# A user's guide to knot and link theory *

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**Abstract.**

We define simple invariants of knots or links (linking number, Arf-Casson invariants and Alexander-Conway polynomials) motivated by interesting results whose statements are accessible to a non-specialist or a student (e.g. Theorems 1.1.3 and 1.2.2). We show how the simplest invariants naturally appear in an attempt to unknot a knot or unlink a link. Then we present certain 'skein' recursive relations for the simplest invariants, which allow to introduce stronger invariants. We state the Vassiliev-Kontsevich theorem in a way convenient for calculating the invariants themselves, not only the dimension of the space of the invariants. We also present coloring invariants although we cannot explain in an elementary way how they appear.

We give rigorous definitions of the main notions in a way not obstructing intuitive understanding. No prerequisites are required for this text.

**Recommendations for participants.**

If a mathematical statement is formulated as a problem, then the objective is to prove this statement. If such a problem is labeled 'theorem' ('lemma', 'corollary', etc.), then this statement is considered more important. Usually we formulate as a problem a beautiful or important statement before giving a sequence of problems which constitute its proof. In this case, in order to prove this statement, one may need to solve some of the subsequent problems. We do not want to deprive you of the pleasure of finding the right moment when you finally are ready to prove the statement. In general, if you are stuck on a certain problem, try looking at the next ones. They may turn out to be helpful. *Remarks* are formally not used later.

When an important notion is defined, it is marked in **bold letters** so that it would be easier to find its definition later. We suggest to all students working on the project to *consult* the jury on any questions on the project. Students who successfully work or complete a part of the project will get interesting *extra problems (subsections marked by stars; they are not yet elaborated).*

For every solution which has been written down and marked with either '+' or '+.' a student (or a group of students) gets a 'bean'. The jury may also award extra beans for beautiful solutions, solutions of hard problems, or solutions typeset in TEX. The jury has infinitely many beans. One may submit a solution in oral form, but one loses a bean with each 5 attempts (successful or not).

Please notify us if you already know solutions of some of the problems. If you confirm your knowledge by presenting some of them, then you will be allowed to use them in solutions of other problems, without earning before a plus-mark (plus-sign) for their solutions.

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# 1 Problems before the semifinal

## 1.1 Main definitions and results on knots

We start with informal description of the main notions (rigorous definitions are given after Problem 1.1.1). You can imagine a *knot* as a thin elastic string whose ends have been glued together, see fig. 1. As in this figure, knots are usually represented by their 'nice' plane projections called *knot diagrams*. Imagine laying down the rope on a table and carefully recording how it crosses itself (i.e. which part lies on top of the other). It should be kept in mind that the projections of the same knot on different planes can look quite dissimilar.

A **trivial knot** is the outline (the boundary) of a triangle.

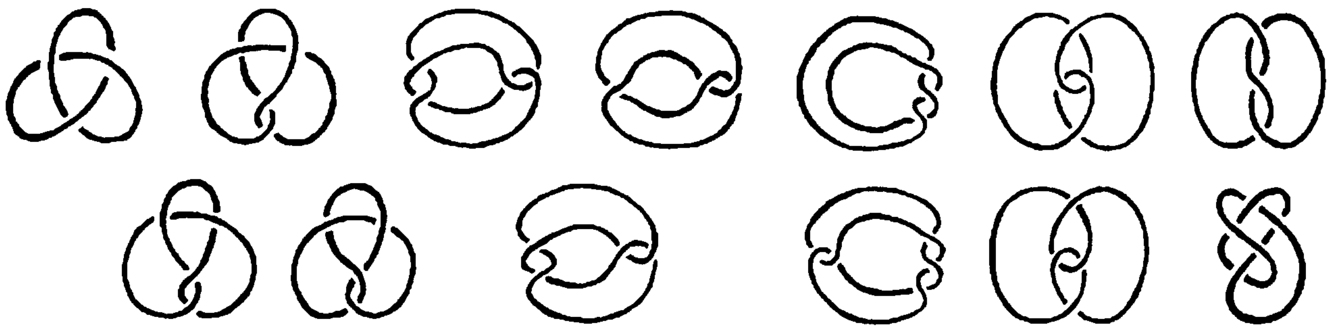

Figure 1: Knots isotopic to the trefoil knot (top row) and to the figure eight knot (bottom row)

By an *isotopy* of a knot we mean its continuous deformation in space as a thin elastic string; no self-intersections are allowed throughout the deformation. Two knots are *isotopic* if one can be transformed to the other by an isotopy. As a proof that specific knots are isotopic we accept a big clear picture, or an experiment with rope that can be repeated by a Jury member.

**Problem 1.1.1.** (a) Some two knots represented in the top row of fig. 1 are isotopic to the leftmost knot in this row. For one of these two knots decompose your isotopy into Reidemeister moves shown in fig. 5.

(b)* All the knots represented in the top row of fig. 1 are isotopic to each other.

(c,d*) The same is true for the knots represented in the bottom row of fig. 1.

(e) All knots with the same knot diagram are isotopic.

**Remark.** Here we justify the necessity of a rigorous definition of isotopy.

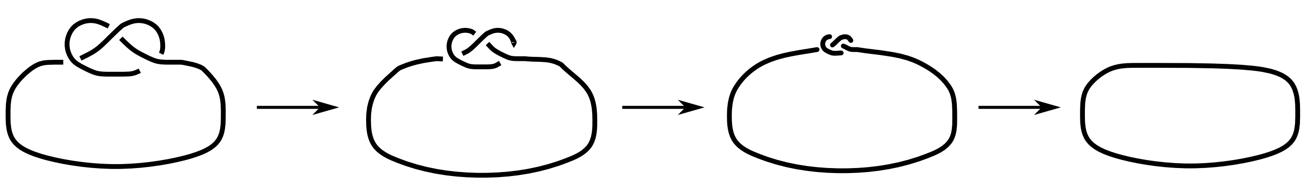

Figure 2: A (non-ambient) isotopy between the trefoil knot and the trivial knot

In fig. 2 we see an isotopy between the trefoil knot and the trivial knot. Is it indeed an isotopy? This is the so called 'piecewise linear non-ambient isotopy' which is *different* from the 'piecewise linear ambient isotopy' defined and used later. (The first notion better reflects the idea of continuous deformation without self-intersections, but is hardly accessible to high school students, cf. [Sk16i].) In fact, any two knots are piecewise linear non-ambient isotopic!

The usual problem with intuitive definitions is not that it is hard to make them rigorous, but that this can be done in several ways.

A **knot** is a spatial closed non-self-intersecting polygonal line.

A **plane diagram** of a knot is its generic[1] projection onto a plane[2], together with the information which part of the knot 'goes under' and which part 'goes over' at any given crossing.

**Problem 1.1.2.** For any knot diagram there is a knot projected to this diagram. (Such a knot need not be unique; see though problem 1.1.1.e.)

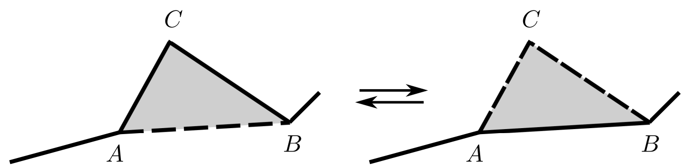


Figure 3: Elementary move

Suppose that two sides $AC$ and $CB$ of a triangle $ABC$ are edges of a knot. Moreover, assume that the knot and (the part of the plane bounded by) the triangle $ABC$ do not intersect at any other points. An **elementary move** $ACB \to AB$ is the replacement of the two edges $AC$ and $CB$ by the edge $AB$, or the inverse operation $AB \to ACB$ (fig. 3).[3] Two knots $K, L$ are called (piecewise linearly ambiently) **isotopic** if there is a sequence of knots $K_1, \dots, K_n$ such that $K_1 = K$, $K_n = L$ and every subsequent knot $K_{j+1}$ is obtained from the previous one $K_j$ by an elementary move.

**Theorem 1.1.3.** *(a) The trivial knot is not isotopic to the trefoil knot.*
*(b) The trivial knot is not isotopic to the figure eight knot.*
*(c) The trefoil knot is not isotopic to the figure eight knot.*
*(d) There is an infinite number of pairwise non-isotopic knots.*

This is proved using *Arf* and *Casson invariants*, see §1.4 and §2.1, or using *proper colorings,* see §2.2 (so you need not spend much time on proving this result right now).

**Remark.** This remark might be useful as both a hint and a warning to problems 1.1.4 and 1.1.5.

In the following paragraph we prove that *if a knot lies in a plane, then the knot is isotopic to the trivial knot.*

Denote the knot in a plane by $M_1M_2 \dots M_n$. Take a point $Z$ outside the plane. Then $M_1M_2 \dots M_n$ is transformed to the trivial knot $M_1ZM_n$ by the following sequence of elementary moves:

$$M_1M_2 \to M_1ZM_2, \quad ZM_2M_3 \to ZM_3, \quad ZM_3M_4 \to ZM_4, \quad \dots, \quad ZM_{n-1}M_n \to ZM_n.$$

The following result shows that intermediate knots of an isotopy from a knot lying in a plane to the trivial knot can be chosen also to lie in this plane.

*Schoenflies Theorem.* Any closed polygonal line without self-intersections in the plane is isotopic (in the plane) to a triangle.

This is a stronger version of the following celebrated result.

*Jordan Theorem.* Every closed non-self-intersecting polygonal line $L$ in the plane $\mathbb{R}^2$ splits the plane into exactly two parts, i.e. $\mathbb{R}^2 - L$ is not connected and is a union of two connected sets.

A subset of the plane is called *connected*, if every two points of this subset can be connected by a polygonal line lying in this subset.

[1]A polygonal line in the plane is *generic* if there is a polygonal line $L$ with the same union of edges such that no three vertices of $L$ belong to any line and no three segments joining some vertices of $L$ have a common interior point.

[2]A university-mathematics terminology is 'a generic image under projection onto a plane'.

[3]If the triangle $ABC$ is degenerate, then elementary move is either subdivision of an edge or inverse operation.

For an algorithmic explanation why the Jordan Theorem (and so the Schoenflies Theorem) is non-trivial, and for a proof of the Jordan Theorem, see §1.3 'Intersection number for polygonal lines in the plane' of [Sk18], [Sk].

**Problem 1.1.4.** Suppose that there is a point on a knot such that if we go around the knot starting from this point, then on some plane diagram we first meet only overcrossings, and then only undercrossings. Then the knot is isotopic to the trivial knot.[4]

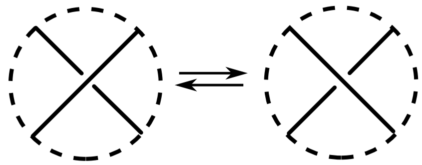

Figure 4: Crossing change

A **crossing change** is change of overcrossing to undercrossing or vise versa, see fig. 4.

Clearly, after any crossing change on the diagrams of the trefoil knot and the figure eight shown in fig. 1 we obtain a diagram of a knot isotopic to the trivial knot.

