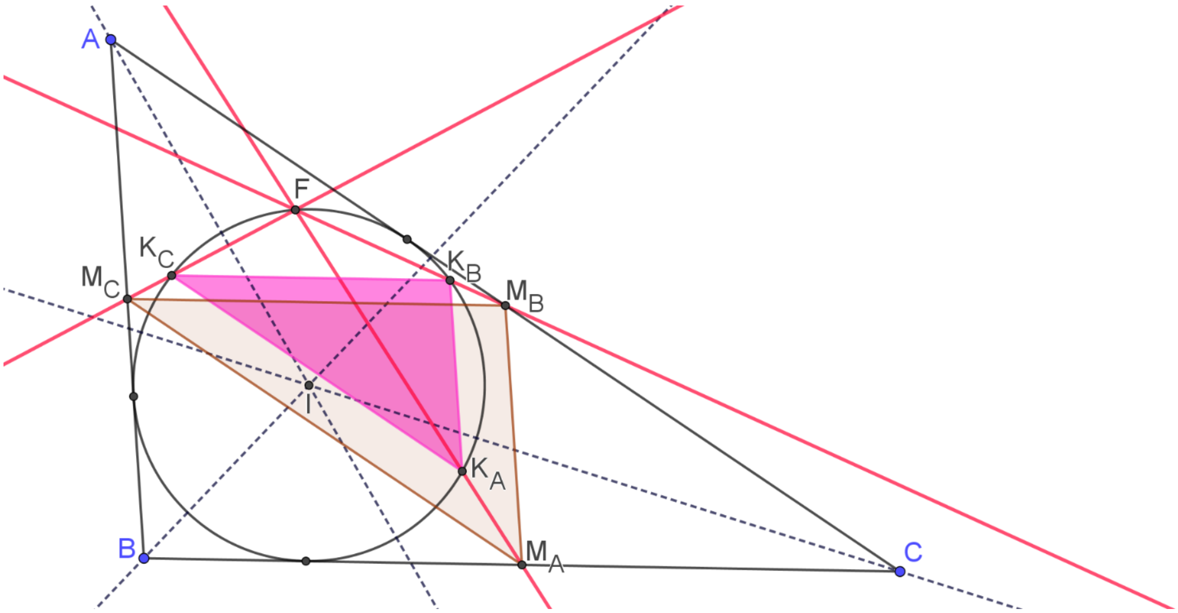

Figure 4.

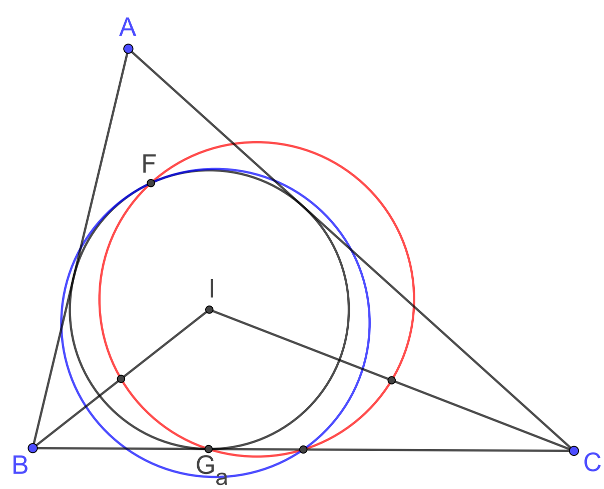

Figure 5.

1.12 By the previous problem, the point F lies on the Euler circle of AIC. The midpoint X of AI also lies on this circle. It suffices to prove that $\angle(G_bF, FX) = \angle(G_bF, F\widetilde{G}_a)$. We have $\angle(G_bF, FX) = \angle(G_bM_b, M_bX) = \angle(G_bC, CI) = \angle(G_bG_a, G_aI) = \angle(G_bG_a, G_a\widetilde{G}_a) = \angle(G_bF, F\widetilde{G}_a)$ (see fig. 6).

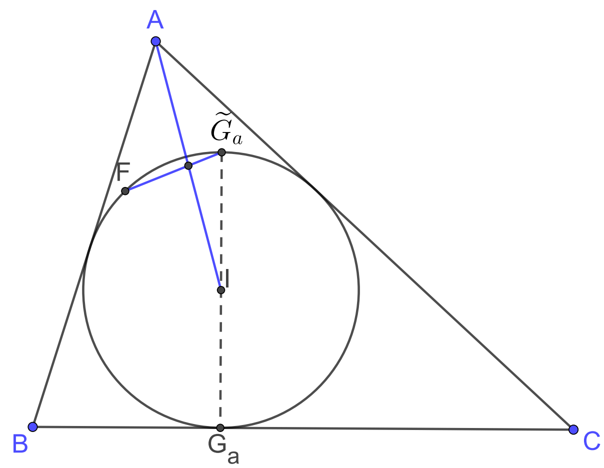

Figure 6.

2. The inverse images of the Feuerbach points

2.1 The inversion in λ_a takes the Euler circle ε to H_bH_c, while takes ε_a to $S_{ab}S_{ac}$. Since F is a common point of ε and ε_a, F'_a is the common point of lines H_bH_c and $S_{ab}S_{ac}$, which are radical axis of λ_a and ε, λ_a and ε_a (see fig. 7).

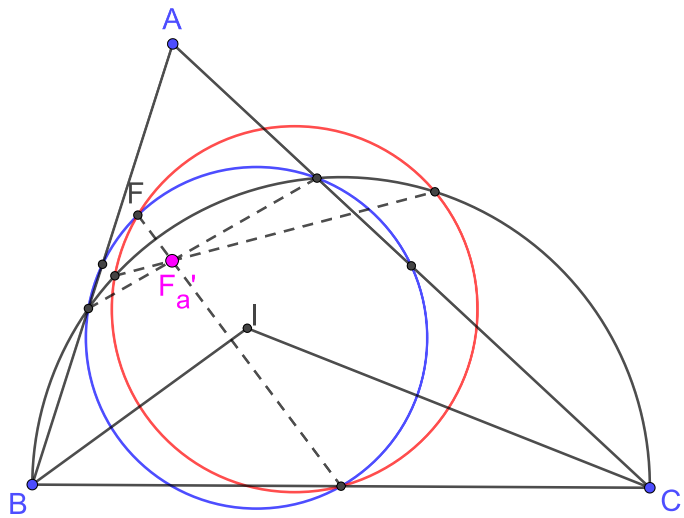

Figure 7.

2.2 First, we prove that F, F'_a, K_b, K'_b are concyclic (similarly, for F, F'_a, K_c, K'_c). Consider the inversion Inv_{K_a} with center K_a taking F'_a to F. Hence $\mathrm{Inv}_{K_a} \circ \mathrm{Inv}_{\lambda_a}$ takes ε to a circle ω' passingh through F and K_a. This transformation preserves angles between lines and circles. Note that the centers of both inversions lie on FF'_a. Therefore, circles ε and ω' form equal angles with FF'_a. This means that ε and ω' touch at F. It follows that ω and ω' coincide, sine there exists a unique circle passing through K_a and touching ε at F.

Further, note that K'_b that is the point of intersection of ray K_aK_b with H_bH_c is the image of K_b under Inv_{K_a}. Hence F, F'_a, K_b, and K'_b are concyclic.

