Invariants of graph drawings in the plane

presented by A. Enne, A. Ryabichev, A. Skopenkov and T. Zaitsev *

Contents

The main results of this text are

• polynomial algorithm for recognition planarity of graphs (Proposition 1.3.6) and explanation how to invent it (§2.1);

• Radon, Tverberg and Özaydin theorems for the plane (linear and topological) 1.1.1.c, 1.1.5.c, 2.2.1, 3.2.2, 3.1.4.d, 3.2.5 and their elementary proofs (§§1.1, 1.1, 3.2; for Tverberg and Özaydin in particular cases).

The elementary proofs we present do not involve configuration spaces and cohomology obstructions. However, the main contents of this text is *introduction into algebraic topology* (more precisely, into the theories of configuration spaces and cohomology obstructions) *motivated by algorithmic, combinatorial and geometric problems.* We shall introduce some ideas of solution of Topological Tverberg Conjecture posed in 1966 and finally solved in 2015, see surveys [BZ, Sk16].

*We are grateful to I. Bogdanov, G. Chelnokov, R. Karasev, Yu. Makarychev and M. Tancer for useful discussions, and to E. Morozov, A. Shchepinova and E. Streltsova for translation of certain parts of this text.

A. Enne: Petrozavodsk State University.

T. Zaitsev: Higher School of Economics.

A. Ryabichev: Higher School of Economics, Independent University of Moscow.

A. Skopenkov: Moscow Institute of Physics and Technology, Independent University of Moscow. Supported in part by RFBR, Grants No. 15-01-06302, the D. Zimin Dynasty foundation, and the Simons-IUM fellowship. Email: skopenko@mccme.ru. http://www.mccme.ru/~skopenko.

The above two directions are linked together by the notion of *the van Kampen number*, whose generalizations is required for both. Idea of a more general *van Kampen obstruction (=invariant)* is necessary to find a polynomial algorithm for recognition of planarity and to formulate the Özaydin theorem.

No specific knowledge is required to solve these problems. All necessary definitions are presented here. But you would need some cleverness, i.e. mathematical culture (which will be improved as a result of solving these problems).

We shall present some beautiful research problems whose solutions we do not know.

$$\begin{tikzcd}
1.2 \arrow[r] & 1.3 \arrow[r] & 1.5 \arrow[r] \arrow[drr] & 2.1 \arrow[r] & 3.1 \\
1.4 \arrow[r] \arrow[urr] & 2.2 & 1.1 \arrow[l, bend right] \arrow[r] & 2.3 \arrow[r] & 3.2
\end{tikzcd}$$

Here is the diagram of (the most essential) dependence of the sections. Therefore, you can start with 1.1, 1.2 or 1.4.

Important statements are named 'Theorems', 'Lemmas' etc. These are also problems to solve, unless the opposite is explicitly indicated. We usually *formulate* the result before its *proof*.[1] In this case the subsequent problems are required to prove the statement. This is always explicitly indicated in the hints or right in the text.

Conventions

If a problem is a statement then a proof of this statement is required in this problem. If a problem is marked with a star (like $(b)^*$), then the problem is more complicated. You can postpone its solution until solving other problems.

For every written solution that the jury mark withs either '+' or '+.' a student (or a group of students) gets a 'bean'. The jury may award extra beans for beautiful solutions, solutions of hard problems, or (some) solutions typeset in TEX. The jury has infinitely many beans. One may submit a solution in the oral form, giving a bean with each 5 attempts (successful or not).

If you are stuck on a certain problem, we suggest to look at the next ones. They may turn out to be helpful. We suggest to all the students working on the project to *consult* the jury on any questions on the project. Students who brilliantly work on the project will get several *extra problems.*

Please let us know if you are familiar with some of more basic problems. If you confirm this familiarity by telling us rigorous proofs of some of them, you would be allowed not to receive plus-marks for all of them.

[1] Often the opposite happens, when the formulations of beautiful results or important problems, for which the theory is developed, are given *after* long time of learning this theory (or not given at all). This approach promotes the wrong view that mathematics studies unmotivated notions and theories. Such a promotion belittles the value of mathematics.

1 Problems before the semifinal

1.1 Radon and Tverberg theorems for the plane

1.1.1. (a) There exist 4 points in the plane such that for any their decomposition into two pairs the segment joining the points of the first pair does not intersect the segment joining the points of the second pair.

(b) There exist 4 points in the plane such that none of them belongs to the triangle with vertices at the others.

(c) **Radon theorem for the plane.** For any 4 points in the plane either one of them belongs to the triangle with vertices at the others, or they can be decomposed into two pairs such that the segment joining the points of the first pair intersects the segment joining the points of the second pair.

The *convex hull* of a finite set of points in the plane is the smallest convex polygon that contains them.

The Radon theorem can be reformulated as follows: *any* 4 *points in the plane can be decomposed into two disjoint sets whose convex hulls intersect.*

1.1.2. (Cf. Problem 2.2.2) For any 4 points in the plane such that no line contains more than 2 points there exists exactly one (Radon) decomposition of these points into two sets such that the convex hulls of these sets do not intersect.

1.1.3. (a) There exist 6 points in the plane which cannot be decomposed into three disjoint sets whose convex hulls have a common point.

(b) There exist 7 points in the plane such that for any their numbering from 1 to 7 point 1 does not belong to the triangles 234 and 567.

(c) There exist 7 points in the plane such that for any their numbering from 1 to 7 the intersection point of the segments 12 and 34 does not belong to the triangle 567.

1.1.4. (a) The vertices of any convex octagon can be decomposed into three disjoint sets whose convex hulls have a common point.

(b)* The same problem with octagon replaced by heptagon.

(c) Any 15 points in the plane can be decomposed into three disjoint sets whose convex hulls have a common point.

(d)* The same problem with 7 points instead of 15.

Hints for problems 1.1.4.d and 1.1.5.c are given in problem 2.3.8.

1.1.5. (a) For any r there exist $3r-3$ points in the plane which cannot be decomposed into three disjoint sets whose convex hulls have a common point.

(b) For any r there exist N such that any N points in the plane can be decomposed into r disjoint sets whose convex hulls have a common point.

(c)* **Tverberg theorem for the plane.** For any r and any $3r-2$ points in the plane can be decomposed into r disjoint sets whose convex hulls have a common point.