**Lemma 1.1.5.** Every plane diagram of a knot can be transformed by crossing changes to a plane diagram of a knot isotopic to the trivial knot.[5]

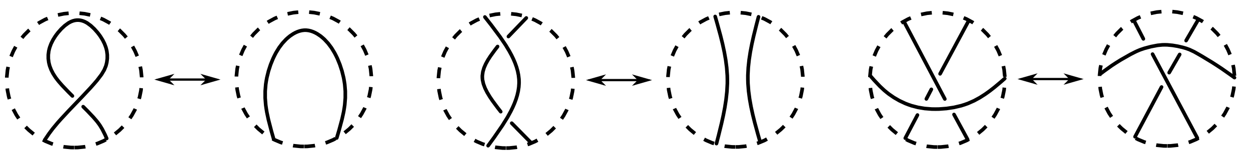

Figure 5: Reidemeister moves.
The plane diagrams are identical outside the disks bounded by dashed circles. No other sides of the plane diagrams except for the pictured ones intersect the disks. (Same for fig. 6, 4, 9 and 10.)

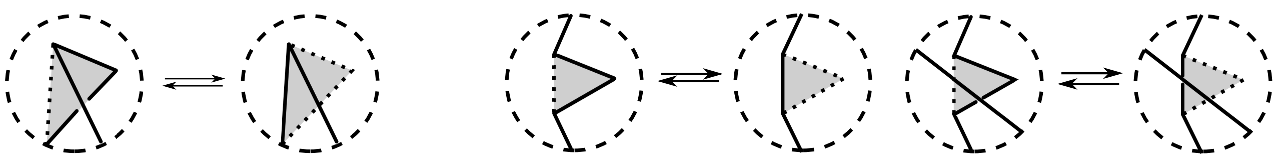

Figure 6: (Left) To a rigorous definition of the first Reidemeister move
(Middle, right) Plane isotopy moves

In this text instead of knots up to isotopy we shall study plane diagrams of knots up to (equivalence generated by) **Reidemeister moves** shown in fig. 5[6] and *plane isotopy moves* shown in fig. 6 (middle, right). I.e. we shall use without proof the following result.

**Theorem 1.1.6** (Reidemeister)**.** * *Two knots are isotopic if and only if some plane diagram of the first knot can be obtained from some plane diagram of the second one by Reidemeister moves and plane isotopy moves.*

[4]This problem would be a motivation for introduction of the Arf invariant (§1.4). The proof illustrates in low dimensions one of the main ideas of the celebrated Zeeman's proof of the higher-dimensional Unknotting Spheres Theorem, see survey [Sk16c, Theorem 2.3].

[5]This simple lemma will be used for inductive construction of invariants using skein relations, see below.

[6]A rigorous definition of the first Reidemeister move is easily given using fig. 6 (left). The other Reidemeister moves have analogous rigorous definitions. The participants are not required to use these rigorous definitions in solutions. You can use informal description of Reidemeister moves in fig. 5 and so ignore plane isotopy moves.

## 1.2 Main definitions and results on links

A **link** is a collection of pairwise disjoint knots, which are called the *components* of the link. Ordered collections are called ordered or colored links, while non-ordered collections are called non-ordered or non-colored links. In this text we abbreviate ‘ordered link’ to just ‘link’.

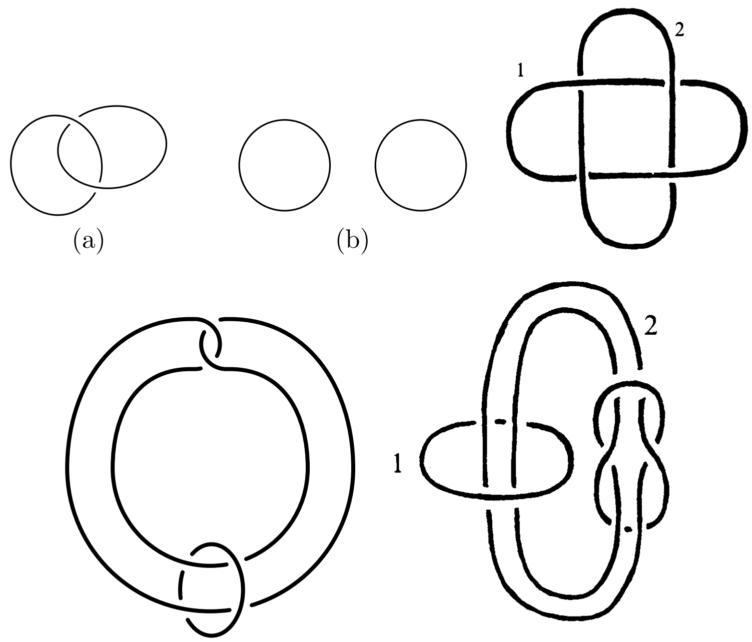


Figure 7: The Hopf link, the trivial link and another three links

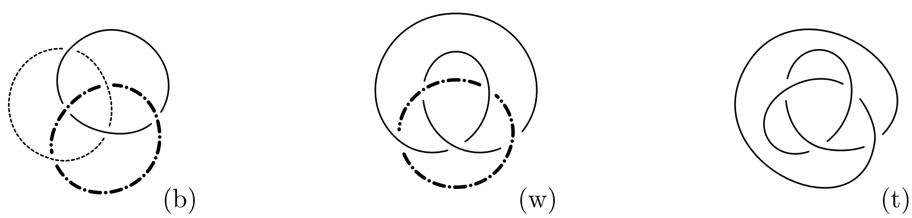


Figure 8: The Borromean rings, the Whitehead link and the trefoil knot

A **trivial link** (with any number of components) is a link formed by triangles in parallel planes.

Plane diagram, isotopy, etc. for links are defined analogously to knots.

The analogues of Lemmas 1.1.6 and 1.1.5 for links are correct.

**Problem 1.2.1.** (a) The Hopf link is isotopic to the link obtained from the Hopf link by switching the components.

(b) The Hopf link is isotopic to some link whose components are symmetric with respect to some straight line.

(c) The fourth link in fig. 7 is isotopic to the Whitehead link in fig. 8.w.

(d,e*) The same as in (a,b) for the Whitehead link.

(f)* The Borromean rings link is isotopic to a link whose components are permuted in a cyclic way under the rotation by angle $2\pi/3$ with respect to some straight line.

**Theorem 1.2.2.** *(a) The Hopf link is not isotopic to the trivial link.*

*(b) The Whitehead link is not isotopic to the trivial link.*

*(c) The Hopf link is not isotopic to the Whitehead link.*

*(d) The Borromean rings link is not isotopic to the trivial link.*

Parts (a) and (c) are proved using *linking number modulo 2*, invent it yourself or see §1.3. Parts (b) and (d) are proved using either *the Alexander-Conway polynomials*, see §2.3, or 'triple linking' (Massey-Milnor) number and 'higher linking' (Sato-Levine) number [Sk, §4.4-§4.6]. Part (d) can also be proved using *proper colorings*, see §2.2.

## 1.3 The Gauss linking number modulo 2 via plane diagrams

**Problem 1.3.1.** Let $A, B, C, D, E, F, O$ be points in space, no four of which lie in one plane. The following three conditions are equivalent.

(i) The outline of $DEF$ intersects the part of the plane $ABC$ bounded by the triangle $ABC$ at exactly one point.

(ii) The segment $BC$ passes below (like in the construction of link diagram) exactly one side of $DEF$ as seen from $A$.

(iii) The outline of $ABC$ passes below an odd number of sides of $DEF$ as seen from $O$.

Suppose that there is an isotopy between two 2-component links, and the second component is fixed throughout the isotopy. Then the trace of the first component is a self-intersecting cylinder disjoint from the second component. If after the isotopy the components are unlinked, then the cylinder can be completed to a self-intersecting disk disjoint from the second component. This observation, together with problem 1.3.1 and [Sk, the Projection lemma 4.2.4], motivates the following definition.

The **linking number modulo 2** $\mathrm{lk}_2$ of the plane diagram of a 2-component link is the number modulo 2 of crossing points on the diagram at which the first component passes above the second component.

**Problem 1.3.2.** (a) Find the linking number modulo 2 for the plane diagrams in fig. 7, for pairs of Borromean rings and for the Whitehead link (fig. 8).

(b) The linking number modulo 2 is preserved under Reidemeister moves.

By (b) the **linking number modulo 2** of a 2-component link (or even of its isotopy class) is well-defined by setting it to be the linking number modulo 2 of any plane diagram of the link.

We shall use without proof the following *Parity lemma:* any two closed polygonal lines in the plane whose vertices are in general position intersect at an even number of points. For a discussion and a proof see §1.3 'Intersection number for polygonal lines in the plane' of [Sk18], [Sk].

**Problem 1.3.3.** (a) Switching the components of a link preserves the linking number modulo 2.

(b) There is a 2-component link which is not isotopic to the trivial link but which has zero linking number modulo 2.

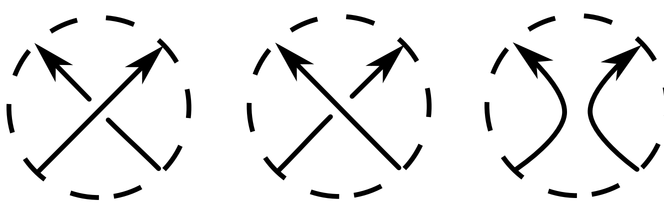

Figure 9: Knots $K_+, K_-, K_0$

Denote by $D_+, D_-, D_0$ any three diagrams of oriented (knots or) links differing as shown in fig. 9 (for a convention on figures see caption to fig. 5). We also denote by $K_+, K_-, K_0$ any three links who have diagrams $D_+, D_-, D_0$. If an invariant (like $\mathrm{lk}_2$) is defined for non-oriented links (or knots), then its value on a link is assigned to the link with any orientation.

**Theorem 1.3.4.** *There is a unique mod2-valued isotopy invariant* $\mathrm{lk}_2$ *of (non-oriented) 2-component links that assumes value 0 on the trivial link and such that (for any links* $K_+$ *and* $K_-$ *having plane diagrams differing as shown in fig. 9)*

$$\mathrm{lk}_2 K_+ - \mathrm{lk}_2 K_- = \begin{cases} 1 & \textit{if at the crossing point different components cross each other;} \\ 0 & \textit{if at the crossing point one component crosses itself.} \end{cases}$$

**Problem 1.3.5.** * If the linking number modulo 2 of two (disjoint outlines of) triangles in space is zero, then the link formed by the triangles is isotopic to the trivial link.

**Theorem 1.3.6** (Conway–Gordon–Sachs)**.** * *If no 4 of 6 points in 3-space lie in the same plane, then there are two linked triangles with vertices at these 6 points. That is, the part of the plane bounded by the first triangle intersects the outline of the second triangle exactly at one point.*

## 1.4 The Arf invariant

Take a plane diagram of a knot and a point $P$ on the diagram different from crossing points. Call $P$ a *basepoint*. A non-ordered pair of crossing points $A$ and $B$ is called **skew** (or $P$-skew) if going around the diagram in some direction starting from $P$ and marking only crossings at $A$ and $B$, we first mark overcrossing at $A$, then undercrossing at $B$, then undercrossing at $A$, and at last overcrossing at $B$.

The $P$-*Arf invariant* $\mathrm{arf}_P$ of the plane diagram is the parity of the number of all skew pairs of crossing points.