To complete with S_{ab} and S_{ac} use inscribed angles. First, $\angle K_aFK_b = \angle K_aK_cK_b$. Further, let FK_b meet ε_a again at R. We have $\angle M_aS_{ac}R = \angle M_aFR$. Lines M_aS_{ac} and K_aK_c are parallel. Since $\angle M_aS_{ac}R = \angle M_aFR$, we have $S_{ac}R \parallel K_bK_c \parallel BC$. it follows $\angle M_aFR = \angle S_{ac}S_{ab}G_a$, which means that F, F'_a, K_b, K'_b, S_{ab} are concyclic. For S_{ac} the proof is analogous (see fig. 8).

2.3 From the solution of the previous problem ψ_{ab} and λ_a are orthogonal. Therefore, M_aS_{ab} touches ψ_{ab}.

2.4 Since M_aS_{ab} and M_aS_{ac} touch ψ_{ab} and ψ_{ac}, respectively, and $M_aS_{ab} \parallel K_aK_b$ $M_aS_{ac} \parallel K_aK_c$, we have $K_bS_{ab} = K'_bS_{ab}$ and $K_cS_{ac} = K'_cS_{ac}$ (equal chords between parallel lines). Hence $\angle K_bF'_aS_{ab} = \angle K'_bF'_aS_{ab}$, QED.

2.5 F'_a lies on K_bK_c, by the previous problem.

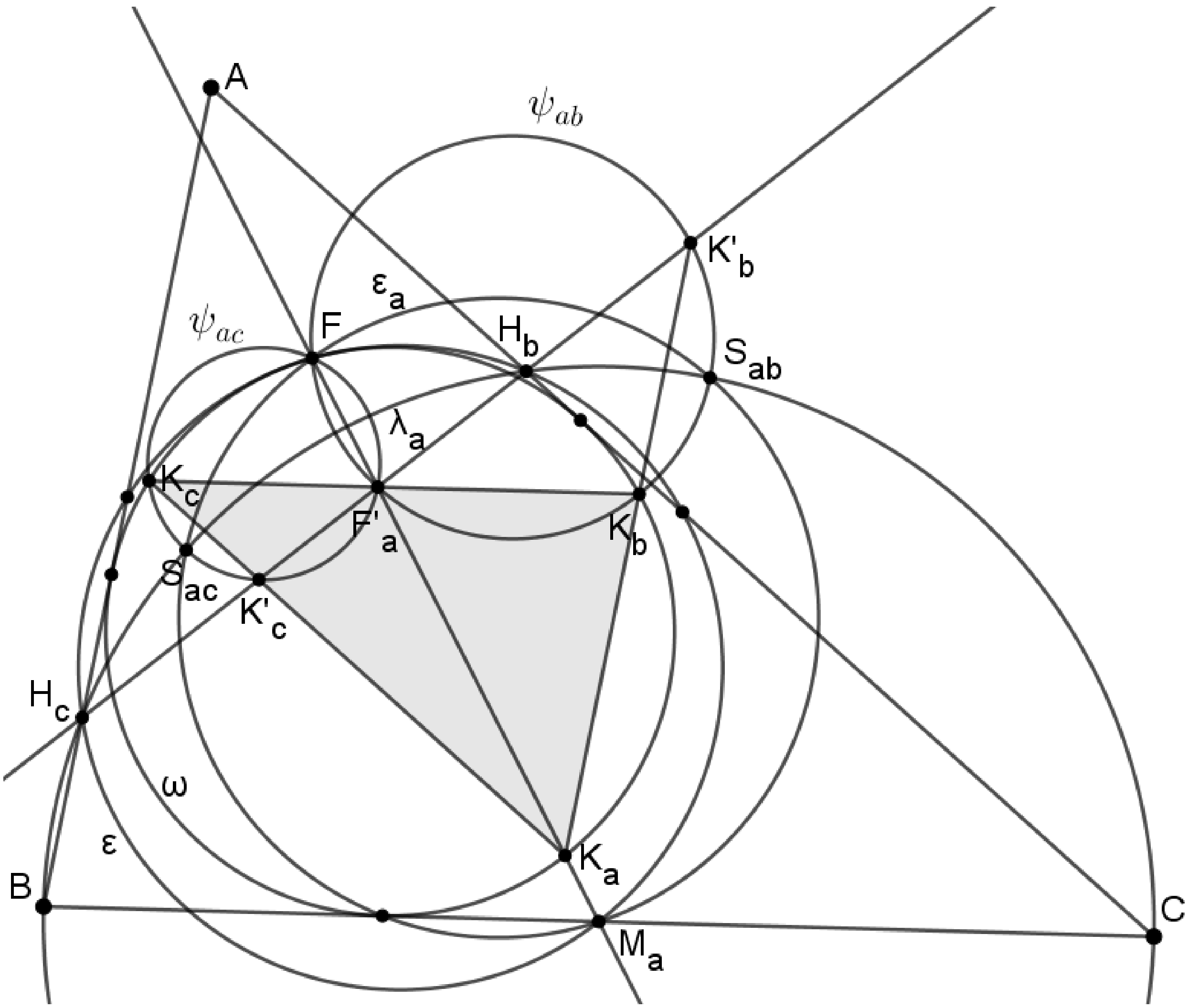

Figure 8.

Note that L_bK_b and L_cK_c are tangent to the incircle at points K_b and K_c respectively. We use the following claim, which is a degenerate case of Brianchon's theorem:

Let $K_cG_bK_bG_c$ be a cyclic quadrilateral. Tangents to its circumcircle through its vertices form another quadrilateral. Then the diagonals of these quadrilaterals have a common point.

So the lines K_bK_c, G_bG_c, L_bL_c are concurrent.

By problem **1.3** the line $G'_bG'_c$ passes through their common point.

2.6 Let us prove for ψ'_{ab}; for ψ'_{ac} the proof is similar. $\angle M_aFM_b = \angle M_aM_cM_b = \angle ACB = \angle G_aS_{ab}T_a$ (the last equality follows since C, G_a, G_b, S_{ab} lie on the circle with diameter CI). By the problem **1.11**, $\angle G_aFM_a = \angle G_aS_{ab}M_a$. Hence $\angle T_aFM_b = \angle M_aFM_b + \angle G_aFM_a = \angle G_aS_{ab}T_a + \angle G_aS_{ab}M_a = \angle T_aS_{ab}M_b$ (see fig. 9).

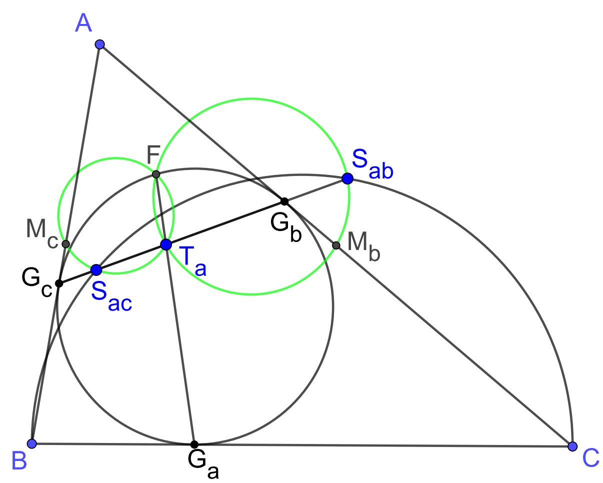

Figure 9.