1.2 Linear realizations of graphs

In this and the following subsections we present two formalizations for the notion of realizability of graphs in the plane. Both formalizations are important; the second one uses the first one. (The formalizations turn out to be equivalent by Fary Theorem 1.3.4; their higher-dimensional generalizations are not.)

By *k points in the plane* we mean a k-element subset of the plane; so these k points are assumed to be pairwise distinct.

Proposition 1.2.1. [2] *(a) From any 5 points in the plane one can choose two disjoint pairs such that the segments with the vertices at these pairs intersect.*

Moreover, if no 3 of these points belong to a line, then the number of intersection points of interiors of segments joining the points is odd.[3]

(b) Two triples of points in the plane are given. Then there exist two intersecting segments without common vertices and such that each segment joins the points from distinct triples.

Proposition 1.2.1 is easily proved by analyzing the convex hull of the points.

Theorem 1.2.2 (General Position). *For each n there exist n points in 3-space such that the segments joining the points have disjoint interiors.*

1.2.3. For any five points in the plane numbered 1, 2, 3, 4, 5 if the segments

(a) jk, $1 \le j < k \le 5$, $k \ne 2$, have disjoint interiors then the points 1 and 2 lie on opposite sides of the triangle 345, cf. figure 1, right;

(b) jk, $1 \le j < k \le 5$, $(j,k) \notin \{(1,2),(1,3)\}$, have disjoint interiors then
EITHER the points 1 and 2 lie on opposite sides of the triangle 345,
OR the points 1 and 3 lie on opposite sides of the triangle 245.

(c) jk, $1 \le j < k \le 5$, $(j,k) \notin \{(1,2),(1,3),(1,4)\}$, have disjoint interiors then
EITHER the points 1 and 2 lie on opposite sides of the triangle 345,
OR the points 1 and 3 lie on opposite sides of the triangle 245,
OR the points 1 and 4 lie on opposite sides of the triangle 235.

(d) Oups... You have already guessed how this problem is formulated and how boolean functions appear in the study of embeddings.

Cf. problem 1.5.6. This illustrates some ideas of NP-hardness of recognizing realizability of 2-hypergraphs in $\mathbb{R}^4$ [Sk', §4].

A **graph** (V,E) is a finite set V together with a collection $E \subset \binom{V}{2}$ of two-element subsets of V.[4] The elements of this finite set are called *vertices*. The selected pairs of vertices are called *edges*.

A *complete graph* is a graph in which every pair of vertices is connected by a unique edge. The complete graph on n vertices is denoted by K_n. A *complete bipartite graph* with two parts consisting of n and m vertices respectively is denoted by $K_{m,n}$: in this graph for any

[2] These are 'linear' versions of the nonplanarity of the complete graph K_5 on 5 vertices and the bipartite graph $K_{3,3}$, see fig. 1, left. But they can be proved easier because the Parity Lemma 1.4.4 is not required for the proof.

[3] The first sentence indeed follows by the 'moreover' part: put the points in general position by a small shift so that no intersection points of segments with disjoint vertices are added. (Or else, for non-general-position points Proposition 1.2.1 is obvious: if points A, B, C among given 5 points belong to one line, B between A and C, and D is any other given point, then segments AC and BD intersect).

[4] The commonly term is *a graph without multiple edges or loops* or *a simple graph.*
The graphs G_1 and G_2 are called *isomorphic* if there is a 1–1 correspondence $f : V_1 \to V_2$ between the set V_1 of vertices of G_1 and the set V_2 of the vertices of G_2 such that *vertices $A, B \in V_1$ are adjacent in G_1 if and only if $f(A), f(B) \in V_2$ are adjacent in G_2.*

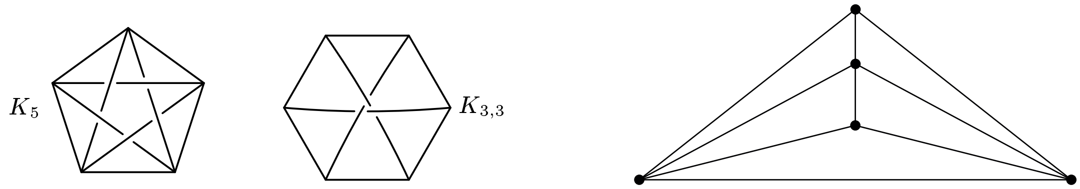

Figure 1: (Left) Nonplanar graphs K_5 and $K_{3,3}$.
(Right) A planar drawing of K_5 without one of the edges.

two vertices from different parts there is the edge between them and every edge connects the vertices from different parts. See Fig. 1.

Informally speaking, a graph (V, E) is called **linear realizable** in $\mathbb{R}^d$ (e.g. in the plane or in 3-space) if there exists a planar drawing without self-intersection of this graph such that every edge is drawn as line segment. Formally, a graph (V, E) is called **linear realizable** in $\mathbb{R}^d$ if there exists an embedded set of segments in $\mathbb{R}^d$ whose vertices correspond to V and whose segments correspond to E such that this segments have disjoint interiors. Such a set is called a *linear realization* of the graph.

The following results are classical:

- K_4 and K_5 without one of the edges are linearly realizable in the plane (figure 1, right).
- neither K_5 nor $K_{3,3}$ is linearly realizable in the plane (Proposition 1.2.1);
- every graph is linearly realizable in 3-space (General Position Theorem 1.2.2).

1.2.4. (a) There is an algorithm for recognition of linear realizability of graphs in the plane.

(Rigorous definition of the notion of algorithm is complicated, we do not give it here. We will accept your solutions based on intuitive understanding of algorithms. To be more precise, (a) means that there is an algorithm for calculating the function that for any graph checks whether the graph is linearly realizable in the plane or not. All other statements on algorithms in this paper can be formalized analogously.)

(b) Give an upper bound of the number of steps in your algorithm for given number of vertices in the graph. [5]

For a solution the following two problems will be useful.

A finite set of points in the plane is *in general position* if no three of these points lie on one line and no three line segments with ends at these points have a common interior points.

1.2.5. (a) If a graph is linear realizable in the plane, then there is a linear realization whose vertices are in general position.

(b) Into how many parts the plane is split by n general position lines (i.e. every two lines are not parallel and for every three lines they do not have a common point)?

Let A, A' and M be two points and a subset in the plane. Sets $M \cup A$ and $M \cup A'$ are called *elementary isotopic* if the segment AA' is disjoint with any line passing through some two points of M. Two subsets of the plane are called *isotopic* if they can be joined by a sequence of subsets, in which every two consecutive subsets are elementary isotopic.