**Problem 1.4.1.** (a) If the $P$-Arf invariant of a plane diagram is non-zero, then $P$ is not a point as in problem 1.1.4.

(b,c,d) Find the $P$-Arf invariant (of some plane diagram) of the trivial, the trefoil and the figure eight knots (for your choice of a basepoint $P$).

(e) The $P$-Arf invariant is independent of the choice of a basepoint $P$.

By (e) the *Arf invariant* of a plane diagram is well-defined by setting it to be the $P$-Arf invariant for any basepoint $P$.

(f) The Arf invariant of a plane diagram is preserved under Reidemeister moves.

By (f) the **Arf invariant** (Arf number) arf of a knot (or even of isotopy class of a knot) is well-defined by setting it to be the Arf invariant of any plane diagram of the knot.

**Problem 1.4.2.** (a) If in fig. 9 $K_+, K_-$ are plane diagrams of knots, then $K_0$ is a plane diagram of 2-component link and $\mathrm{arf}\, K_+ - \mathrm{arf}\, K_- = \mathrm{lk}_2\, K_0$.

(b) There is a knot which is not isotopic to the trivial knot but which has zero Arf invariant.

**Theorem 1.4.3.** *There is a unique mod2-valued isotopy invariant* arf *of (non-oriented) knots that assumes value 0 on the trivial knot and such that*

$$\mathrm{arf}\, K_+ - \mathrm{arf}\, K_- = \mathrm{lk}_2 K_0.$$

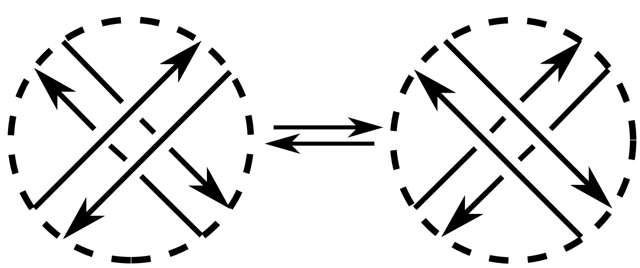

Figure 10: Pass move

**Problem 1.4.4.** Two knots are called *pass equivalent* if some plane diagram of the first knot (with some orientation) can be transformed to some plane diagram of the second knot (with some orientation) using Reidemeister moves and *pass moves* of fig. 10.

(a) If two knots are pass equivalent, then their Arf invariants are equal.

(b)* The eight figure knot is pass equivalent to the trefoil knot.

(c)* If the Arf invariants of two knots are equal, then the knots are pass equivalent.

**Theorem 1.4.5.** * *Take any 7 points in space, no four of which belong to any plane. Take* $\binom{7}{2} = 21$ *segments joining them. Then there is a closed polygonal line formed by taken segments and non-isotopic to the boundary of a triangle.*

## 1.5 Oriented knots and links

You know what is oriented polygonal line, so you know what is oriented knot (fig. 11).

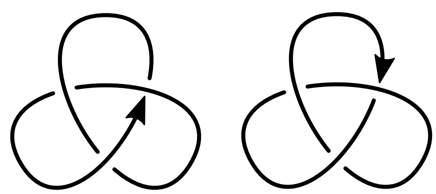

Figure 11: Two trefoil knots with the opposite orientations

Both the informal notion and rigorous definition of *isotopic* oriented knots are given analogously to isotopic knots.

**Problem 1.5.1.** Isotopic oriented polygonal lines without self-intersections on the plane and on the sphere are defined analogously to isotopic oriented knots in space.

(a) An oriented spherical triangle is isotopic on the sphere to the same triangle with the opposite orientation.

(b) The analogue of (a) for the plane is false.

**Problem 1.5.2.** (a) Two trivial; (b) Two trefoil; (c) Two figure eight
knots with the opposite orientations are isotopic.

**Theorem 1.5.3.** (H. Trotter, 1964) *There exists an oriented knot which is not isotopic to the same knot with the opposite orientation.*

This is proved using *the Jones poynomial* [PS96], [CDM12]; the proof is outside the scope of this text.

The **connected sum** # of *oriented* knots is defined in fig. 12.[7]

This is not a well-defined operation on oriented knots. So we denote by $K\#L$ any of the connected sums of $K$ and $L$.

**Problem 1.5.4.** For any oriented knots $K, L, M$ and the trivial oriented knot $O$ we have

(a) $K\#O = K$. (b) $K\#L = L\#K$. (c) $(K\#L)\#M = K\#(L\#M)$.

(d) $\mathrm{arf}(K\#L) = \mathrm{arf}\,K + \mathrm{arf}\,L$ (here knots $K, L$ are non-oriented).

(The rigorous meaning of (a) is 'there is a connected sum of $K$ and $O$ isotopic to $K$'. Analogous rigorous meanings have (b) and (c). See though Remark below.)

[7]More precisely, consider disjoint oriented plane diagrams of the two oriented knots. Find a rectangle in the plane where one pair of sides are edges of each knot, but the rectangle is otherwise disjoint from the knots, and the edges are oriented around the outline of the rectangle in the same direction. Now join the two diagrams together by deleting these edges from the knots and adding the edges that form the other pair of sides of the rectangle. The resulting connected sum diagram inherits an orientation consistent with the orientations of the two original diagrams.

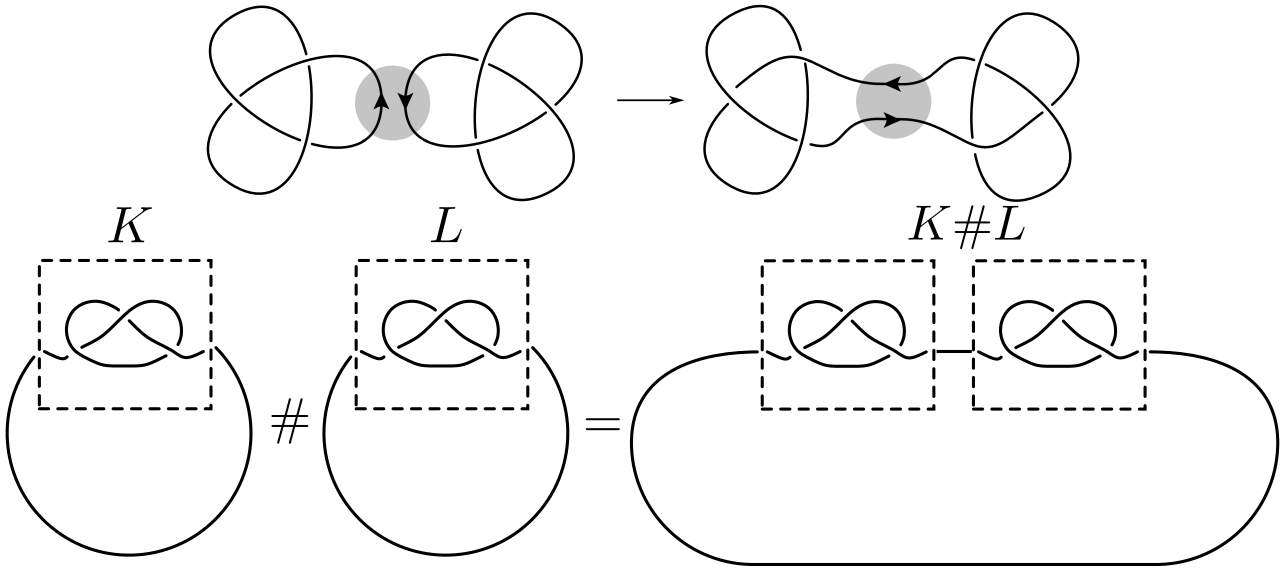


Figure 12: Connected sum of knots

**Remark.** An *isotopy class* of a knot is the set of knots isotopic to this knot. The oriented isotopy class $[K\#L]$ of the connected sum of two oriented isotopy classes $[K],[L]$ of oriented knots $K,L$ is independent of the choices used in the construction, and of the representatives $K,L$ of $[K],[L]$. Hence the connected sum of oriented isotopy classes of oriented knots is well-defined by $[K]\#[L]:=[K\#L]$, see [Sk15p, Remark 2.3.a]. For isotopy classes of non-oriented knots the connected sum is not well-defined.

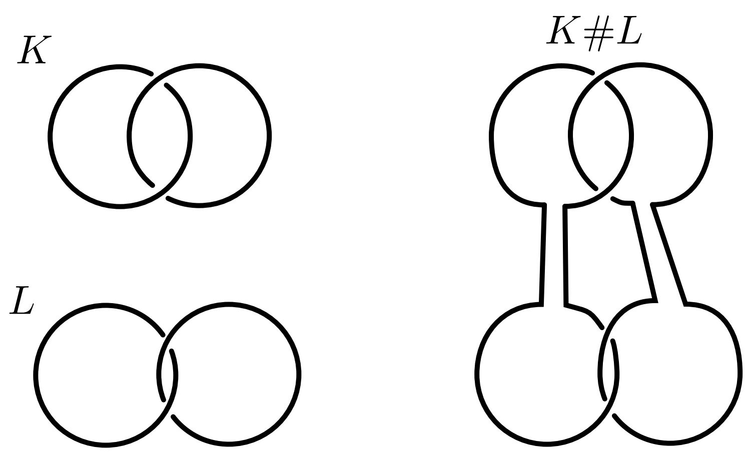


Figure 13: Connected sum of links

The connected sum # of links (ordered or not, oriented or not) is defined analogously to the connected sum of knots, see fig. 13. This is not a well-defined operation on links, and problem 1.5.6 shows that this does not give a well-defined operation on their isotopy classes. So we denote by $K\#L$ any of the connected sums of $K$ and $L$.

**Problem 1.5.5.** (a,b,c,d) Prove the analogues of problem 1.5.4.a,b,c,d for links.

**Problem 1.5.6.** There are two isotopic pairs $(K,L)$ and $(K',L')$ of

(a) non-ordered; (b)* ordered

2-component links (oriented or not) such that some connected sums $K\#L$ and $K'\#L'$ are not isotopic.

## 1.6 The Gauss linking number via plane diagrams

Let $(\overrightarrow{AB}, \overrightarrow{CD})$ be ordered pair of vectors (oriented segments) in the plane intersecting at a point $P$. Define **the sign** of $P$ to be $+1$ if $ABC$ is oriented clockwise and to be $-1$ otherwise (fig. 14).

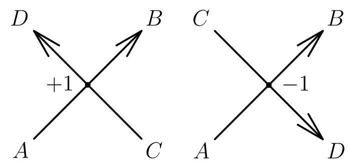


Figure 14: The sign of intersection point

The **linking number** lk of the plane diagram of an oriented 2-component link is the sum of signs of all those crossing points on the diagram at which the first component passes above the second component. At every crossing point the *first* (the *second*) vector is the oriented edge of the first (the second) component.

**Problem 1.6.1.** (a,b) Find the linking number for (some plane diagram of) the Hopf link and pairs of Borromean rings, for your choice of orientation on the components.

(c) The linking number is preserved under Reidemeister moves.

By (c) the **linking number** of an oriented 2-component link (or of its isotopy class) is well-defined by setting it to be the linking number of any plane diagram of the link.

The *absolute value of the linking number* of a (non-oriented) 2-component link (or of its isotopy class) is well-defined by taking any orientations on the components.