2.7 F lies on circles $(M_aM_bM_c)$ and $(S_{ab}M_aS_{ac})$.

2.8 Follows from two previous problems (see fig. 10).

2.9 Using the circle ψ'_{ab} from the problem **2.6** we have: $\angle A^{\triangle}P_bF = \angle T_aM_bF = \angle T_aS_{ab}F = \angle A^{\triangle}S_{ab}F$. It follows that $A^{\triangle}$ lies on the circle $\psi^{\triangle}_{ab}$. Similarly, prove that $A^{\triangle}$ lies on $\psi^{\triangle}_{ac}$ (see fig. 11).

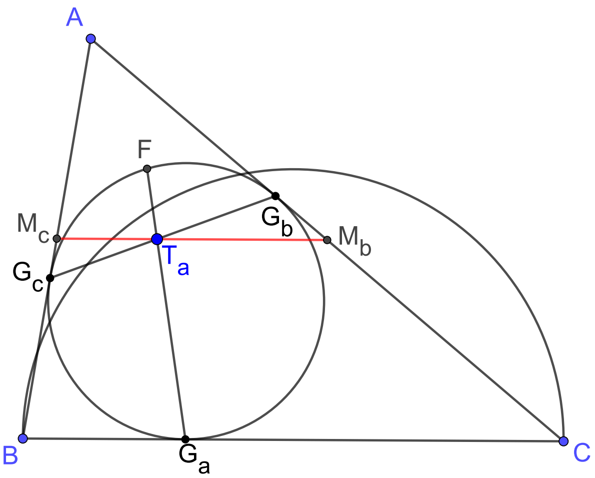

Figure 10.

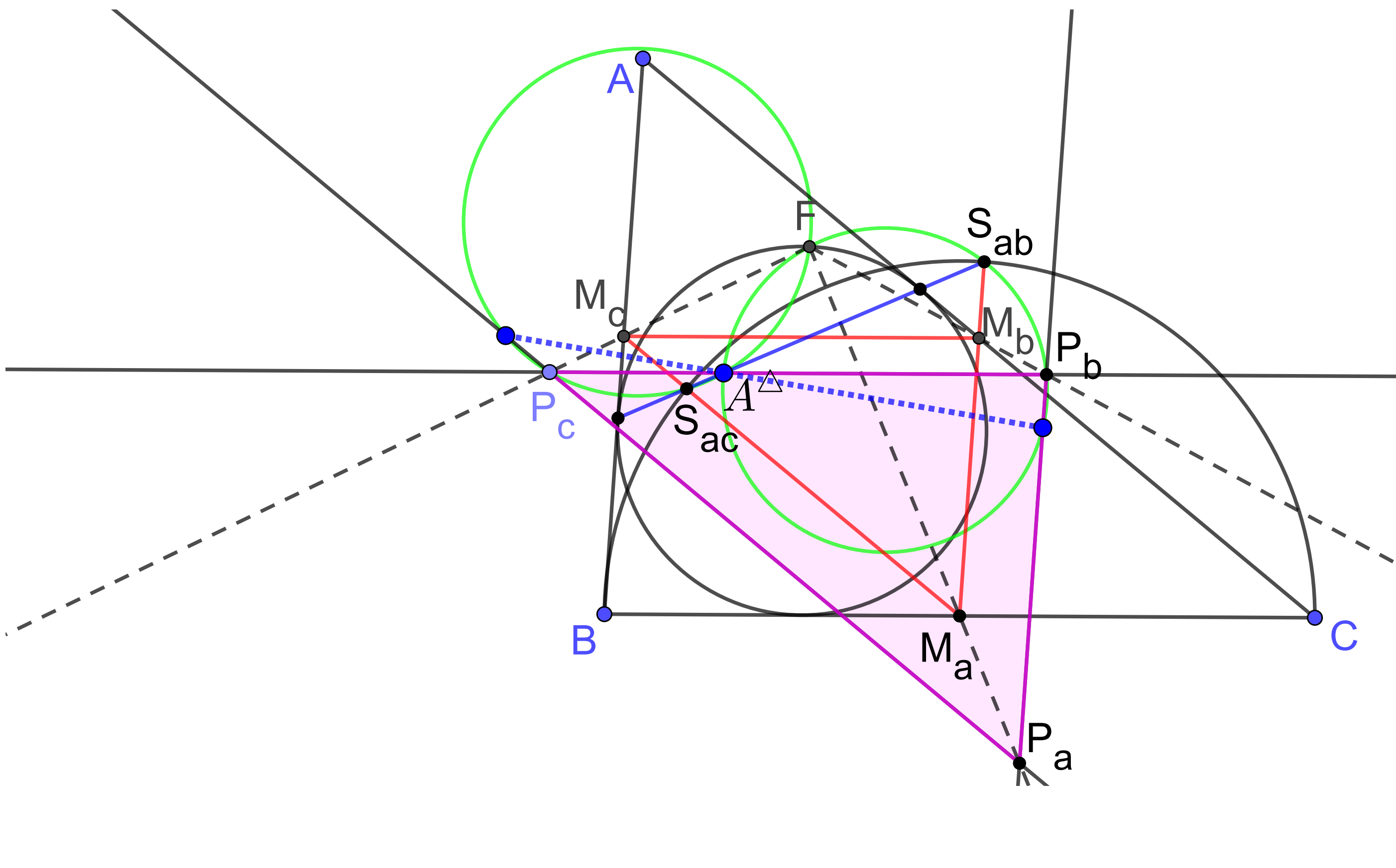

Figure 11.

2.10 First, we prove that H_b, G_b, S_{ac}, S_{ca} are concyclic. Note that BS_{ca} and H_bS_{ac} are parallel, since both these lines are perpendicular to the bisector of $\angle BAC$. Similarly, $BS_{ac} \parallel G_bS_{ca}$. Hence $BS_{ac}H_bS_{ca}$ is a parallelogram, and $H_bS_{ac} = BS_{ca} = G_bS_{ca}$. Hence $S_{ac}H_bG_bS_{ca}$ is an isosceles trapezoid, thus it is inscribed (see fig. 8).

Now let us prove that the Feuerbach point also lies on this circle. By Archimedes lemma, $\angle H_bFG_b = \frac{1}{2}\angle H_bFM_b = \frac{1}{2}|\angle A - \angle C|$. Further, $\angle H_bS_{ac}S_{ca} = \angle S_{ca}G_bC = \frac{\pi}{2} - \frac{\angle C}{2}$, and $\angle G_bS_{ac}S_{ca} = \angle AG_bS_{ac} = \frac{\pi}{2} - \frac{\angle A}{2}$, hence

$$\angle H_bS_{ac}G_b = |\angle H_bS_{ac}S_{ca} - \angle G_bS_{ac}S_{ca}| = \frac{1}{2}|\angle A - \angle C| = \angle H_bFG_b,$$

which imples that F lies on the circumcircle of the trapezoid $H_bG_bS_{ca}S_{ac}$

It is easy to see that the point I is the orthocenter of the triangle $S_{ac}S_{ca}G_b$. It follows that the reflection of I in the line $S_{ac}S_{ca}$ lies on the circle $(S_{ac}S_{ca}G_b)$, QED.