1.2.6. (a) For every n there is a finite number of n-element subsets of the plane such that every n-element subset of the plane is isotopic to one of them.

(b)* (Riddle) Give an upper bound of the number of this n-element subsets.

[5]The 'complexity' in the number of edges is 'the same' as the complexity in the number of vertices, because for a planar graph with n vertices and e edges we have $e \leq 3n - 6$ and there are planar graphs with n vertices and e edges such that $e = 3n - 6$.

A criterion for linear realizability of graphs in the plane follows from the Fary Theorem 1.3.4 below and any planarity criterion (e.g. Kuratowski Theorem 1.3.3 below).

1.3 Main results on graph planarity

Informally speaking, a graph is called planar if this graph can be drawn in the plane without 'self-intersections'. Formally, a graph is called **planar** (or piecewise linear realizable in the plane) if there exists a set of polygonal lines in the plane, such that the end vertices of these polygonal lines correspond to the vertices of the graph, the polygonal lines correspond to the edges of the graph, and the interiors of the polygonal lines do not intersect.

For example, the graphs K_5 and $K_{3,3}$ (pic. 1) are not planar. The proof for K_5 is given in the proof of the (stronger) Proposition 1.5.1, see also Problem 1.5.5.

1.3.1. (a) There is an algorithm for recognition of planarity of graphs. (You can use without proof Kuratowski theorem 1.3.3 below.)

(b) Give an upper bound of the number of steps in your algorithm for given number of vertices in the graph.

(c)* There is an algorithm for recognition of planarity of graphs that is polynomial in the number of the vertices n in the graph, i.e. the number of steps in this algorithm does not exceed Cn^k for some numbers C and k. (Hint: see Proposition 1.3.6.)

Informally speaking, a *subgraph* of a given graph is a part of this graph. Formally, graph G is called the **subgraph** of graph H if any vertex of the graph G is the vertex of the graph H, and any edge of the graph G is the edge of the graph H. (Two vertices of the graph G connected by an edge in the graph H are not necessarily connected by an edge in the graph G.)

It is clear that any subgraph of a planar graph is planar.

The *subdivision of edge* operation for a graph is shown in the pic. 2. Two graphs are called *homeomorphic* if one can be transformed to the other using subdivisions of edges and inverse operations. This is equivalent to the existence of a graph that can be obtained from each of these graphs by subdivisions of edges. A motivation for this definition is given in the solutions.

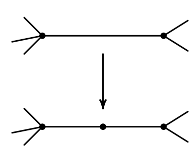

Figure 2: Subdivision of edge

1.3.2. (a) There is a graph such that the degree of any vertex of this graph is more than two, this graph does not have a subgraph isomorphic to K_5, but there is a subgraph of this graph homeomorphic to K_5.

(b) A graph G is planar if and only if there is some graph H such that G homeomorphic to H and H is linear realizable in the plane.

It is clear that homeomorphic graphs are planar or not simultaneously.

Theorem 1.3.3 (Kuratowski). *A graph is planar if and only if this graph have no subgraphs homeomorphic to K_5 or $K_{3,3}$ (Fig. 1).*

Theorem 1.3.4 (Fary). *A planar graph can be drawn in the plane such that all edges will be corresponded to segments. Formally, if a graph is planar (i.e. piecewise linear realizable in the plane), then it is linear realizable in the plane.*

It is not required to prove these theorems (for proofs see e.g. [**?**, Sk05, ST07]). But Kuratowski Theorem is allowed to prove other results.

A note on algorithms. (This note can be skipped without compromising the understanding of the rest of the material.)

Despite the beauty and simplicity of the Kuratowski and Fari theorems, the corresponding algorithms for recognizing graph planarity are slow. Therefore, other ways of recognizing planarity are of interest. Another (exponential) algorithm is based on consideration of *thickenings* [Sk15, §1]. There is even an algorithm which is linear in the number of vertices [BM04, HT74]. This linear algorithm is complicated. The following criterion 1.3.6.a is interesting because it gives polynomial algorithm.

1.3.5. (a) There are 5 cavaliers, ranked by nobility, and five their wives. Two cavaliers and the wife of the third one participate in a dance, but not all such trios are decent. Prove that for some two disjoint pairs of cavaliers the sum of the following three numbers is odd:

• the number of cavaliers from the first pair, lying between the cavaliers from the second pair on the scale of decency,

• the number of wives of the cavaliers from the first pair, who can dance with the second pair,

• the number of wives of the cavaliers from the second pair, who can dance with the first pair.

(Decency is the property of an unordered pair of cavaliers and a wife. For example, if A, B can dance with the wife of C and B, C can invite wife of A, it is not necessary that C, A can dance with the wife of B.)

(b) Analogous problem for 6 cavaliers of whom 3 are English and 3 are French, all 6 are ranked by nobility in an arbitrary way, and one Englishman, one Frenchman and wife of any of the remaining four cavaliers participate in the dance (but not all such trios are decent).

Proposition 1.3.6. *(a)* Suppose that an arbitrary ordering of the vertices of a graph is given. The graph is planar if and only if there are vertices $V_1, \dots, V_s$ and edges $e_1, \dots, e_s$ such that $V_i \not\in e_i$ for each $i = 1, \dots, s$, and for each non-adjacent edges x, y*

the number of ends of x, lying between ends of y (for the above ordering),

has the same parity as

the number of those $i = 1, \dots, s$ for which either $V_i \in x$ and $e_i = y$, or $V_i \in y$ and $e_i = x$.

(b) There is an algorithm for recognizing the solvability of a system of k linear equations with k variables, the algorithm polynomial in k.

The 'only if' part of (a) follows by the Kuratowski Theorem 1.3.3 and assertions 1.3.5. You will be able to prove the 'if' part of (a) after studying §2.1, where it is also explained how to invent the statement (it involves a reformulation of *the van Kampen cohomology obstruction*). Formally, the 'if' part of (a) follows by statements 2.1.1.b, 2.1.2 and 2.1.7.b.

Proposition 1.3.6.b is proved using *exclusion of variables*, see details in [CLR, Vi02].

1.4 Intersection number for polygonal lines in the plane

1.4.1. Take 14 general position points in the plane, of which 7 are red and another 7 are yellow. Can the number of intersection points of the red segments (i.e. the segments between the red points) with the yellow segments be equal to 7?