We shall use without proof the following *Triviality lemma:* for any two closed oriented polygonal lines in the plane whose vertices are in general position the sum of signs of their intersection points is zero. For a discussion and a proof see §1.3 'Intersection number for polygonal lines in the plane' of [Sk18], [Sk].

**Problem 1.6.2.** (a) Does switching the components of a link preserve or negate the linking number?

(b) Reversing the orientation of either of the components negates the linking number.

(c) Draw an oriented 2-component link whose linking number is $-5$.

(d) For any of the connected sums $K\#L$ of oriented 2-component links $K, L$ we have $\mathrm{lk}(K\#L) = \mathrm{lk}\,K + \mathrm{lk}\,L$.

(e) There is a 2-component link which is not isotopic to the trivial link but which has zero linking number.

**Theorem 1.6.3.** *There is a unique integer-valued isotopy invariant* lk *of oriented 2-component links that assumes value 0 on the trivial link and and such that*

$$\mathrm{lk}\,K_+ - \mathrm{lk}\,K_- = \begin{cases} 1 & \textit{if at the crossing point different components cross each other;} \\ 0 & \textit{if at the crossing point one component crosses itself.} \end{cases}$$

**Problem 1.6.4.** Two links (oriented or not) are called *link homotopic* if one link can be continuously deformed to the other so that no intersections of different components appear throughout the deformation (but self-intersections of the components may appear). Or, equivalently, some plane diagram of the first link can be transformed to some plane diagram of the second link using Reidemeister moves and crossing changes of one component.

(a) The Whitehead link is link homotopic to the trivial link.

(b) If two oriented 2-component links are link homotopic, then their linking numbers are equal.

(c)* If the linking numbers of two 2-component links are equal, then the links are link homotopic.

(d)* The Borromean rings link is not link homotopic to the trivial link.

# 2 Problems after the semifinal

Subsection 2.2 only use the material of §§1.1 and 1.2. Subsection 2.3 only use the material of §§1.1, 1.2 and 1.5.

## 2.1 The Casson invariant

The **sign** of a crossing point of an oriented plane diagram of a knot is defined after figure 14; the first (the second) vector is the vector of overcrossing (of undercrossing). Clearly, the sign is independent of the orientation of the diagram, and so is defined for non-oriented diagram.

The **sign** of a $P$-skew pair of crossing points in a plane diagram of a knot (for any basepoint $P$) is the product of the signs of the two crossing points.

The *$P$-Casson invariant* of a plane diagram is the sum of signs over all $P$-skew pairs of crossing points.

**Problem 2.1.1.** (a) Draw a plane diagram of a knot and a basepoint $P$ such that $P$-Casson invariant is $-5$.

(b,c,d,e,f) Same as problems 1.4.1.b,c,d,e,f for the Casson invariant.

(g) Find and prove the analogue of problem 1.4.2.a for the Casson invariant.

(h,i) Same as problems 1.4.2.b and 1.5.4.d for the Casson invariant.

By (e,f) the **Casson invariant** (Casson number) $c_2$ of a plane diagram, of a knot, or even of isotopy class of a knot, is well-defined by setting it to be the $P$-Casson invariant of any plane diagram of the knot for any basepoint $P$.

**Theorem 2.1.2.** *There is a unique integer-valued isotopy invariant $c_2$ of (non-oriented) knots that assumes value 0 on the trivial knot and for which*

$$c_2(K_+) - c_2(K_-) = \operatorname{lk} K_0.$$

*(The number $\operatorname{lk} K_0$ is well-defined because change of the orientation on both components of an oriented link does not change the linking number.)*

## 2.2 Proper colorings

A *strand* in a plane diagram (of a knot or link) is a connected piece that goes from one undercrossing to the next. A **proper coloring** of a plane diagram (of a knot or link) is a coloring of its strands in one of three colors so that at least two colors are used, and at each crossing, either all three colors are present or only one color is present. A plane diagram (of a knot or link) is **3-colorable** if it has a proper coloring.

**Problem 2.2.1.** For each of the following knots or links take any diagram and decide if it is 3-colorable.

(a) the trivial knot. (b) the trefoil knot. (c) the figure eight knot.

(d-j) links in fig. 7 and 8.

**Problem 2.2.2.** (a) The 3-colorability of a plane diagram is preserved under the Reidemeister moves.

(b) Neither of links in fig. 7 and 8 (except the trivial link) is isotopic to the trivial link.

**Problem 2.2.3.** * The $5_1$ knot is not isotopic to the trivial knot.

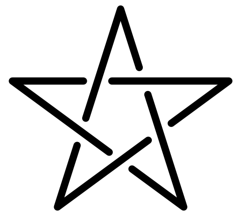

Figure 15: The $5_1$ knot

## 2.3 Alexander-Conway polynomials

**Problem 2.3.1.** * (a) There is a unique mod2-valued isotopy invariant arf of oriented 3-component links that assumes value 0 on the trivial link and for which

$$\operatorname{arf}(K_+) - \operatorname{arf}(K_-) = \begin{cases} \operatorname{lk}_2 K_0 & \text{at the crossing point different components cross each other;} \\ 0 & \text{at the crossing point one component crosses itself.} \end{cases}$$

(Here $\operatorname{lk}_2 K_0$ is defined because $K_0$ is a 2-component link.)[8]

(b) There is a unique mod2-valued isotopy invariant $a_3$ of oriented 2-component links that assumes value 0 on the trivial link and for which

$$a_3(K_+) - a_3(K_-) = \operatorname{arf} K_0.$$

(Here $\operatorname{arf} K_0$ is defined because $K_0$ is either a knot or a 3-component link.)

(c) There is a unique mod2-valued isotopy invariant $a_3$ of oriented 4-component links that assumes value 0 on the trivial link and for which

$$a_3(K_+) - a_3(K_-) = \begin{cases} \operatorname{arf} K_0 & \text{at the crossing point different components cross each other;} \\ 0 & \text{at the crossing point one component crosses itself.} \end{cases}$$

(Here $\operatorname{arf} K_0$ is defined because $K_0$ is a 3-component link.)

Proof of the existence in problems 2.3.1 and theorem 2.3.3 is outside the scope of this text. See an elementary proof in [Ka06', §2-§5], [Ka06]. You can earn a plus-mark (plus-sign) for proving the uniqueness, and solve other problems assuming the existence. For a relation to proper colorings see [Ka06', §6].

**Problem 2.3.2.** Calculate (for your choice of orientation on the components)

(a) the arf invariant of the Borromean rings;

(b,c,d*) the $a_3$ invariant of the Hopf link, of the Whitehead link, and of *4-Borromean rings*, i.e. of any link of your choice for which every 3-component sublinks are isotopic to the trivial link, but the entire link is not isotopic to the trivial link.

**Theorem 2.3.3.** * *(a) There is a unique infinite sequence $c_{-1} = 0, c_0, c_1, c_2, \dots$ of $\mathbb{Z}$-valued isotopy invariants of oriented* non-ordered *links that assume values $c_0 = 1$ and $c_1 = c_2 = \ldots = 0$ on the trivial knot and for which*

$$c_n(K_+) - c_n(K_-) = c_{n-1}(K_0)$$

*whenever $n \geq 0$.*

[8] Theorem 1.4.3 is the analogue of problem 2.3.1 for 1-component links (knots). The definition of arf given in §1.4 applies to knots only and here the point is to extend it to 3-component links.

*(b) There is a unique infinite sequence $c_{-1} = 0, c_0, c_1, c_2, \dots$ of $\mathbb{Z}$-valued isotopy invariants of oriented* ordered *links that assume values $c_0 = 1$ and $c_1 = c_2 = \dots = 0$ on the trivial knot and such that for any $n \ge 0$ we have*

$$c_n(K_+) - c_n(K_-) = c_{n-1}(K_0),$$

*where $K_0$ is $K_0$ from fig. 9 with some ordering of the components.*

Actually two versions of theorem 2.3.3 are equivalent. You can use theorem 2.3.3.b without proof.[9]

The polynomial $C(K)(t) := c_0(K) + c_1(K)t + c_2(K)t^2 + \dots$ is called the *Conway polynomial*, see problem 2.3.5.e. Introduction of this polynomial allows to calculate all the invariants $c_n$ as quickly as one of them. The formula in theorem 2.3.3 is equivalent to

$$C(K_+) - C(K_-) = tC(K_0).$$

The polynomial $C(K)(t) := c_0(K) + c_1(K)t + c_2(K)t^2 + \dots$ is called the *Conway polynomial*, see problem 2.3.5.e. Introduction of this polynomial allows to calculate all the invariants $c_n$ as quickly as one of them. The formula in theorem 2.3.3 is equivalent to

$$C(K_+) - C(K_-) = tC(K_0).$$

**Problem 2.3.4.** Calculate the Conway polynomial of the following links (for your choice of orientation on the components).

(a) the trivial link with 2 components; (b) the trivial link with $n$ components;
(c) the Hopf link; (d) the trefoil knot; (e) the figure eight knot;
(f) the Whitehead link; (g) the Borromean rings; (h) the $5_1$ knot.

**Problem 2.3.5.** (a) We have $c_0(K) = 1$ if $K$ is a knot and $c_0(K) = 0$ otherwise (i.e. if $K$ has more than one component).

(b) For a knot $K$ we have $c_{2j+1}(K) = 0$ and $c_2$ is the Casson invariant.

(c) For a 2-component link $K$ we have $c_{2j}(K) = 0$ and $c_1$ is the linking coefficient.

(d) For a $k$-component link $K$ we have $c_j(K) = 0$ if either $j \le k - 2$ or $j - k$ is even.

(e) For every knot or link all but a finitely many of the invariants $c_n$ are zeroes.

**Problem 2.3.6.** (a) Change of the orientations of all components of a link (in particular, change of the orientation of a knot) preserves the Conway polynomial.

(b) There is a 2-component link such that change of the orientation of its one component changes the degree of the Conway polynomial (so this change neither preserves nor negates the Conway polynomial).

(c) For any of the connected sums $K\#L$ of knots $K, L$ we have $C(K\#L) = C(K)C(L)$.

A link is *split* if it is isotopic to a link whose components are contained in disjoint balls.

**Problem 2.3.7.** (a,b,c) No link of theorem 1.2.2 is split.

(d) The Conway polynomial of a split link is trivial.

[9] It is not clear which of the two versions is stated in [CDM12, §2.3.1], so we present both versions and deduce the stronger version from the weaker version.

## 2.4 Vassiliev-Goussarov invariants (sketch) *

Denote by

- $\Sigma$ the set of isotopy classes of singular knots [PS96, 4.1],
- $\delta_n$ the set of all chord diagrams that have $n$ chords [PS96, 4.8];
- $\sigma(K)$ the *chord diagram* of a singular knot $K$ [PS96, 4.8], [CDM12, 3.4.1] (not to be confused with *Gauss diagrams* for a non-singular knot $K$ [CDM12, 1.8.4]).