2.11 Let us move $P_aP_bP_c$ linearly. Points $A^{\triangle}, B^{\triangle}, C^{\triangle}$ move keeping cross-ratio. $X_a = FA^{\triangle} \cap BC$ (and similarly, X_b, X_c) move keeping cross-ratio. The intersection points of perpendiculars to the sidelines through $A^{\triangle}, B^{\triangle}, C^{\triangle}$ with OI also move keeping cross-ratio. To prove that they coincide, it suffices to find 3 particular positions. Two positions: $P_aP_bP_c = K_aK_bK_c$ and $P_aP_bP_c = M_aM_bM_c$ follow from the previous problems. The third position is a case, where $P_aP_bP_c$ degenerates to 3 lines passing through F parallel to the sidelines. In this position all perpendiculars are lines at infinity. The statement on a circle follows from the problem **3.1**.

Alternatively, one could derive all the statements from the main theorem (see problem **3.4**).

3. Generalized poles of a triangle and Kulanin's theorem

3.1 From problems **0.11**, **0.10**, and the first statement of the problem **0.8**, it follows that the isogonal conjugate P' of P traces along a rectangular hyperbola passing through the vertices of ABC. By the second statement of the problem **0.8**, it suffices to prove that the pedal circles of P' pass through the center of this hyperbola. Using problems **1.9** and **1.8**, reduce to the known statement: the Euler circle of a triangle inscribed into a rectangular hyperbola passes through the center of this hyperbola (see fig. 12).

3.2 Let P move linearly along the line ℓ. Then P_a, P_b, P_c move linearly along the sidelines. Hence $A_{op} = F_\ell P_a \cap M_bM_c$ moves linearly along M_bM_c. It suffices to prove that P_b, P_c, A_{op} are collinear. For this purpose we check 3 positions of P. The case $P = O$ is trivial. Two cases, where P lie on the circumcircle, follow from Kulanon's theorem (the pedal circle of P degenerates to the Simson line) (see fig. 13).

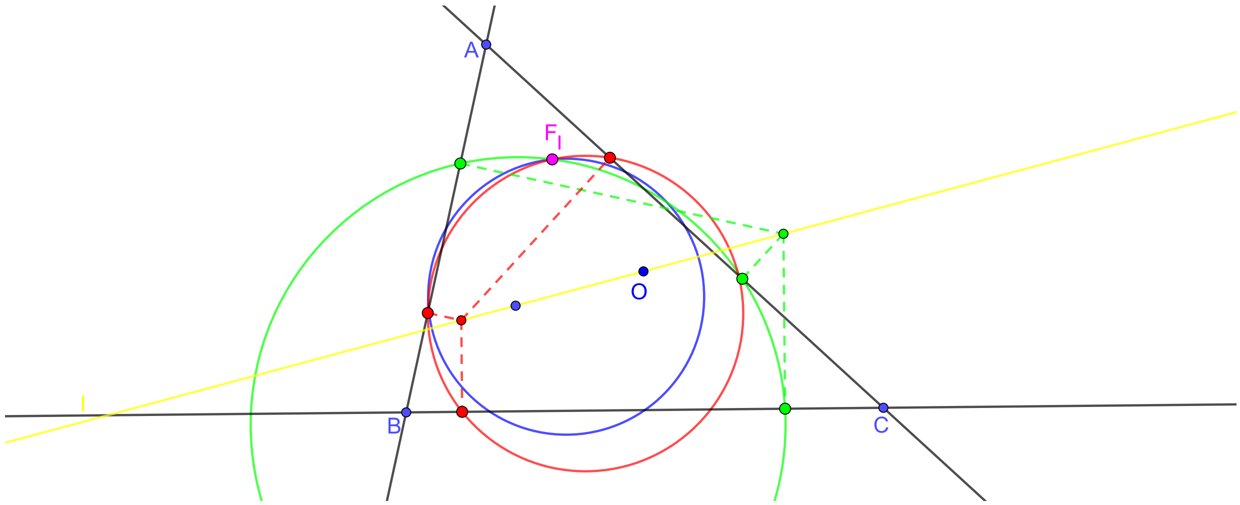

Figure 12.

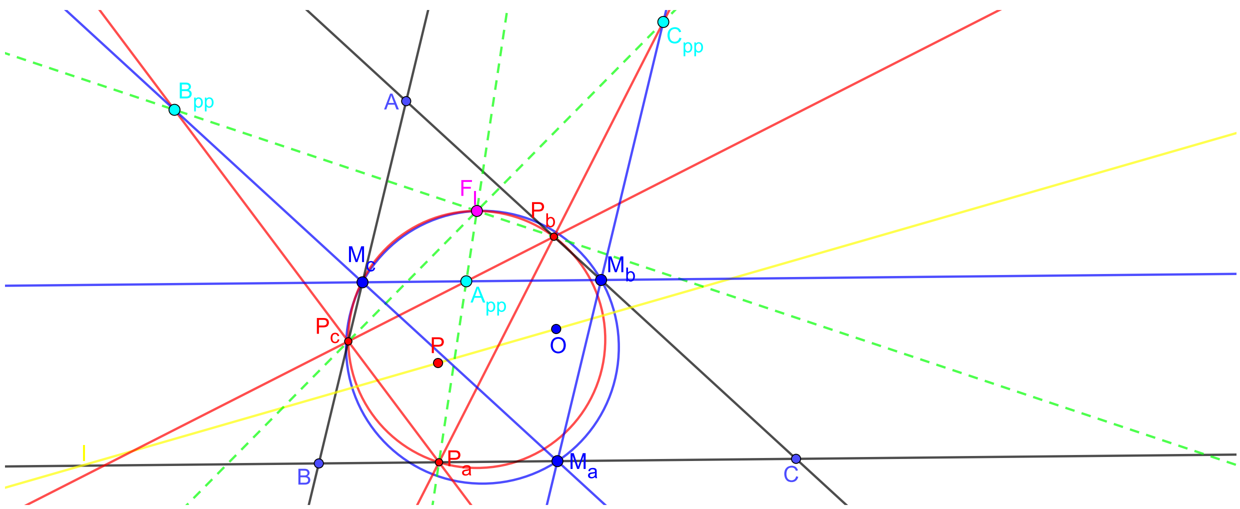

Figure 13.

3.3 Let P move linearly along the line ℓ. Let t be the length of the oriented segment OP. Introduce the Cartesian coordinates with origin F_ℓ, and by $(X_a(t), Y_a(t))$, $(X_b(t), Y_b(t))$, $(X_c(t), Y_c(t))$ denote coordinates of A_{pq}, B_{pq}, C_{pq}, respectively. We show that each of coordinate functions is a rational function of the form $f(t)/g(t)$ with $deg f = 2$, and $deg g = 1$.

For instance, consider A_{pq} (similar reasoning for points B_{pq} and C_{pq}). It is clear that P_b and P_c move linearly, i.e. its coordinates are linear functions of t. The line through the points $P_b(x_1, y_1)$ and $P_c(x_2, y_2)$ is $(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$. It is of the form $\alpha x + \beta y + \gamma = 0$, where α and β are linear in t, while γ is quadratic in t. Finally, intersect $P_b P_c$ with the constant line $F_\ell Q_a$, find the coordinates of A_{pq}.

Now let us prove that the directions of $A_{pq}B_{pq}$, $B_{pq}C_{pq}$, $C_{pq}A_{pq}$ are constant (though these lines do not move linearly). This condition is equivalent to equalities

$$F_\ell A_{pq} = \alpha \cdot F_\ell C_{pq} \quad \text{and} \quad F_\ell B_{pq} = \beta \cdot F_\ell C_{pq},$$

where α and β are some constants.