Hint. To solve the problem use the following fact: *For every 3 red and 3 yellow general position points in the plane the number of intersection points of the red segments with the yellow segments is even.*

If we prove the Jordan Theorem 1.4.3.b below for a triangle, this fact would follow because the outline of the yellow triangle comes *into* the red triangle as many times as it comes *out*. The following proof is easier and can be generalized to higher dimensions [SS].

The intersection of the convex hull of the red triangle and the outline of the yellow triangle is a finite union of polygonal lines (non-degenerate to points). The outlines of the triangles intersect at the endpoints of the polygonal lines. The number of endpoints is even, so the fact follows.

1.4.2. * (a) A closed non-self-intersecting polygonal line in the plane and two points outside this line are given. Find an algorithm checking if we can connect these points by a polygonal line which does not intersect the given polygonal line.

(b) The same question if we can see only the part of the polygonal line that lies in some square containing two given points.

To solve Problems 1.4.2.ab and 1.4.3.bc prove and use Parity Lemma 1.4.4. Only this lemma is used in the following sections.

A subset of the plane is called *connected*, if every two points of this subset can be connected by a polygonal path lying in this subset.

1.4.3. (a)* Every non-self-intersecting polygonal line separates the plane into no more than two parts.

(b) **Jordan Theorem.** *Every non-self-intersecting polygonal line L separates the plane, i.e. $\mathbb{R}^2 - L$ is not connected.* [6]

(c) Any two polygonal lines in square connecting its opposite vertices intersect.

Two polygonal lines in the plane are called *in general position*, if their vertices are in general position.

Lemma 1.4.4 (Parity)**.** *Any two general position closed polygonal lines in the plane intersect each other at an even number of points.* [7]

This lemma will be reduced to its particular case, which is proved in the hint to the problem 1.4.1. If one of the polygonal lines is a triangle, the lemma can be proved analogously to that particular case. We present a different proof of this 'intermediate' case, which generalizes to a proof of the general case. The proof is by reduction to the particular case, using *singular cone* idea which formalizes in a short way the *motion-to-infinity* idea [BE82, §5].

Proof of the Parity Lemma 1.4.4 for the case when one of the polygonal lines b is the outline of a triangle. Denote another polygonal line by a. Take a point A such that $\partial(AMN)$ and b are in general position for each edge MN of the polygonal line a. Denote by ∂T the outline of a triangle T. Then (see Fig. 3)

$$|a \cap b| = \sum_{MN} |MN \cap b| \underset{2}{\equiv} \sum_{MN} |\partial(AMN) \cap b| \underset{2}{\equiv} 0.$$

[6]If you deduce the Jordan Theorem from the Euler Formula, think how to prove the Euler Formula.

[7]This is not trivial because the polygonal lines may have self-intersections and because the Jordan Theorem 1.4.3.b is not obvious. It is not reasonable to deduce the Parity Lemma from the Jordan Theorem or the Euler Theorem because this will make a vicious circle.

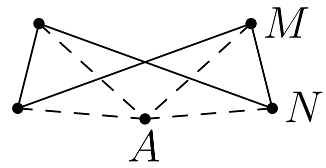

Figure 3: Singular cone idea

Here the summation is over edges MN of a, and the last congruence follows by the particular case of triangles. $\square$

1.4.5. (a) Take a closed polygonal line L in the plane. Take any point $A \not\in L$ and color it in the color 0. For any point $P \not\in L$ let the color 0 or 1 of P be the parity of the number of intersection points of a *general position* polygonal line (define what it is!) joining P to A. Prove that this coloring is well-defined.

(b) The complement to a closed polygonal line in the plane (with general position vertices) has a chess-board coloring (so that the adjacent domains have different colors, Fig. 4).

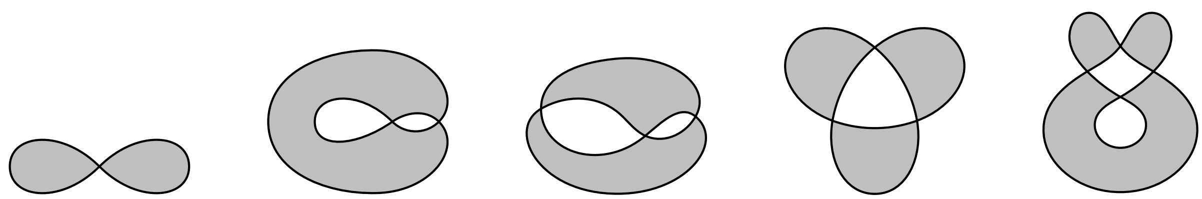

Figure 4: The modulo two interiors of some closed polygonal lines

The **modulo two interior** of a closed polygonal line in the plane is the union of black domains for a chess-board coloring (provided the infinite domain is white).

1.5 Self-intersection invariant for graph drawings

For some given graph we will consider its plane drawings such that the drawings of edges are polygonal lines (possibly with intersections). Let us explain this formally.

A linear map $f : K \to \mathbb{R}^2$ of a graph $K = (V, E)$ to the plane is a map $f : V \to \mathbb{R}^2$. *The image* $f(AB)$ *of the edge* AB is the segment $f(A)f(B)$.

A piecewise-linear map $f : K \to \mathbb{R}^2$ of a graph $K = (V, E)$ to the plane is a collection of (non-closed) polygonal lines corresponding to the edges of K, whose endpoints correspond to the vertices of K. *The image* $f(AB)$ *of the edge* AB is the corresponding polygonal line.

Clearly, a piecewise-linear map of a graph K to the plane is 'the same' that a linear map of some graph homeomorphic to K.

So a graph is planar if there exists its piecewise-linear map to the plane such that the images of all the edges of the graph are disjoint (except for their common endpoints) and non-self-intersecting polygonal lines.

Proposition 1.5.1. *(a) For every piecewise-linear map of the graph* K_5 *to the plane there exist two non-adjacent edges with intersecting images.*

(b) The same for a continuous map* $K_5 \to \mathbb{R}^2$.

To illustrate the *van Kampen obstruction* idea (see 1.3.6.a) consider the proof of the following 'quantitative version' of non-planarity of the graph K_5 (similar to the 'moreover' part of the Proposition 1.2.1.a). We will prove that for 'almost every' piecewise-linear map of K_5 to the plane the number of the intersection points of non-adjacent edges is odd. Think how to formalize the words 'almost every' and compare your formalization with the definition given below.