**Theorem 2.4.1** (Vassiliev-Kontsevich, [PS96], [CDM12])**.** *For any map $\lambda : \delta_n \to \mathbb{R}$ there exists a map $v : \Sigma \to \mathbb{R}$ having properties (1)-(3) below if and only if $\lambda$ satisfies to the 1-term and the 4-term relations [PS96, (4.5),(4.6)].*

*(1) For any singular knots $K_+, K_-$ and $K^0$ from [PS96, (4.1)] (notice the difference with fig. 9) we have*

$$v(K_+) - v(K_-) = v(K^0),$$

*($2_n$) $v(K) = 0$ for each singular knot that has more than $n$ double points, and*

*(3) $v(K) = \lambda(\sigma(K))$ for each singular knot that has exactly $n$ double points.*

A map $v : \Sigma \to \mathbb{R}$ such that (1) holds is called a *Vassiliev-Goussarov invariant.*

A map $v : \Sigma \to \mathbb{R}$ such that ($2_n$) holds is called a *map of order at most $n$.*

**Problem 2.4.2.** (a) The map $v$ of theorem 2.4.1 is unique up to Vassiliev-Goussarov invariant of order at most $n-1$. More precisely, the difference between maps $v, v' : \Sigma \to \mathbb{R}$ satisfying to (1), ($2_n$) and (3), satisfies to (1) and ($2_{n-1}$).

(b) Prove the 'only if' part of theorem 2.4.1.

(0),(1),(2),(3)* Prove the 'if' part of theorem 2.4.1 for $n = 0, 1, 2, 3$.

Hint: for $n = 2$ use theorem 2.1.2, for $n = 3$ use the coefficient of $h^3$ in $J(e^h)$, where $J$ is the Jones polynomial in $t$-parametrization [CDM12, 2.4.2, 2.4.3].

In the remaining problems theorem 2.4.1 can be used without proof. Assertion '$v(K) = x$ for any singular knot $K$ whose chord diagram is $a$' is shortened to '$v(a) = x$'.

**Problem 2.4.3.** (a) There exists a unique Vassiliev-Goussarov invariant $v_2 : \Sigma \to \mathbb{R}$ of order at most 2 such that

- $v_2(O) = 0$ for the trivial knot $O$, and
- $v_2(1212) = 1$ ((1212) is the 'non-trivial diagram with 2 chords' [PS96, Figure 4.4], 3rd diagram of the first line).

Warning: in this problem it is allowed to use theorem 2.4.1 but not theorem 2.1.2.

(b,b',c,d) Calculate $v_2$ for the right trefoil, left trefoil, figure eight knot and the $5_1$ knot.

**Problem 2.4.4.** (a) There exists a unique Vassiliev-Goussarov invariant $v_3 : \Sigma \to \mathbb{R}$ of order at most 3 such that

- $v_3(O) = 0$ for the trivial knot $O$ and for the left trefoil $O$, and
- $v_3(123123) = 1$ ((123123) is the 'non-trivial most symmetric diagram with 3 chords', [PS96, Figure 4.4], 5th diagram of the second line).

(b,c,d*) Calculate $v_3$ for the right trefoil, figure eight knot and the $5_1$ knot.

Hints: Problems 2, 3, 4ab, Results/Theorems 11, 13, 14 from [PS96, §4].

**Problem 2.4.5.** (a) There exists a unique Vassiliev-Goussarov invariant $v_4 : \Sigma \to \mathbb{R}$ of order at most 4 such that

- $v_4(O) = 0$ for the trivial knot $O$, for the left trefoil $O$, and for the right trefoil $O$,
- $v_4(12341234) = 2$, $v_4(12341432) = 3$ and $v_4(12341423) = 5$ [PS96, Problem 4.4.b].

(c*,d*) Calculate $v_4$ for the figure eight knot and the $5_1$ knot.

## Hints and solutions for problems before the semifinal

**1.1.1.** (a,b,c,d) 'Probably the best way of solving this problem is to make a model of the trefoil knot and the figure eight knot by using a shoelace and then move it around from one position to the other. Fig. 16 gives some hints concerning transformations of the figure eight knot.' [Pr95, §2]

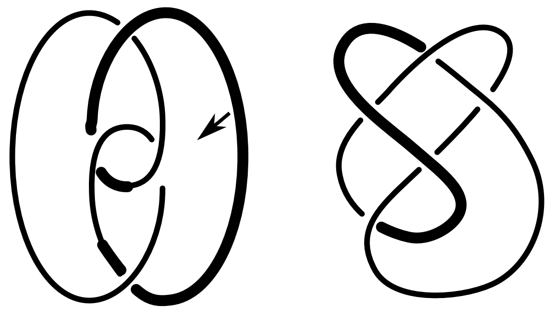

Figure 16: Isotopy of the figure eight knot

(e) Let us consider two knots with coinciding plane diagrams, in a horizontal plane $\pi$. For each point $X$ in the space let $p(X)$ be the line containing $X$, perpendicular to $\pi$. Let $h(X)$ be the height of $X$, relative to $\pi$, which is positive ($h(X) > 0$) if $X$ is in the upper half-space and negative ($h(X) < 0$) if $X$ is in the lower half-space. To each point $A$ of the first knot associate a point $A$ of the second knot by the following procedure. There are two cases:

*Case 1: The projection of the point $A$ on $\pi$ is not a crossing point on the plane diagram.* In this case $p(A)$ intersects the first knot only at the point $A$. Since the plane diagrams coincide, the line $p(A)$ intersects the second knot also at a single point. Define $A'$ to be this point.

*Case 2: The projection of the point $A$ on $\pi$ is a crossing point of the plane diagram.* In this case the line $p(A)$ intersects the first knot in an additional point $B$. Since the plane diagrams coincide, the line $p(A)$ intersects the second knot in two points $C$ and $D$, where we assume that $h(C) > h(D)$. If $h(A) > h(B)$, we define $A' = C$, and in the opposite case $A' = D$.

For each point $A$ of the first knot and each number $t \in [0, 1]$ let $A(t)$ be the point on the line $p(A)$ with the height $h(A(t)) = (1 - t)h(A) + th(A')$. By construction $A(0) = A$, $A(1) = A'$ and the transformation of the first knot, which moves $A(0)$ in the direction of $A(1)$ with constant speed, so that at the time $t$ it occupies the position $A(t)$, is the required isotopy.

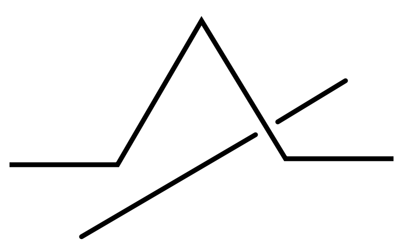

Figure 17: The bridge over some crossing point

**1.1.2.** See fig. 17. For each crossing point of the plane diagram, on the upper edge of the crossing, choose two points, close to the intersection and on the opposite sides of the intersection. Replace the line segment between the two chosen points by a 'bridge' rising above the plane diagram, which connects these two points. After replacing all crossing points by the corresponding bridges, we obtain the required knot.

**1.1.3.** (a) Use the results of problems 1.4.1, 2.1.1, or 2.2.1.

(b) Use the results of problems 1.4.1 or 2.1.1.

(c) Use the result of problem 2.2.1.

(d) Take any of the connected sums of $n$ trefoil knots. By assertions 2.1.1.c,i the Casson invariant of this knot is $n$. It follows that for different values of $n$ the corresponding knots are not isotopic.

**1.1.4.** Choose a knot projected to the given plane diagram in the same way as in assertion 1.1.2. Suppose that all 'bridges' lie in the upper half-space w.r.t. the projection plane. By the assumption there are points $X$ and $Y$ on the knot which divide the knot into two polygonal lines $p$ and $q$ such that

• $q$ lies in the projection plane and passes only through undercrossings;

• $p$ is projected to polygonal line $p'$ which passes only through overcrossings.

Take a point $Z$ in the upper half-space, and a point $T$ in the lower half-space. Let us construct an isotopy between the given knot and the closed polygonal line $XZYT$, which is isotopic to the trivial knot. The isotopy consists of 3 steps, all of them keeping $X, Y$ fixed.

*Step 1. An isotopy between $q$ and $XTY$.* Suppose that $q = A_0A_1 \dots A_n$, where $A_0 = X$ and $A_n = Y$. Then the isotopy is given by

$$A_0A_1 \to A_0TA_1, \quad TA_1A_2 \to TA_2, \quad TA_2A_3 \to TA_3, \quad \dots TA_{n-1}A_n \to TA_n.$$

*Step 2. An isotopy between $p$ and $p'$.* Remove all the 'bridges' by elementary moves.

*Step 3. An isotopy between $p'$ and $XZY$.* This is done analogously to step 1.

**1.1.5.** Follows by assertion 1.1.4.

*Another idea of the proof (cf. [PS96, Theorem 3.8]).* Denote by $\pi$ the horizontal plane containing the plane diagram. For each point $X$ in the space, $p(X)$ and $h(X)$ are defined in the solution of the problem 1.1.1.e. Let $l$ be a line in the plane, which passes through a vertex $A_0$ of the plane diagram, while the whole diagram is contained in one of the two half-planes determined by $l$. Let $A_0, A_1, \dots A_n$ be all vertices of the plane diagram, in the order of their appearance, while we move along the diagram in some direction. Choose points $B_0, \dots, B_n$ so that $A_i \in p(B_i)$ for $i = 1, \dots, n$, and $h(B_i) < h(B_j)$ for $i < j$. Let $B_{n+1}$ be a point, whose projection on $\pi$ is close to $A_0$ and $h(B_{n+1}) > h(B_n)$. We claim that the knot $B_0 \dots B_nB_{n+1}$ is isotopic to the trivial knot. Indeed, by the choice of the line $l$, the projection of the knot onto any plane, perpendicular to the line $l$, is a closed polygonal line without self-intersections. It remains to modify crossing in the plane diagram so that they are in agreement with the projection of the constructed knot to the plane $\pi$.

**1.1.6.** See [PS96, §1.7].

*Remark.* Since [PS96, §1.6] does not contain as rigorous definition of Reidemeister moves as that of plane isotopies,[10] the argument in [PS96, §1.7] does not constitute a rigorous proof. We believe that a rigorous proof can be recovered using rigorous definition of Reidemeister moves.

**1.2.1.** (a) This follows by (b) (or can be proved independently).

(d) This follows by (e) (or can be proved independently).

(e) See figure 18.

[10]This also shows that having plane isotopy in the statement [PS96, §1.7] does not make the statement rigorous, and thus should be avoided. On an intuitive level, plane isotopies should better be ignored. With the alternative rigorous definition below, plane isotopies can be expressed via Reidemeister moves and so should better be ignored in the statement.

Let us present an alternative rigorous definition of the first Reidemeister move. The other Reidemeister moves have analogous rigorous definitions. On the plane take a closed non-self-intersecting polygonal line $L$ whose interior (see the Jordan Theorem in remark after theorem 1.1.3) intersects a knot diagram $D$ by a non-self-intersecting polygonal line $M$ joining two points on $L$. Let $N$ be a closed non-self-intersecting polygonal line in the interior of $L$ such that $N \cap L = \emptyset$, $N \cap M$ is one point and $M \cup N$ cam be made a generic (self-intersecting) polygonal line. *The first Reidemeister move* is replacement of $M$ to $M \cup N$ in $D$, with any 'information' at the appearing crossing.