Let us prove the first equalilty (similarly, for the second one) in coordinates:

$$X_a(t) = \alpha_1 \cdot X_c(t) \qquad Y_a(t) = \alpha_2 \cdot Y_c(t)$$

(where α_1 and α_2 are constants). Recall that each function in these relation has the form $f(t)/g(t)$ with $deg f = 2$, and $deg g = 1$. Multiplying by denominators, we obtain cubic equations in t. Thus it suffices to find 4 different values of t, or, equivalently, 4 different positions of P, for which the condition holds

First, consider cases $P = O$ and $P = Q$. the previous problem implies that $\{A_{pq}, C_{pq}\}$ coincide, and do not coincide with F_ℓ. Defining constants α_1 and α_2 by $\alpha_1 = X_a(0)/X_c(0)$ and $\alpha_2 = Y_a(0)/Y_c(0)$, we have equalities

$$X_a(t) = \alpha_1 \cdot X_c(t) \quad \text{and} \quad Y_a(t) = \alpha_2 \cdot Y_c(t)$$

which are true for two distinct values of t.

Secondly, consider two positions of P that are antipodal points of the the circumcircle Ω of ABC, lying on ℓ. In this case $A_{pq} = C_{pq} = F_\ell$, i.e. their coordinates are 0, and required equalities hold.

Thus we have proved that all lines $A_{pq}C_{pq}$ are parallel, and the same is true for lines $A_{pq}B_{pq}$ and $B_{pq}C_{pq}$. Hence, while P is moving, $A_{pq}B_{pq}C_{pq}$ traces a family of triangles homothetic with center F_ℓ.

Now it remains to show that the triangles formed by limes through A_{pq}, B_{pq}, C_{pq} parallel to R_bR_c, R_cR_a, R_aR_b, respectively, is homothetic to $R_aR_bR_c$ with homothety center F_ℓ. For this purpose, consider the case $P = R$. We have $A_{pq} = A_{rq}$, $B_{pq} = B_{rq}$, and $C_{pq} = C_{rq}$ lying on the lines R_bR_c, R_cR_a, and R_aR_b, respectively. For an arbitrary position of P consider the homothety with center F_ℓ taking A_{rq} to A_{pq}. By the arguments above, this homothety takes B_{rq} to B_{pq}, and it takes C_{rq} to C_{pq}. Hence R_bR_c maps to the line through A_{pq} parallel to R_bR_c. Similarly, for the lines R_cR_a and R_aR_b (see fig. 14).

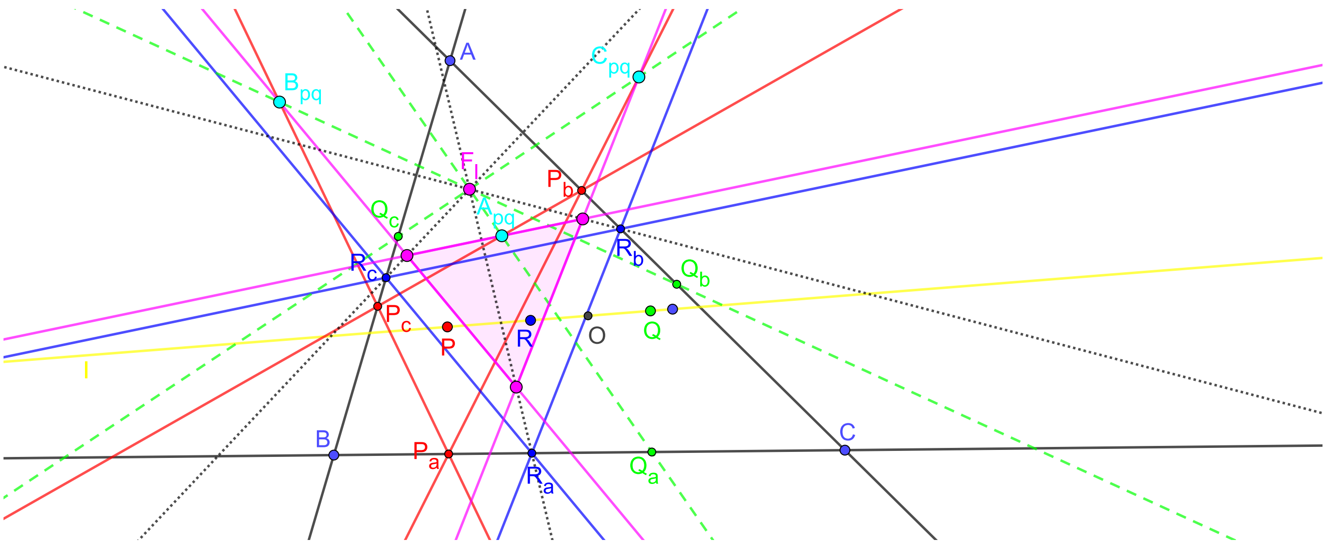

Figure 14.

3.4 The problems **2.5** and **2.8** could be proved without using the main theorem, thus we just mention that the construction in the problem **2.5** is the case $P = I$, $Q = O$ of the main theorem, while the construction in the problem **2.8** is the case $P = Q = I$, $R = O$. Now derive the problem **2.11**. Let the perpendicular to BC through its intersection point with $FA^{\triangle}$ meets OI at Q. The line P_bP_c passes through $A^{\triangle} = A_{iq}$. Applying the mein theorem to I, Q, $R = O$. obtain that P_aP_b and P_aP_c pass through C_{iq} and B_{iq}, respectively. Therefore, $B^{\triangle} = B_{iq}$ and $C^{\triangle} = C_{iq}$. Hence the feet of perpendiculars from Q onto AB and AC are the intersection points of AB and AC with $FC^{\triangle}$ and $FB^{\triangle}$, respectively. The statement about the circle follows from Kulanin's theorem. (see fig. 15, 16, 17).