A linear map of a graph to the plane is called *a general position linear map* if the images of all the vertices are in general position. A piecewise-linear map f of a graph K to the plane is called **a general position map** if there exist a graph H homeomorphic to K and a general position linear map of H to the plane such that this map 'corresponds' to the map f.

1.5.2. For every general position map of a graph to the plane the images of any two non-adjacent edges intersect in a finite number of points.

For a general position map $f : K \to \mathbb{R}^2$ **the van Kampen number** (or the self-intersection invariant) $v(f)$ is the parity of the number of intersection points of the pairs of images of non-adjacent edges. Obviously, if K is planar then $v(f) = 0$ for some general position map $f : K \to \mathbb{R}^2$.

1.5.3. (a) There exists a general position map $f : K_5 \to \mathbb{R}^2$ such that $v(f) = 1$.

(b) If a graph K is a disjoint union of two cycles length 3, then $v(f) = 0$ for every general position map $f : K \to \mathbb{R}^2$.

(c) There exist a planar graph K and a general position map $f : K \to \mathbb{R}^2$ such that $v(f) \neq 0$.

Lemma 1.5.4. *For every general position map* $f : K_5 \to \mathbb{R}^2$ *the van Kampen number* $v(f)$ *is odd.*

1.5.5. Are the statements 1.5.1, 1.5.3.a and the Lemma 1.5.4 correct when we replace the graph K_5 by the graph $K_{3,3}$?

1.5.6. (a) Remove the edge joining the vertices 1 and 2 form the graph K_5. Then for every general position embedding of the obtained graph to the plane the images of the vertices 1 and 2 are separated by the image of the triangle 345. (This statement is meaningful because of Jordan Theorem 1.4.3.b.)

(b,c,d) State and prove analogue of problems 1.2.3.bcd for piecewise-linear embeddings.

Invariants of graph drawings in the plane

presented by A. Enne, A. Ryabichev, A. Skopenkov and T. Zaitsev

Hints and solutions for some problems before the semifinal[8]

1.1.1. (a) Consider a triangle and any point inside it.

(b) Consider a square.

(c) See proof of statement 1.1.2

1.1.2. The convex hull of 4 points in general position is either a triangle or a quadrilateral. In the case of a triangle, the Radon point is the point that is not a vertex of the convex hull. In the case of a quadrilateral, the Radon point is the intersection point of the diagonals.

1.1.3. (a) Consider a pair of points at each vertex of a triangle (or a similar set of distinct points).

(b) Consider a convex 7-gon.

(c) Consider an equilateral triangle ABC. Denote by O its center. Define points A_1, B_1, C_1 as the images of points A, B, C under homothetic transformation with the center O and the ratio $1/2$.

Let us show that this set of points satisfies the required condition. Observe that for each pair of intersecting segments there exists an isometry mapping from this pair to one of the following pairs: $\{AB_1, BA_1\}, \{AO, A_1B_1\}, \{AO, BA_1\}$. Note also that $A_1 = AO \cap A_1B_1 = AO \cap A_1B$ belongs neither to $\triangle BC_1C$ nor to $\triangle B_1C_1C$, and $AB_1 \cap BA_1$ does not belong to $\triangle\ OC_1C$.

1.1.4. (a) *(Written by N. Volkov.)* Consider an octagon with vertices denoted by $A_1 \ldots A_8$ (in clockwise order, see Fig. 5). Decompose the set of its vertices into three disjoint sets: $\{A_1, A_3, A_5, A_7\}$, $\{A_2, A_6\}$ and $\{A_4, A_8\}$. Convex hulls of these sets are quadrilateral $A_1A_3A_5A_7$ and segments A_2A_6 and A_4A_8, respectively. Let us show that these convex hulls have a common point.

Consider intersection of the segment A_2A_6 and the quadrilateral $A_1A_3A_5A_7$. Since the octagon is convex, the quadrilateral $A_1A_2A_3A_6$ is also convex. Segments A_2A_6 and A_1A_3 intersect because they are the diagonals of a convex quadrilateral. We obtain analogous assertion for the segments A_2A_6 and A_5A_7. Since our figures are convex, the intersection of A_2A_6 and $A_1A_3A_5A_7$ is a segment whose ends belong to sides A_1A_3 and A_5A_7 of the quadrilateral $A_1A_3A_5A_7$. For the same reason, the intersection of A_4A_8 and $A_1A_3A_5A_7$ is a segment whose ends belong to sides A_1A_7 and A_3A_5 of the quadrilateral $A_1A_3A_5A_7$. It follows from convexity of $A_1A_3A_5A_7$ that segments with ends on its opposite sides intersect. This proves that segments A_2A_6, A_4A_8 and the quadrilateral $A_1A_3A_5A_7$ have a common point.

(b) Consider a convex 7-gon $A_1 \ldots A_7$. Then draw a segment A_4A_7. Define points A, B, C, D, E, F as the intersection points of segments A_5A_3, A_5A_2, A_5A_1, A_6A_3, A_6A_2, A_6A_1 with the segment A_4A_7, respectively. Let us consider the following two cases.

The case when points A, B, C, D, E, F lie on $\overrightarrow{A_4A_7}$ in this very order (see Fig. 6). In this case define X as $A_1A_5 \cap A_3A_6$. Let us prove that X belongs to $\triangle\ A_2A_4A_7$. Since the point C lies between points A_4 and D, we must also have that the point X lies on the same side as point A_1 with respect to the segment A_4A_7. But points X and A_3 lie on the different sides with respect to A_2A_4 because of convexity of the 7-gon. The same argument applies to A_2A_7. Hence point X belongs to $\triangle A_2A_4A_7$.

[8]The computer versions of Figures 3, 4 and 10 was prepared by I. Derkach, V. Kovirshina and Yu. Tikhonov.