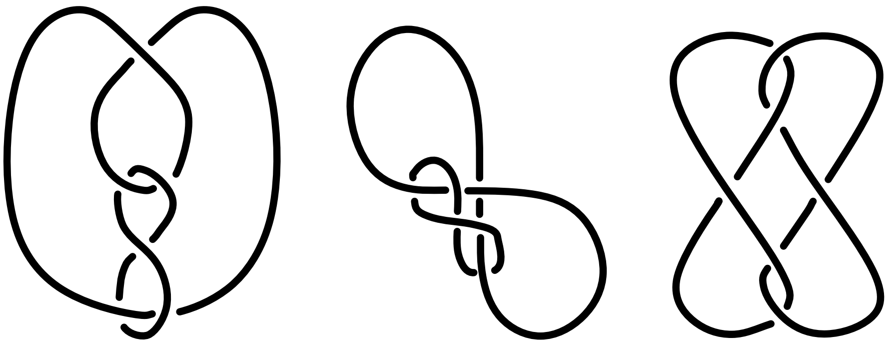

Figure 18: Isotopy of the Whitehead link

(f) Take three ellipses given by the following three systems of equations:

$$\begin{cases} x = 0 \\ y^2 + 2z^2 = 1 \end{cases}, \qquad \begin{cases} y = 0 \\ z^2 + 2x^2 = 1 \end{cases} \qquad \text{and} \qquad \begin{cases} z = 0 \\ x^2 + 2y^2 = 1. \end{cases}$$

See figure 19. Take the quadrilaterals circumscribed around these ellipses and symmetric w.r.t. the coordinate axes. Then the straight line is given by $x = y = z$.

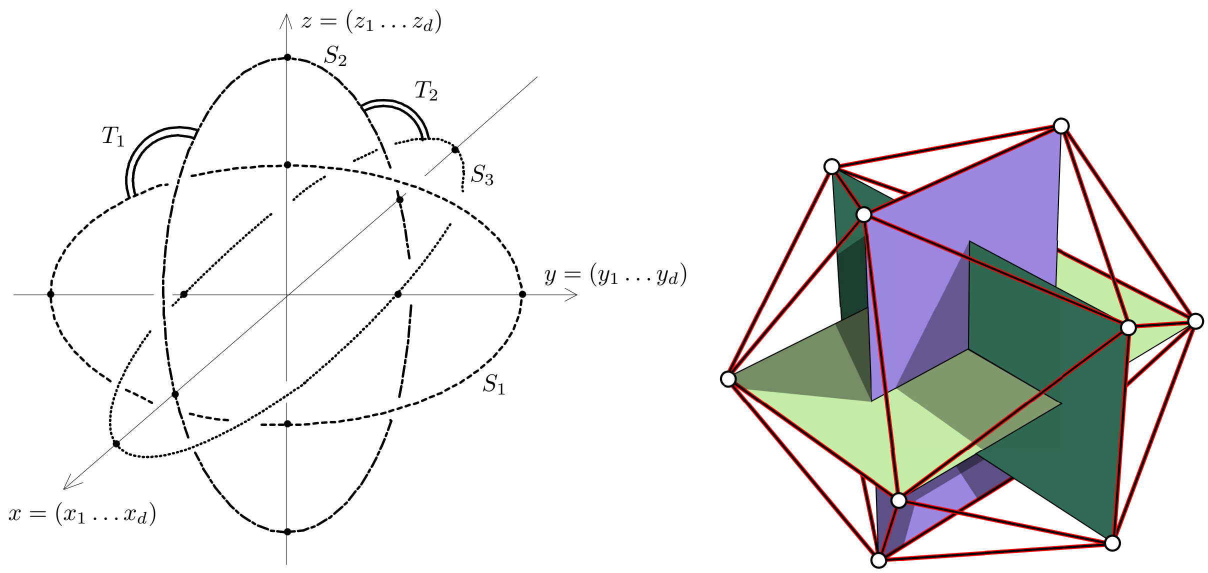


Figure 19: Borromean rings

**1.2.2.** (a,c) Use results of problem 1.3.2.

(b,d) Use, for example, results of problems 2.2.1 and 2.2.2.

**1.3.1.** In the plane passing through $A, B, C$ denote by $\langle ABC \rangle$ the part bounded by the triangle $ABC$.

(i⇔ii) A segment $XY$ passes above the segment $BC$ as seen from $A$ if and only if $XY$ intersects $\langle ABC \rangle$.

The outline of $DEF$ is the union of segments $DE, EF$ and $DF$. Hence i⇔ii.

(i⇔iii) The outline of $DEF$ is either disjoint from the surface of tetrahedron $OABC$ or intersects the surface at exactly two points. Hence the following three properties are equivalent:

- the outline of $DEF$ intersects $\langle ABC \rangle$ at exactly one point;
- the outline of $DEF$ intersects exactly one of the triangles $\langle OAB \rangle$, $\langle OBC \rangle$ and $\langle OAC \rangle$.
- the outline of $ABC$ passes below exactly one side of $DEF$ as seen from $O$.

**1.3.2.** (a) *Answers* (from left to right): 1, 0, 0, 0.

(b) Prove the statement separately for every Reidemeister move. For moves I and III the number of crossing points where the first component passes above the second one does not change. For move II this number changes by 0 or $\pm 2$.

**1.3.3.** (a) Take a plane diagram of a link. By the Parity lemma (stated before problem 1.3.3) the number of crossing points where the first component passes above the second one has the same parity as the number of crossing points where the second component passes above the first one. This completes the proof.

(b) An example is the fourth link in fig. 7. One can prove that this link is not isotopic to the trivial one using linking number, see §1.6.

**1.3.4.** Suppose that f is another invariant aside from $\mathrm{lk}_2$ satisfying the assumptions. Then $\mathrm{f} - \mathrm{lk}_2$ is an isotopy invariant assuming zero value on the trivial link and invariant under crossing changes. By the analogue of lemma 1.1.5 for links any plane diagram of a link can be obtained from the diagram of a link isotopic to the trivial link by some crossing changes. Hence $\mathrm{f} - \mathrm{lk}_2 = 0$.

**1.3.5.** The proof should not be hard, and we encourage you to supply the details.

**1.3.6.** See [Sk14, §1, Theorem 1.1].

**1.4.1.** (a) If $P$ is a point on the plane diagram as in problem 1.1.4, then there are no $P$-skew pairs of crossings. Hence the $P$-Arf invariant is zero.

(b) *Answer: 0.* The trivial knot has no crossings and no skew pairs of crossings. Therefore the Arf invariant of this knot is 0 for any choice of the basepoint.

(c) *Answer: 1.* The trefoil knot has 3 crossings. For any basepoint $P$ exactly one pair of crossings is $P$-skew. Hence $P$-Arf invariant of the trefoil knot is 1.

(d) *Answer: 1.*

(e) It suffices to show that the Arf invariant remains unchanged when the basepoint moves through one crossing on the plane diagram. Let $P_1$ and $P_2$ be two basepoints such that the segment $P_1P_2$ contains exactly one crossing point $X$. Consider two cases.

*Case 1: $P_1P_2$ passes through undercrossing.* Then $X$ does not form either $P_1$-skew or $P_2$-skew pair with any other crossing. Hence $P_1$- and $P_2$-Arf invariants of the diagram are equal.

*Case 2: $P_1P_2$ passes through overcrossing.* Then $X$ divides the diagram into two closed polygonal lines $q_1$ and $q_2$ such that $P_1$ lies on $q_1$ and $P_2$ lies on $q_2$. Denote by $n_1$ (respectively, $n_2$) the number of intersections of $q_1$ and $q_2$ for which $q_1$ passes above $q_2$ (respectively, $q_2$ passes above $q_1$). Denote by $N_1$ (respectively, $N_2$) the number of $P_1$-skew (respectively, $P_2$-skew) pairs formed by $X$ and some intersection of $q_1$ and $q_2$. Then

$$\mathrm{arf}_{P_1} D - \mathrm{arf}_{P_2} D = N_1 - N_2 = n_1 - n_2 \underset{2}{\equiv} n_1 + n_2 \underset{2}{\equiv} 0,$$

where $D$ is the given plane diagram. Here

• the first equality holds because a pair of crossings is either $P_1$-skew or $P_2$-skew (but not both) if and only if the pair is formed by $X$ and some intersection of $q_1$ and $q_2$;

• the second equality holds because $N_1 = n_1$ and $N_2 = n_2$; indeed, an intersection of $q_1$ and $q_2$ forms a $P_1$-skew (respectively, $P_2$-skew) pair with $X$ if and only if at this intersection $q_1$ passes above (respectively, below) $q_2$;

• $\underset{2}{\equiv}$ is congruence modulo 2;

• the last congruence follows by Parity lemma for $q_1$ and $q_2$.

(f) Prove the statement for each Reidemeister move separately. The common idea is to cleverly choose a basepoint.

*Type I move.* Take basepoints before and after the move as in fig. 20 (left). Check that the crossing $A$ does not form a $P$-skew pair with any other crossing.

*Type II move.* Take basepoints before and after the move as in fig. 20 (middle). Check that neither of the crossings $A$ and $B$ forms a $P$-skew pair with any other crossing.

*Type III move.* Take basepoints before and after the move as in fig. 20 (right). Check that neither of the crossings $A$, $B$ forms a $P$-skew pair with any other crossing and that neither of the crossings $A'$, $B'$ forms a $P'$-skew pair with any other crossing. Then check that a crossing $X$ distinct from $A$, $B$, $C$ forms $P$-skew pair with $C$ if and only if $X$ forms $P'$-skew pair with $C'$.

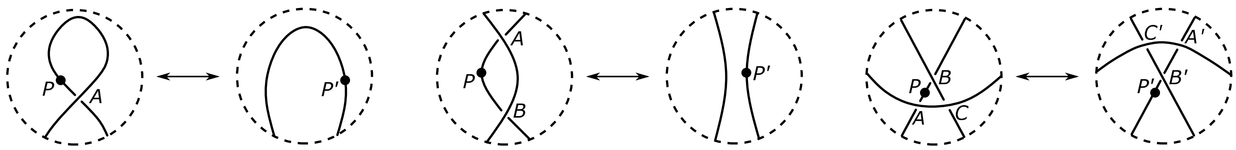


Figure 20: Arf-invariant does not change under Reidemeister moves

**1.4.2.** (a) Take basepoints $P_+$, $P_-$ as in fig. 21. Check that the crossing $A_-$ does not form a $P_-$-skew pair with any other crossing in $K_-$. Then check that the number of crossings which form a $P_+$-skew pair with $A_+$ in $K_+$ equals $\mathrm{lk}_2 K_0$ modulo 2.

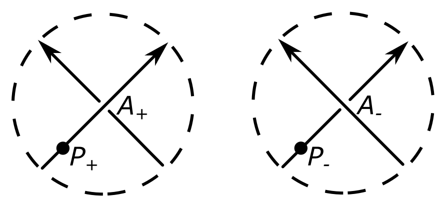


Figure 21: To the proof of skein relation for Arf invariant

(b) Take any of the connected sums of the two trefoil knots. By assertions 1.4.1.c and 1.5.4.d the Arf invariant of this knot is 0. However by assertions 2.1.1.c,i the Casson invariant of this knot is 2, hence this knot is not isotopic to the trivial knot.