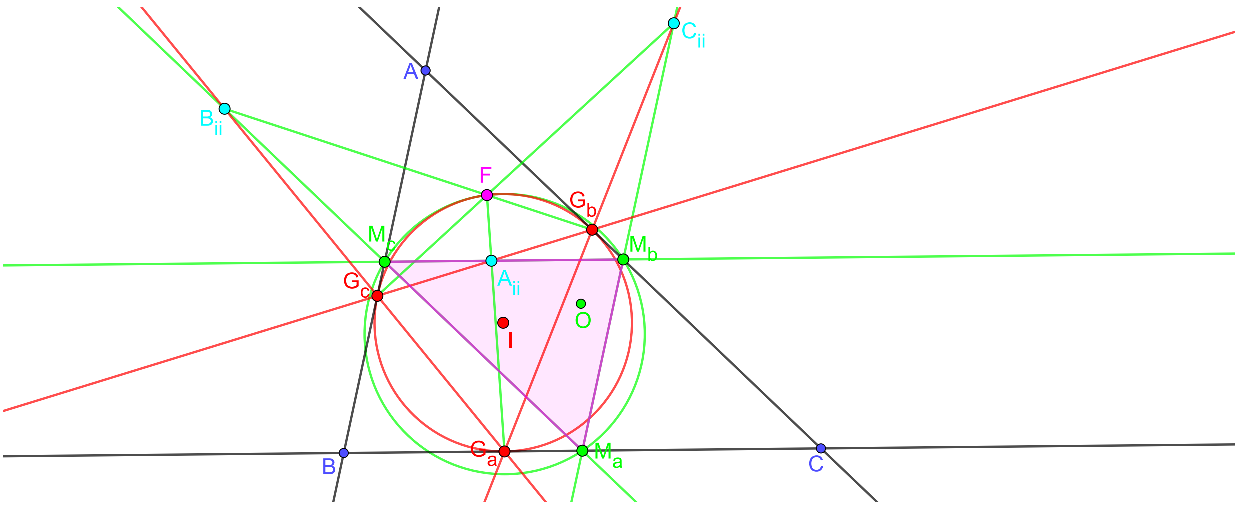

Figure 15. $P = Q = I$**,** $R = O$

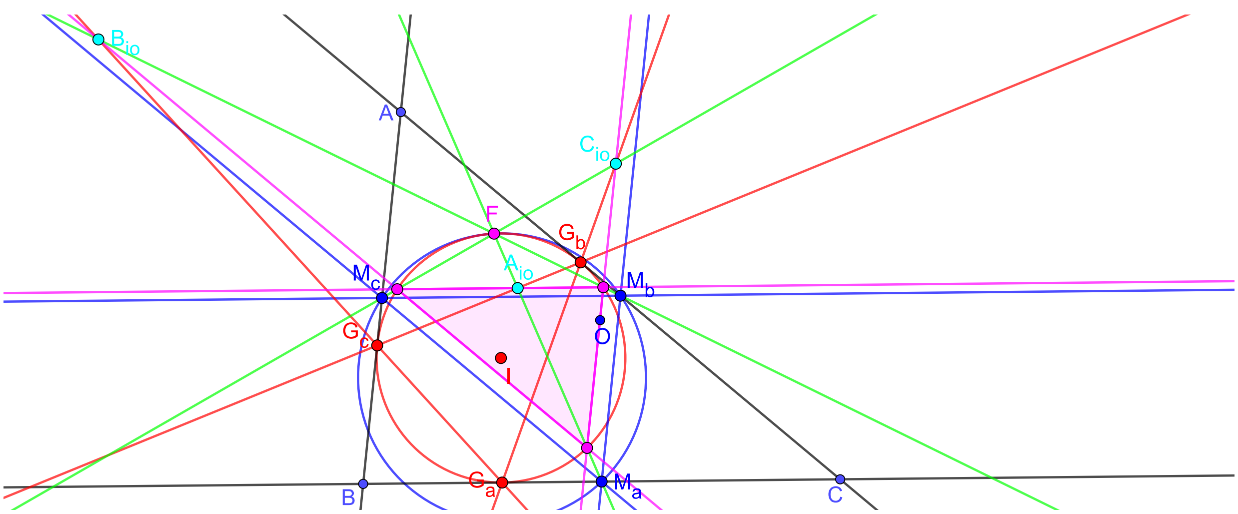

Figure 16. $P = I$**,** $Q = R = O$

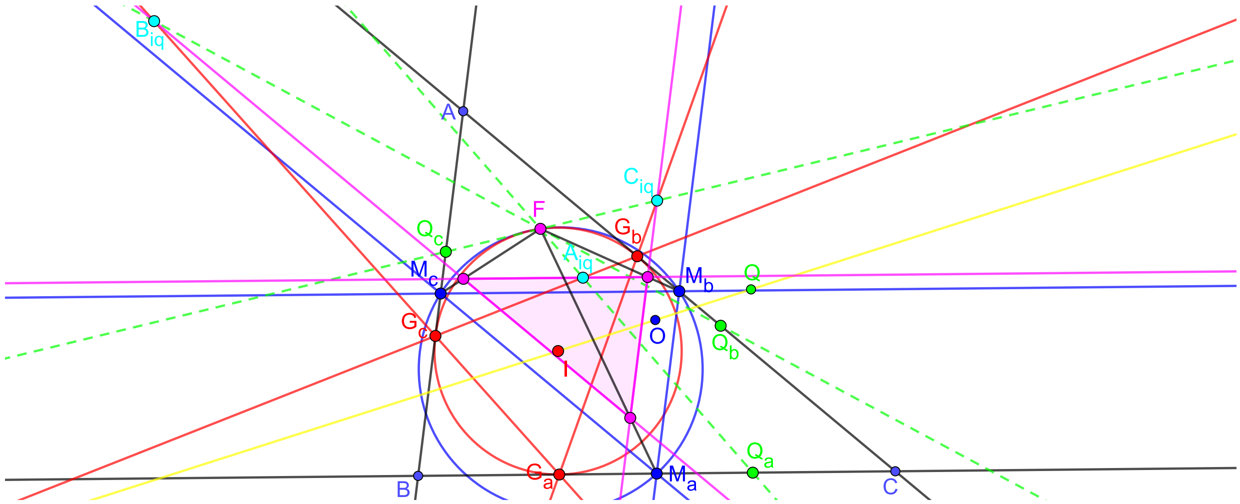

Figure 17. $P = I$**,** $R = O$

3.5 The main theorem is equivalent to the equality $FA_{rq}/FA_{pq} = FB_{rq}/FB_{pq} = FC_{rq}/FC_{pq}$. Similarly, we have $FA_{rp}/FA_{qp} = FB_{rp}/FB_{qp} = FC_{rp}/FC_{qp}$. Let R be a point such that $R_bR_c \parallel A_{pq}A_{qp}$. Then there exists a homothety with center F taking R_bR_c to $A_{pq}A_{qp}$. This homothety takes A_{rq} to A_{pq}, and takes A_{rp} to A_{qp}. From the equalities above it follows that this homothety takes $R_aR_bR_c$ to the triangle formed by $A_{pq}A_{qp}$, $B_{pq}B_{qp}$, $C_{pq}C_{qp}$, and the required statement follows.

3.6 It is known that the lines symmetric to ℓ in the midlines of the triangle intersect at some point X. The point P symmetric to X in M_bM_c lies on ℓ and on the circle (AM_bM_c). (Since the reflections in the sidelines of a line through the orthocenter meet on the circumcircle). Using the problem **3.2** and the Simson line of P with respect to AM_bM_c, we have $PF_\ell \perp BC$. It is clear that $PX \perp BC$. Similarly, for the other reflections of X in the midlines. Hence $F_\ell = X$ (see fig. 18).

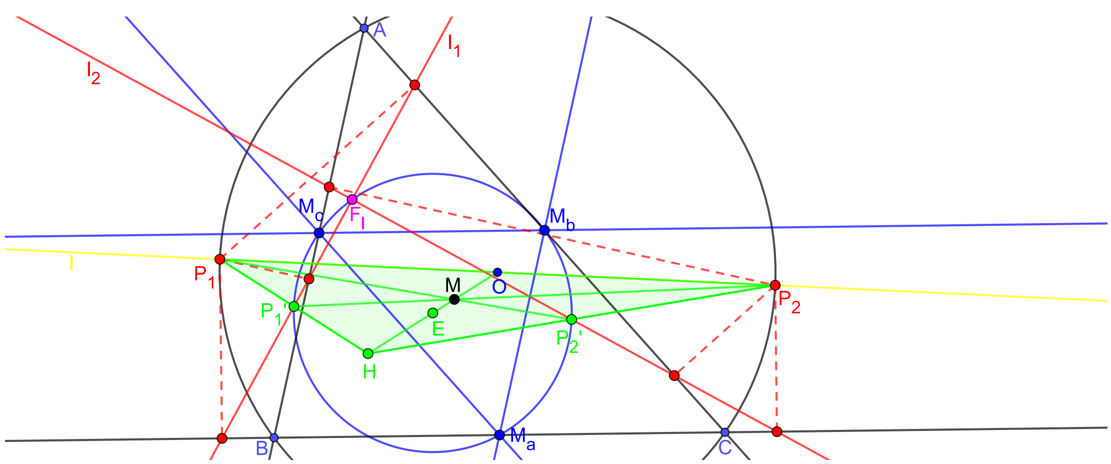

Figure 18.