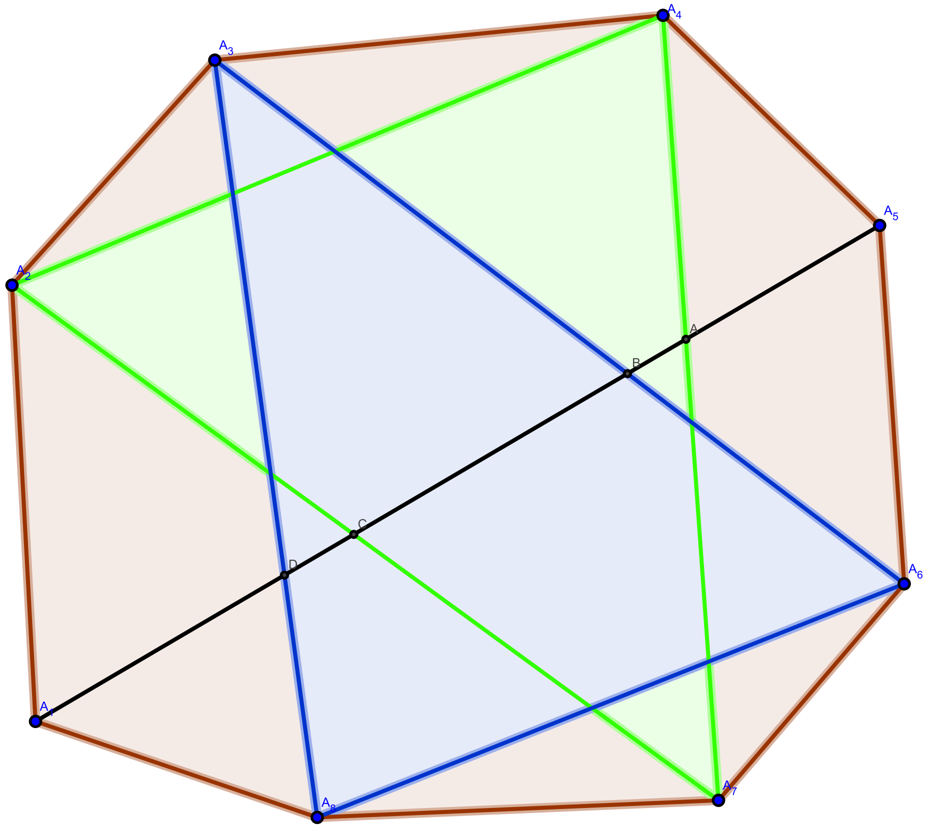

Figure 5: The convex octagon.

The case when points A, B, C, D, E, F lie on $\overrightarrow{A_4A_7}$ in any other order. It is obvious that points A, B, C lie on $\overrightarrow{A_4A_7}$ in this very order. Similar assertion is true for points D, E, F and for pairs of points (A, D), (B, E), (C, F). If point C lies between D and E, then point C belongs to $\triangle A_2A_3A_6$. Therefore we can assume that point C lies between E and F. For the same reason point D lies between A and B. This implies that points lie on $\overrightarrow{A_4A_7}$ in order $ADBECF$. Then, point E lies between points A and C, therefore it belongs to $\triangle A_1A_3A_5$.

(c) Let Z be the given set of 15 points. Consider the vertices of the convex hull of Z. If the number of the vertices is at least eight, then we can decompose these points into three sets using (a).

Otherwise, if convex hull consists of less than 8 points, then let the first set of our decomposition S_1 be the vertices forming a boundary of the convex hull of set Z. Since at least 4 points are left, it follows that we can decompose them into two sets S_2, S_3 with intersecting convex hulls. The intersection point belong to the convex hull of S_1.

1.1.5. (a) *The first solution.* Suppose the contrary. Take a convex $(3r-3)$-gon such that no three of its diagonals have a common point. Among the decomposition sets of its vertices there is either a set consisting of 1 vertex, or 3 sets consisting of 2 vertices. In the first case we obtain a contradiction because the polygon is convex. In the second case we obtain a because the vertices are in general position.

The second solution. Consider $r-1$ points at each vertex of a triangle or a similar set of distinct points.

(b) (*Solution for $N = 9r+1$, written by I. Bogdanov.*) Consider convex hulls for each $6r+1$ of required points. Any three of these convex hulls intersect because they have a common point. Therefore by the Helly theorem they all have a common point (not necessary one of the preceding points). Denote by O this common point. Let us prove that O a common

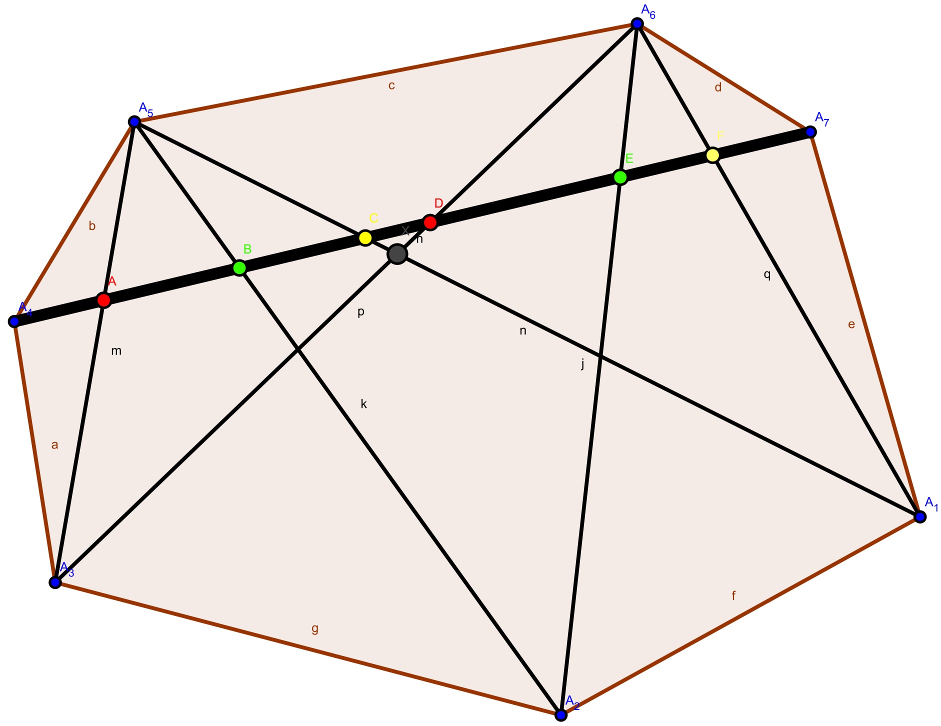

Figure 6: The convex heptagon.

point of our polygons.

(The following elegant property of O we will not use: on both sides of any line passing through O there is at least $3r+1$ of the given points.)

Point O belongs to the convex hull of any $6r+1$ of these given points. In other words, from any $6r+1$ of the given points we can choose three points such that the triangle with vertices at this points contains point O. Let us choose this triangle, remove it, choose new triangle, etc. This can be done at least r times.