**1.4.3.** *Existence.* Follows from assertions 1.4.1.b and 1.4.2.a.

*Uniqueness.* The proof goes along the lines of the proof of theorem 1.3.4. Use lemma 1.1.5 itself instead of its analogue for links.

**1.4.4.** See [Ka87, pp. 75–78].

**1.4.5.** See [CG83, Theorem 2].

**1.5.1.** (b) *First solution.* An oriented polygonal line is called *positive* if the bounded part of the plane is always on the right side of each of its oriented segments (see the Jordan theorem in remark after theorem 1.1.3). Prove that the positivity of an oriented polygonal line is preserved by elementary moves.

*Hint to the second solution.* The positivity can be equivalently defined as follows. We say that an oriented polygonal line $A_1 \dots A_n$ is *positive* if for each of its inner (interior) points $O$ the sum of oriented angles $\angle A_1OA_2 + \angle A_2OA_3 + \dots + \angle A_{n-1}OA_n + \angle A_nOA_1$ is always positive (i.e. the *winding number* of the oriented polygonal line around any interior point is positive).

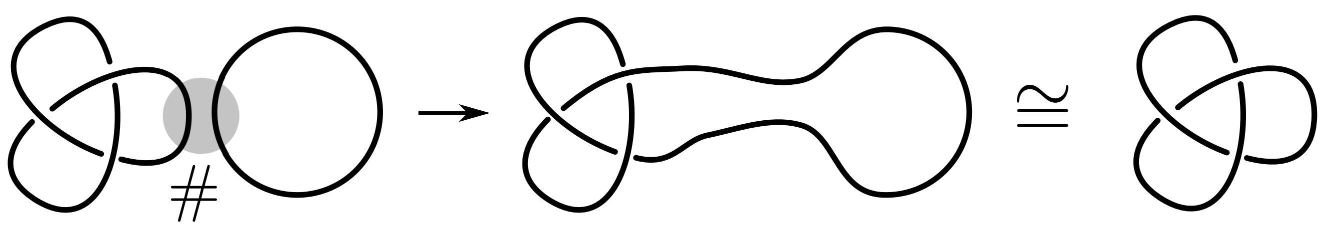


Figure 22: Proof of $K\#O = K$

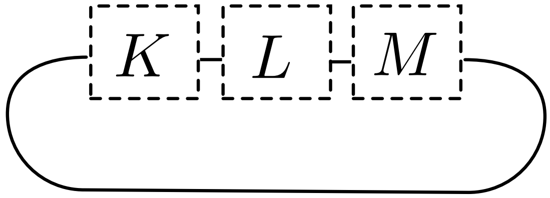


Figure 23: Proof of $(K\#L)\#M = K\#(L\#M)$

**1.5.2.** (a), (b), (c) Each of the three indicated oriented knots is transformed into the oriented knot with the opposite orientation by the rotation through the angle $\pi$ around the 'vertical' axis passing through the 'upper' point of the knot (see the leftmost diagram in fig. 1, the first and the second row for the trefoil and the figure eight knot, respectively). This rotation is included into a continuous family of rotations through the angle $\pi t$, $t \in [0, 1]$, with respect to the same line. This is the required isotopy.

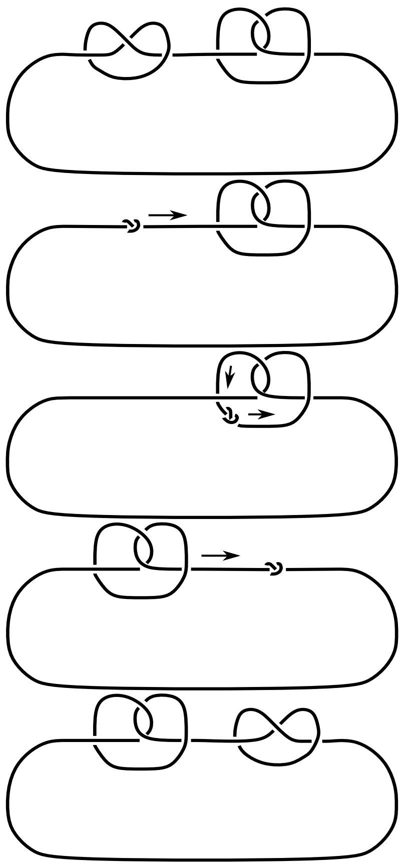

Figure 24: Proof of $K\#L = L\#K$

**1.5.4.** (a) See fig. 22.

(b) It is sufficient to take a small knot of the class $L$ and push it through a knot from the class $K$, see fig. 24.

(c) Isotopic classes of both the left hand and the right hand side of the equality have a common representative exhibited in fig. 23.

(d) Choose basepoint close to the 'place of connection'. Check that all skew pairs of crossings in $K\#L$ are obtained from the skew pairs of crossings in $K$ and in $L$.

**1.5.5.** (d) Check that all crossings of different components in $K\#L$ are obtained from such crossings in $K$ and in $L$.

**1.5.6.** (a) As an example we can take equal links consisting of a trefoil and an unknot in disjoint cubes. Cf. [PS96, Figure 3.16].

(b) See [As]. Fig. 25 presents an alternative example suggested by A. Ryabichev.

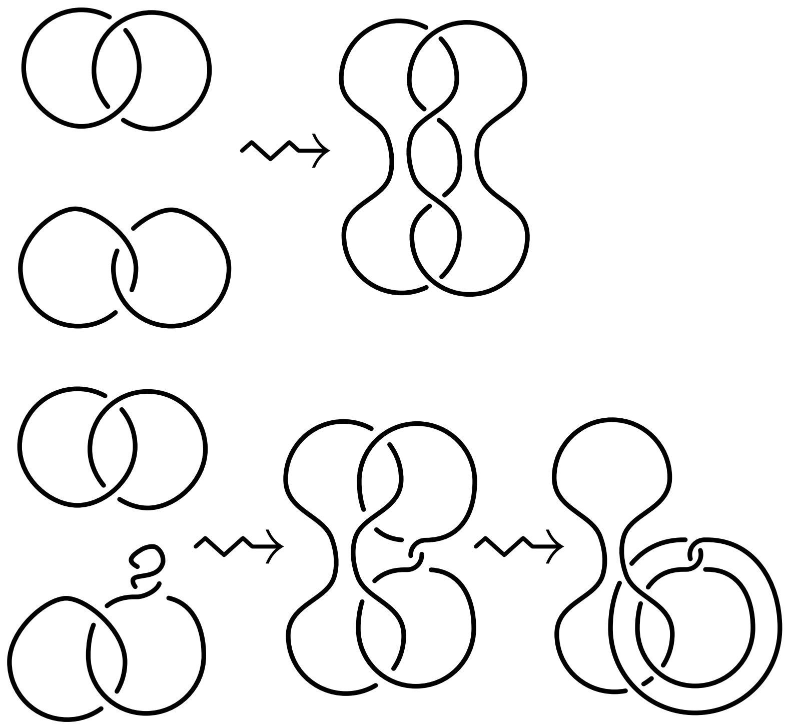

Figure 25: Connected sum of isotopy classes of ordered links is not well-defined

**1.6.1.** *Answers:* (a) 1; (b) 0.

(c) The proof is analogous to assertion 1.3.2. It suffices to check that the signs of all crossing points does not change.

**1.6.2.** (a) The proof is analogous to assertion 1.3.2.b. Take a plane diagram of a link. By the Triviality lemma (stated before problem 1.6.2) the sum of signs of crossing points where the first component passes above the second one has opposite sign to the sum of signs of crossing points where the second component passes above the first one. Switching the components negates the sign of every crossing point. This completes the proof.

(b) Reversing the orientation of either of the components negates the sign of every crossing point.

(c) Take the connected sum of 5 Hopf links oriented so that their linking numbers equal to $-1$.

(d) The proof is analogous to assertion 1.5.4. The signed set of crossing points of plane diagram of $K\#L$ is the union of the signed sets of crossing points of plane diagrams of links $K$ and $L$.

(e) An example is the Whitehead link. The Whitehead link is not isotopic to the trivial link by theorem 1.2.2.

**1.6.3.** The proof is analogous to theorem 1.3.4.

# Hints and solutions for some problems after the semifinal

**2.1.1.** *Answers:* (b) 0; (c) 1; (d) $-1$.

(a) Take any connected sum of five figure eight knots. By (d) and assertion 2.1.1.i below the Casson invariant of this knot is $-5$.

(b) The trivial knot has no crossings, and so no skew pairs of crossings. Therefore the Casson invariant of this knot is 0.

(c) All three crossings of the trefoil knot have the same sign. Since the trefoil knot has exactly one linked pair of crossings (regardless the choice of the base-point), we obtain that the Casson invariant of this knot is 1.

(e) The proof is analogous to assertion 1.4.1.e. Use the Triviality lemma stated after problem 1.6.1. Here and below keep in mind the signs of intersection points!

(f) The proof is analogous to assertion 1.4.1.f.

(g) The proof is analogous to assertion 1.4.2.a.

(h) Take any connected sum of the trefoil knot and the figure eight knot. By assertions 2.1.1.c,d and 2.1.1.i the Casson invariant of this knot is 0. However, by assertions 2.3.4.d,e and 2.3.6.b the Conway polynomial of this knot is $(t^2+1)(t^2-1) \neq 1$. Hence this knot is not isotopic to the trivial knot.

(i) The proof is analogous to assertion 1.5.4.d.

**2.1.2.** The proof is analogous to theorem 1.4.3.

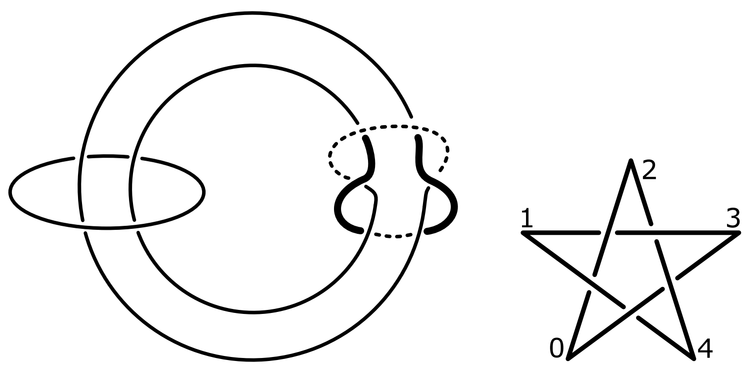


Figure 26: A 3-coloring of a link and 5-coloring of the $5_1$ knot

**2.2.1.** *Answers:* b,e,h — 3-colorable, a,c,d,f,g,i,j — not 3-colorable. For a proper coloring of a diagram of trefoil knot see the attached copy of [Pr95, p. 30, figure 4.3]. For a proper coloring of the last diagram from fig. 7 see fig. 26 left. (This diagram was erroneously stated to be not 3-colorable in [Pr95, §4]. This minor mistake was found by L.M. Bannöhr, S. Zotova and L. Kravtsova.)