3.7 If P coincides to one of the points A_ℓ, B_ℓ, C_ℓ, then F_ℓ lies on these circles, by Kulanin's theorem.

Let A', B', C' be reflections of A, B, C in O. By Pascal theorem, $A'A_\ell \cap B'B_\ell$ lies on the circle (ABC). Hence, lines $A'A_\ell$, $B'B_\ell$, $C'C_\ell$ meet at some point F'_ℓ of the circumcircle. As $\angle A_\ell F'_\ell A = \angle A' F'_\ell A = 90^\circ$, the point F'_ℓ lies on the required circles. H lies on $F_\ell F'_\ell$, since the powers of H with respect to these circles are equal to $HH_a \cdot HA = HH_b \cdot HB = HH_c \cdot HC$ (see fig. 19).

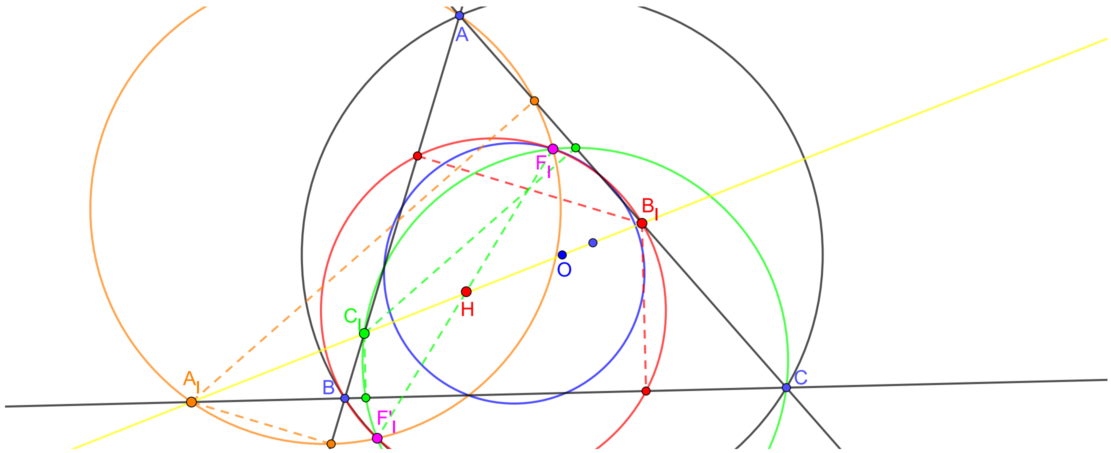

Figure 19.

3.8 Consider the quadrilateral $H_aH_bH_cF_\ell$ inscribed in the Euler circle ε. We set $H_aH_c \cap F_\ell H_b = B_{hh}$, $H_aH_b \cap F_\ell H_c = C_{hh}$, $H_aF_\ell \cap H_bH_c = A_{hh}$. It follws that $A_{hh}B_{hh}C_{hh}$ is an autopolar traingle for ε.

Now let us show that the sidelines of $A_{hh}B_{hh}C_{hh}$ contain A, B, C. Indeed, consider $H_cM_cH_bM_b$ that is a quadrilateral inscribed in ε. By the construction, A lies on the polar line of A_{hh}. But the polar line of A_{hh} is $B_{hh}C_{hh}$, hence A lies on $B_{hh}C_{hh}$. Similarly, for the other vertices (see fig. 20).

Moreover, this problem is a particular case of the next one.

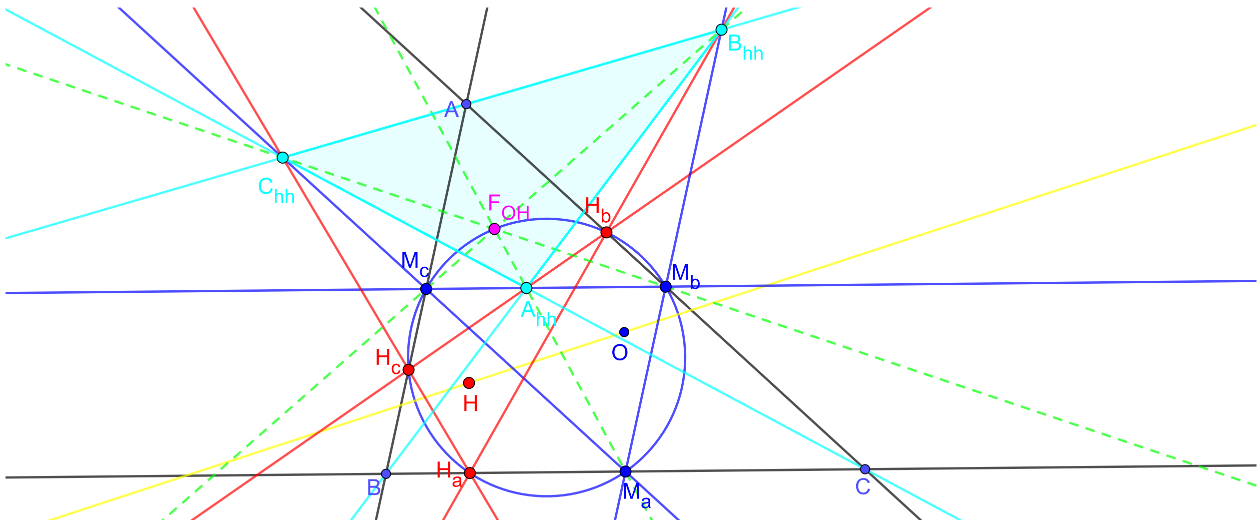

Figure 20.

3.9 a) the cross-ratio of the lines $P_aB, P_aP_c, P_aA, P_aP_b$ equals -1. Projecting otno the line AB_{pq}, we get the harmonic quadruple $AB_{pq} \cap BC$, B_{pq}, A, $AB_{pq} \cap P_aP_b$. Similarly, $AB_{pq} \cap BC$, B_{pq}, A, $AB_{pq} \cap Q_aQ_b$ is a harmonic quadruple. Hence AB_{pq}, P_aP_b, Q_aQ_b are collinear, i.e. A lies on $B_{pq}C_{pq}$.

b,c) By Pappus theorem, $P_bQ_c \cap P_cQ_b$ lies on PQ. Similarly, for $P_aQ_c \cap P_cQ_a$ and $P_aQ_b \cap P_bQ_a$. By Pascal theorem, $P_aQ_bP_cQ_aP_bQ_c$ is inscribed into a conic Ω. Let $F = B_{pq}P_b \cap C_{pq}P_c$. by Pascal theorem and the item a), $FP_bQ_bQ_aQ_cP_c$ is inscribed into Ω. Analogous statement is true for $A_{pq}P_a \cap C_{pq}P_c$. it follows that $A_{pq}P_a$, $B_{pq}P_b$, $C_{pq}P_c$ meet on Ω, and c) follows. Now be could be proved similarly to the previous problem.