1.2.1. (a) Suppose to the contrary that there exists 5 points $OABCD$ in the plane such that one cannot choose two required disjoint pairs. Hence $A \notin OB$ and $B \notin OA$. Then the point A does not belong to the ray OB. For this reason we can assume that the points A, B, C, D are seen from the point O in this order. Then the triangles OAC and OBD intersect at a unique point O. Therefore using the following modification of the Parity Lemma 1.4.4 we get that $AC \cap BD \neq \emptyset$, which is a contradiction.

Assume that the outlines of two triangles in the plane intersect at a unique point, and a line sufficiently close to this point intersects the outlines by four distinct points. Then the points X, Y corresponding to one triangle are unlinked with the points Z, T corresponding to the other triangle, i.e. the segment XY contains either both or none of the points Z, T.

Proof. Denote the point by O and the triangles by $OX'Y'$ and $OZ'T'$ (so that X, Y, Z, T are the intersection points of the line and OX', OY', OZ', OT' respectively). Let $a := \partial(OX'Y')$ and $b := \partial(OZ'T')$. The ststement follows because

$$|XY \cap \{Z,T\}| = |XY \cap b| = |\partial(OXY) \cap b| - 1 \underset{2}{\equiv} |a \cap b| + |\partial(XYY'X') \cap b| - 1 \underset{2}{\equiv} 0.$$

Here the last congruence holds because $|a \cap b| = 1$ and by the Parity Lemma 1.4.4 (in fact, we need the case when b is a triangle).

(b) Analogously to (a).

1.2.2. Choose three points in 3-space that do not belong to one line. Suppose that we have $n \geq 3$ points in general position. Then there is a finite number of planes containing triples of these n points. Hence there is a point that does not lie on any of these planes. Add this point to our set of n points. Since the 'new' point is not in one plane with any three of the 'old' n points, the obtained set of $n+1$ points is in general position. Thus *for each n there exist n points in 3-space that are in general position.*

Take such n points. Denote by A the set of all segments joining pairs of these points. If some two segments from A with different endpoints intersect, then four endpoints of these two segments lie in one plane. If some two segments from A with common endpoint intersect not only at their common endpoint, then the three endpoints of these two segments are on one line. So we obtain a contradiction.

1.2.3. (a) Any five points can be transformed into five points in general position leaving the required properties unchanged. By hypothesis, the number of intersection points of segment 12 and the outline of triangle 345 equals to the number of intersection points of interiors of segments joining the points. This number is odd by Proposition 1.2.1.a.

(b,c,d) Analogously, the assertion follows by Proposition 1.2.1.a.

1.2.4, 1.2.5, 1.2.6. [Ta, Chapters 1 and 6].

Answer for 1.2.4.b: the number of steps is more than exponential.

1.3.1. (a) See details in [Ta, Chapters 1 and 6]. *Another algorithm* can also be obtained using proposition 1.2.4.a and the Fary theorem.

(b) The number of steps is more than exponential time [Ta, Chapters 1 and 6].

(c) This fact follows from Proposition 1.3.6.a,b.

1.3.2. (a) Take 7 vertices 1234567. Suppose that each two of the vertices 12345 except the pairs $\{3,4\}$ and $\{4,5\}$ are connected by an edge, 6 is connected with $\{3,4,7\}$ and 7 is connected with $\{4,5,6\}$. There are only 5 vertices having degree ≥ 4 in this graph, but they does not form a complete graph. But if we remove the edge 67, then the obtained graph will be homeomorphic to K_5.

(b) The statement follows by the definition of planarity. If the graph is planar, then every edge is presented by a polygonal line. Define a new graph as follows: the vertices of a new graph correspond to the vertices of the polygonal line, and the edges of a new graph correspond to the edges of the polygonal line. The proof of the converse implication is analogous.

1.3.5. Let us sum the considered sums over all 15 unordered pairs of disjoint pairs of cavaliers. Prove that this larger sum is odd for any choice of decent trios.

1.4.1. The answer is 'no', because the number of the intersection points is even.

Proof that the number is even. (Analogously to the proof of the Parity Lemma 1.4.4.) Denote the yellow points by $A_1, \ldots, A_7$ and the red points by $B_1, \ldots, B_7$. Take two points C and D in the plane such that all the 16 points are in general position. Then

$$0 \underset{2}{\equiv} \sum_{i<j,\, k<l} |\partial(CA_iA_j) \cap \partial(DB_kB_l)| \underset{2}{\equiv} \sum_{i<j,\, k<l} |A_iA_j \cap B_kB_l|.$$

Here the first equality follows from the fact in the hint, and the second one holds because each of the edges CA_i and DB_j belongs to six triangles, so the intersection points lying on these edges appear in the first sum an even number of times.

1.4.2. Denote the points by A and B. The algorithm is the following.

Draw all the lines passing through A, B and each of the vertices of the given polygonal line. Take a point C outside these lines. Calculate the number of intersection points of the polygonal line ACB with the given polygonal line. The answer is "we can" if this number is even, and "we cannot" otherwise.

If the number is odd the proof is analogous to the one of the Problem 1.4.5. Denote the given polygonal line by L. Suppose there exists a polygonal line L_1 joining A to B such that L and L_1 are disjoint. Replace L_1 by a close polygonal line L_1' such that the vertices of L_1', the vertices of L_2 and the point C are in general position. The word 'close' means that the vertices of L_1' correspond to the vertices of L_1 and the distance between the corresponding vertices is less than sufficiently small ε. Denote the endpoints of L_1' by A' and B'. If we choose ε sufficiently small, then L_1' and L are disjoint and polygonal lines ACB and $A'CB'$ intersect L at the same number of points. Then, applying the Parity Lemma 1.4.4 to the closed polygonal lines $A'CB' \cup L_1'$ and L, we obtain a contradiction, because these polygonal lines intersect each other at an odd number of points.

If the number is even, the proof is analogous to the one of the Problem 1.4.3.a.

1.4.3. (a) [BE82, §6].

(b,c) See Problem 1.4.5.

1.4.4. The general case is reduced to the particular case considered in §1.4 analogously to the above reduction of the particular case of two triangles. Just replace b by the second polygonal line.

Remark. The following generalization of the Parity Lemma 1.4.4 is true. A *1-cycle (modulo 2)* is a finite collection of (non-closed) polygonal lines in the plane such that each point of the plane is the end of an even number of them. Then any two *general position* (define!) 1-cycles in the plane intersect at an even number of points. The proof is obtained from the proof of the Parity Lemma 1.4.4 by replacing 'the closed polygonal lines' by '1-cycles'.