**2.2.2.** (a) See the attached copy of [Pr95, pp. 29-30, Theorem 4.1].

(b) Follows from (a) and assertions 2.2.1.d-j (see the attached copy of [Pr95, p. 30]). The last diagram from fig. 7 is distinguished from the trivial link by the number of proper colorings of a plane diagram. Prove that this number is preserved under the Reidemeister moves.

**2.2.3.** A plane diagram is *5-colorable* if there exists a coloring of its strands in five colors $0, 1, 2, 3, 4$ so that

• at least two colors are used;

• at each crossing if the upper strand has color $a$ and two lower strands have colors $b$ and $c$, then $2a \equiv b + c \pmod 5$.

Similarly to assertion 2.2.2.a one can prove that the 5-colorability of a plane diagram is preserved under Redemeister moves. The $5_1$ knot is 5-colorable, see fig. 26, right. The trivial knot is not. This completes the proof.

**2.3.1.** These are particular cases of mod2 version of theorem 2.3.3.

**2.3.2.** *Answers:* (a, b) 0; (c) 1 (independently of the choice of orientation).

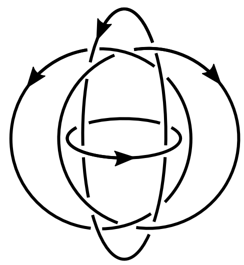

Figure 27: 4-Borromean rings

*Remark.* The invariants arf, $a_3$ for links may depend on the orientation on the components (for $a_3$ see the attached copy of [CDM12, 2.3.4]).

**2.3.3.** The uniqueness is analogous to theorems 2.1.2 and 1.4.3; solve first problem 2.3.4.

*Deduction of (a) from (b).* It suffices to show that all invariants $c_n$ defined in (b) are preserved under changes of the order of the components.

Let $D$ be a plane diagram of some link with two or more components and let $D'$ be a plane diagram obtained from $D$ by a change of the components' order. By $\mathrm{cr}\, D$ denote the number of crossings in $D$. By $u(D)$ denote the minimal possible number of crossing changes needed to obtain a diagram of a link which is isotopic to the trivial one from $D$ (such sequence of crossing changes exists by the analogue of lemma 1.1.5 for links).

The proof is by induction on $\mathrm{cr}\, D$. If $\mathrm{cr}\, D = 0$, then $D$ is a diagram of a link which is isotopic to the trivial one and by assertion 2.3.4.b we have $C(D) = 0$ for any ordering of the components. Suppose that $\mathrm{cr}\, D > 0$; then continue the proof by induction on $u(D)$. If $u(D) = 0$, then $D$ is a diagram of a link which is isotopic to the trivial one; this case is considered above. Suppose that $u(D) > 0$ and let $D_*$ be a link obtained from $D$ by a crossing change and such that $u(D_*) < u(D)$. Suppose $D'_*$ is a link obtained from $D'$ by the change of the same crossing; then

$$\pm(C(D) - C(D_*)) = C(D_0) \text{ and } \pm(C(D') - C(D'_*)) = C(D'_0),$$

where

- $D_0$ is a diagram of a link $K_0$ (with some ordering of the components) from fig. 9 for $D$, $D_*$ being $D_+$, $D_-$ in some order;
- $D'_0$ is a diagram of a link $K_0$ (with some ordering of the components) from fig. 9 for $D'$, $D'_*$ being $D_+$, $D_-$ in some order.

Note that the diagrams $D_*$ and $D'_*$ coincide up to the order of the components. The same is true for the diagrams $D_0$ and $D'_0$. Since $u(D_*) < u(D)$ and $\mathrm{cr}\, D_0 < \mathrm{cr}\, D$, by inductive hypotheses we have $C(D_*) = C(D'_*)$ and $C(D_0) = C(D'_0)$. Then $C(D) = C(D')$.

**2.3.4.** *Answers:* (a, b) 0; (c) $\pm t$; (d) $1 + t^2$; (e) $1 - t^2$; (f) $\pm t^3$; (g) $\pm t^4$; (h) $1 + 3t^2 + t^4$.

*Remark.* The signs in the answers to (c), (f), (g) depend on the orientation on the components.

*Hint.* For examples of such calculations for (a), (c), and (d) see the attached copy of [CDM12, 2.3.2].

**2.3.5.** Let $D$ be a plane diagram of the given link $K$.

(a) For any diagram $D_*$ obtained from $D$ by a crossing change we have $c_0(D) - c_0(D_*) = 0$. I. e. $c_0$ is invariant of crossing changes. By the analogue of lemma 1.1.5 for links the diagram $D$ can be obtained by crossing changes from a diagram of a link isotopic to the trivial one. The assertion follows from the definition of $c_0$ on the trivial knot and assertion 2.3.4.b.

(b) The first part follows from (d). The second part follows from the definition of $c_2$ and theorem 2.1.1.g.

(c) The first part follows from (d). The second part follows from the definitions of $c_0, c_1$ and theorem 1.6.3.

(d) The proof is by induction on $\operatorname{cr} D$. If $\operatorname{cr} D = 0$, then $K$ is isotopic to the trivial link. If $K$ is a knot, then $C(D) = 1$. Otherwise $C(D) = 0$ by assertion 2.3.4.b. Suppose that $\operatorname{cr} D > 0$; then continue the proof by induction on $u(D)$. If $u(D) = 0$, then $K$ is isotopic to the trivial link; this case is considered above. Suppose that $u(D) > 0$ and let $D_*$ be a link obtained from $D$ by a crossing change and such that $u(D_*) < u(D)$. Then we have

$$\pm(c_j(D) - c_j(D_*)) = c_{j-1}(D_0),$$

where $D_0$ is the diagram from fig. 9 corresponding to $D$, $D_*$ being $D_+$, $D_-$ in some order. Note that the link $D_*$ consists of $k$ components and the link $D_0$ consists of $k' = k \pm 1$ components. Therefore if $j \le k - 2$, then $j - 1 \le k'$ and if $j - k$ is even, then $(j-1) - k'$ is even. Since $u(D_*) < u(D)$ and $\operatorname{cr} D_0 < \operatorname{cr} D$, by inductive hypothesis we have $c_j(D_*) = c_{j-1}(D_0) = 0$. Then $c_j(D) = 0$.

(e) Prove more general statement: *for a plane diagram $D$ we have* $\deg C(D) \le \operatorname{cr} D$. The proof is analogous to problem 2.3.5.d.

**2.3.6.** (a) The proof is analogous to problem 2.3.5.d.

(b) See the attached copy of [CDM12, 2.3.4].

(c) The proof goes along the lines of problem 2.3.5.d. Let $D$ and $G$ be plane diagrams of $K$ and $L$. Fix the diagram $G$ and prove that $C(D\#G) = C(D)C(G)$ by induction on $\operatorname{cr} D$.

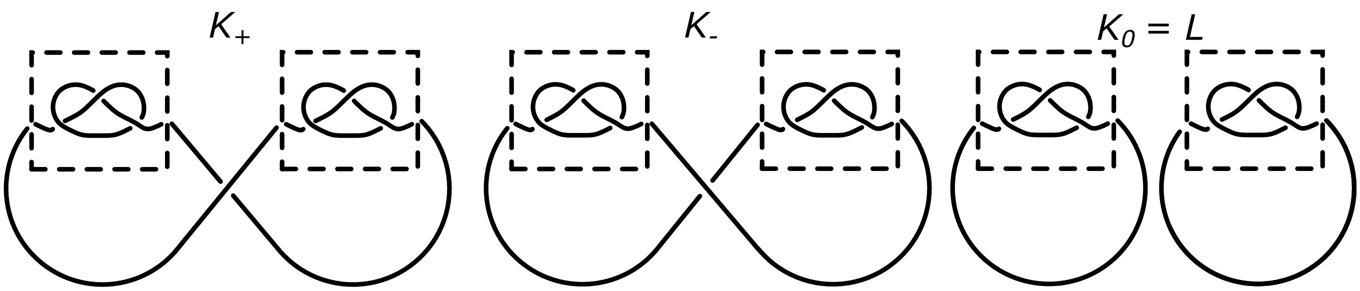


Figure 28: Proof that $C(\text{split link}) = 0$

**2.3.7.** (b,c,d) Follows from 2.3.4.c,f,g above and 2.3.7.e below.

(e) If $L$ is a split link, then there exist links $K_+$, $K_-$, $K_0$ such that

- their plane diagrams differ like in fig. 9;
- the links $K_+$ and $K_-$ are isotopic;
- the link $K_0$ is isotopic to $L$.

We have $C(L) = C(K_0) = \frac{1}{t}(C(K_+) - C(K_-)) = 0$.

# References


[As] *A. Asanau.* A simple proof that connected sum of ordered oriented links is not well-defined, Math. Notes, 2018.

[Be07] * *M. Bestvina.* Knots: a handout for Summer Math Program. http://www.math.utah.edu/~bestvina/HS07/knots-hs07.pdf.

[CDM12] * *S. Chmutov, S. Duzhin, J. Mostovoy.* Introduction to Vassiliev knot invariants, Cambridge Univ. Press, 2012. http://www.pdmi.ras.ru/~duzhin/papers/cdbook.

[CG83] *J. H. Conway and C. M. A. Gordon*, Knots and links in spatial graphs, J. Graph Theory 7 (1983), 445–453.

[Ka06] *L.H. Kauffman.* Formal Knot Theory, Dover Publications, 2006.

[Ka06'] *L.H. Kauffman.* Remarks on Formal Knot Theory, arXiv:math/0605622 (note that the numbers and names of sections in table of contents i p. 1 are incorrect).

[Ka87] *L.H. Kauffman.* On knots. Annals of Mathematics Studies. 115. Princeton University Press. 1987.

[Pr95] * *V. V. Prasolov.* Intuitive topology. Amer. Math. Soc., Providence, R.I., 1995.

[PS96] * *V. V. Prasolov, A. B. Sossinsky* Knots, Links, Braids, and 3-manifolds. Amer. Math. Soc. Publ., Providence, R.I., 1996.

[Sc] *M. Scharlemann.* Crossing changes, http://web.math.ucsb.edu/~mgscharl/crossings.pdf.

[So89] * *A. Sossinsky.* Knots, links and their polynomials. Kvant, 1989, N4, http://kvant.mccme.ru/1989/04/uzly_zacepleniya_i_ih_polinomy.htm.

[Sk15p] *A. Skopenkov.* Classification of knotted tori, Proc. A of the Royal Society of Edinburgh, to appear, arxiv: 1502.04470.

[Sk16c] * *A. Skopenkov,* Embeddings in Euclidean space: an introduction to their classification, http://www.map.mpim-bonn.mpg.de/Embeddings_in_Euclidean_space:_an_introduction_to_their_classification, submitted to Boll. Man. Atl.

[Sk16i] * *A. Skopenkov,* Isotopy, submitted to Boll. Man. Atl. http://www.map.mpim-bonn.mpg.de/Isotopy.

[Sk18] * *A. Skopenkov.* Invariants of graph drawings in the plane, arXiv:1805.10237.

[Sk14] * *A. Skopenkov.* Realizability of hypergraphs and Ramsey link theory, arxiv:1402.0658.

[Sk] * *A. Skopenkov.* Algebraic Topology From Algorithmic Viewpoint, draft of a book, mostly in Russian, http://www.mccme.ru/circles/oim/algor.pdf.

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