3.10 Let X_a move along BC linearly. Then X_b and X_c move along AC and AB, respectively, preserving cross-ratio.

- It suffices to prove for 3 positions of X_a. Cases $X_a = B, C, M_a$ follow from the previous problem;

- $X = BX_b \cap CX_c$ moves along some conic passing through B and C preserving cross-ratio (the proof is analogous to one for the problem **0.11**). This conic also passes through A, since $X = A$, for $X_a = BC \cap B_{hh}C_{hh}$. Thus it suffices to prove that AX passes through X_a, for some 3 positions of X_a. The first position is $X_a = X = B$, the second is $X_a = X = C$, the third is $X_a = M_a$, $X = M$ (and the fourth is $X_a = H_a$, $X = H$).

- X moves along a rectangular hyperbola $ABCHM$. It is known that the circle $(X_aX_bX_c)$ passes through the center of this hyperbola (we do not know an easy proof of this fact; see A. Akopyan, A.Zaslavsky "Geometrical properties of conics", Theorem 4.3). The center of this hyperbola is

F_{OL}, that is the generalized Feuerbach point for the isogonal conjugate on this hyperbola (see fig. 21).

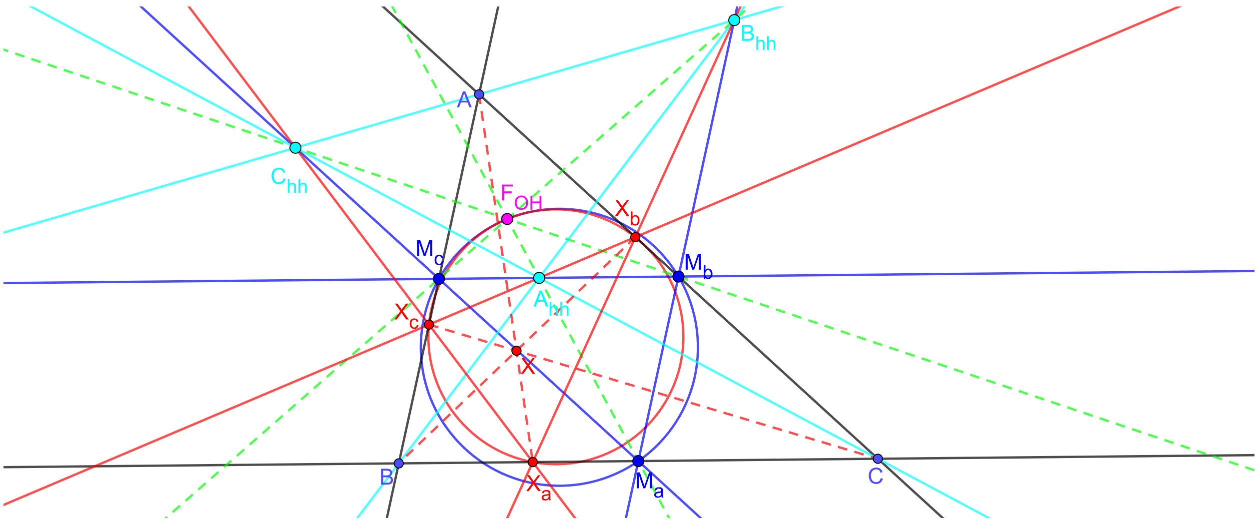

Figure 21.

3.11 We present a sketch of the proof that all traingles $Z_aZ_bZ_c$ are homothetic (though we do not specify the center of homothety).

We need to prove that $BC \parallel Z_bZ_c \parallel A_{pp}A_{qq}$ (here we use the problem 3.2: $A_{pp}A_{qq}$ is the midline). Reduce it to a statement of projective geometry:

Assume that traingles $P_aP_bP_c$ and $Q_aQ_bQ_c$ are given, and let F be an arbitrary point. Let $A_{pp} = P_bP_c \cap FP_a$, while $A_{pq}, B_{qp}, \ldots$ are defined similarly, like in the problem 3.3. Let $Z_b = A_{pq}C_{pq} \cap A_{qp}C_{qp}$ and $Z_c = A_{pq}B_{pq} \cap A_{qp}B_{qp}$. Hence P_aP_b, $A_{pp}A_{qq}$, and Z_bZ_c are concurrent.

Note that cross-ratios of quadruples $A_{qp}Q_c$, $A_{qp}B_{qp}$, $A_{qp}F$, $A_{qp}Q_a$ and $A_{pq}P_b$, $A_{pq}C_{pq}$, $A_{pq}F$, $A_{pq}P_a$ are equal (project with center A_{qp} onto Q_cQ_a, then project with center F onto P_bP_a, then with center A_{pq}. Similarly, cross-ratios of quadruples $A_{qp}Q_c$, $A_{qp}C_{qp}$, $A_{qp}F$, $A_{qp}Q_a$ and $A_{pq}P_b$, $A_{pq}B_{pq}$, $A_{pq}F$, $A_{pq}P_a$ are equal. Now one can apply the following fact:

Let f be a map $l \mapsto f(l)$ from the band of lines through A_{qp} to the band of lines through A_{pq} that preserves cross-ratio. Let m and n be arbitrary lines through A_{qp}. Consider the line through $m \cap f(n)$ and $n \cap f(m)$. Then this line passes through a fixed point, regardless of a choice of m and n.

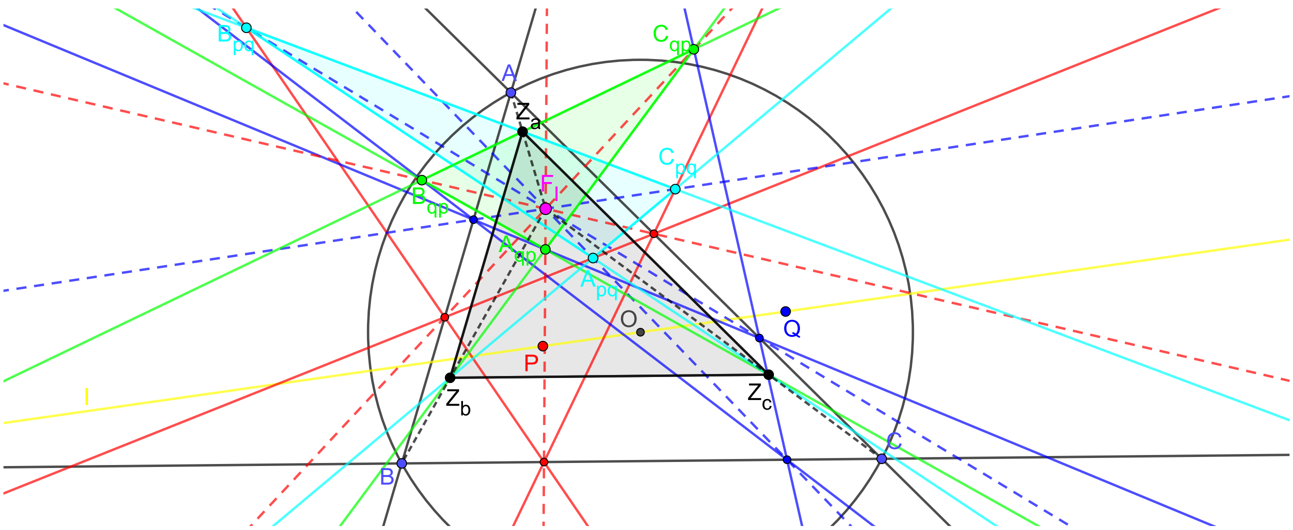

Figure 22.