1.4.5. (a) Suppose there are two general position polygonal lines L_1 and L_2 joining A to P. Replace L_1 by a close polygonal line L_1' such that the intersection points of L_1' and L_2 do not coincide with the self-intersection points and the vertices of L_1' and L_2, except for their common endpoints. If we choose ε sufficiently small, then L_1' and L_1 intersect L at the same number of points. Then, applying the Parity Lemma 1.4.4 to the closed polygonal lines L and $L_1' \cup L_2$, we conclude that $|L \cap L_2| \underset{2}{\equiv} |L \cap L_1'| = |L \cap L_1|$.

(b) When the polygonal line L_1 passes to the adjacent domain, the number of the intersection points considered in (a) increases by 1, therefore the adjacent domains have different colors.

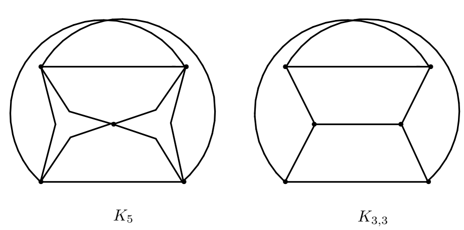

Figure 7: "Almost embedding" $K_5 \to K_{3,3}$

1.5.1. (a) This follows from the Lemma 1.5.4.

Note. Another proof of non-planarity of $K_{3,3}$ follows by Statement 1.5.1.a and Fig. 7.

1.5.2. The image of every edge is a polygonal line, i. e. a finite union of segments. Every two segments in general position intersects at a finite number of points.

1.5.3. (a) Consider the regular pentagon with the diagonals.
(b) This statement follows from the Parity Lemma 1.4.4.
(c) Take two segments with a common interior point.

1.5.4. Suppose the maps at general position $f, f' : K_5 \to \mathbb{R}^2$ are given. Assume these maps are in general position *relative to each other*, cf. the proof of linear analogue 1.2.1.a of Lemma 1.5.4. By statement 1.2.1.a, it suffices to prove the Lemma in the particular case when the maps f and f' coincide on all the edges except one edge σ, see Fig. 8. All the edges of K_5 non-adjacent to σ form a cycle, denote it by Δ (this very property of the graph K_5 is necessary to the proof). Therefore by the Parity Lemma 1.4.4 we have

$$v(f) - v(f') = |(f\sigma \cup f'\sigma) \cap f\Delta| \mod 2 = 0.$$

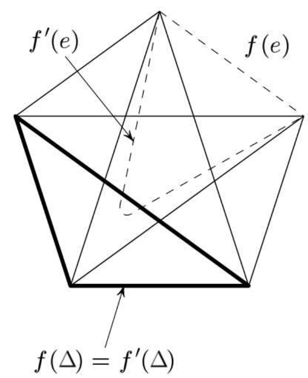

Figure 8: The residue $v(f)$ is independent of f

1.5.5. Analogously to Problem 1.5.1 and Lemma 1.5.4.

1.5.6. Analogously to Problem 1.2.3.

Invariants of graph drawings in the plane

presented by A. Enne, A. Ryabichev, A. Skopenkov and T. Zaitsev

2 Problems after the semifinal

2.1 Polynomial algorithm for recognition of planarity

The idea is to consider natural object (intersection sufficiently small) for any general position map from a graph to the plane, to investigate how this object depends on the map, and so to derive from this object an obstruction to planarity which is independent of the map.

A graph is called $\mathbb{Z}_2$**-planar** if there exists a general position embedding of this graph in the plane such that the number of intersection points of any two non-adjacent edges is even.

2.1.1. (a) The graph obtained from K_5 by the subdivision of an edge is not $\mathbb{Z}_2$-planar.

(b) *Hanani-Tutte Theorem.* A graph is planar if and only if it is $\mathbb{Z}_2$-planar. (Hint: use the Kuratowski Theorem 1.3.3.)

2.1.2. A graph and an arbitrary ordering of its vertices are given. Then there exists a general position embedding of this graph into the plane such that for any pair of non-adjacent edges σ, τ the number of intersection points of their images has the same parity as the number of ends of σ that lie between the ends of τ.

Let $f : K \to \mathbb{R}^2$ be a general position embedding of a graph K. Take any pair of non-adjacent edges σ, τ. By assertion 1.5.2 the intersection $f\sigma \cap f\tau$ consists of a finite number of points. Assign to the unordered pair $\{\sigma, \tau\}$ the residue

$$|f\sigma \cap f\tau| \mod 2.$$

Denote by K^* the set of all unordered pairs of non-adjacent edges of the graph K. The obtained map $K^* \to \mathbb{Z}_2$ is called **intersection cocycle**. In other words, we obtained a 'partial matrix', i. e., a symmetric arrangement of zeroes and units in some cells of a $e \times e$-matrix, where e is the number of edges of the graph K. Those cells of this matrix correspond to the pairs of non-adjacent edges.

2.1.3. (a) Find the intersection cocycle for a linear general position map $K_4 \to \mathbb{R}^2$ such that the images of vertices form a convex quadrilateral.

(b) The same for the graph K_5.

2.1.4. Suppose $f : K \to \mathbb{R}^2$ is a general position map of a graph K and σ, τ is a pair of non-adjacent edges.

(a) For any pair $(x, y) \in \partial(\sigma \times \tau) := (\partial\sigma \times \tau) \cup (\sigma \times \partial\tau)$ we have $f(x) \neq f(y)$.

(b)* The number $|f\sigma \cap f\tau|$ has the same parity as the number of rotations of the vector $f(x) - f(y)$ while one goes around the border $\partial(\sigma \times \tau)$ of the rectangle $\sigma \times \tau$.

2.1.5. How does the intersection cocycle change

(a) if in Problem 2.1.2 we change the order of vertices?

(I-V) under the Reidemeister moves in Fig. 9.I-V? (The graph drawing changes in the disk as in Fig. 9.I–V, while out of this disk the graph drawing remains unchanged. Drawings of all edges distinct from the pictured edges do not intersect the disk. The vertex A is not an end of the edge τ in Fig. 9.V.)

Suppose A is a vertex which is not an end of an edge σ. **An elementary coboundary** of the pair (A, σ) is the map $\delta(A, \sigma) : K^* \to \mathbb{Z}_2$ which assigns 1 to any pair $\{\sigma, \tau\}$ with $\tau \ni A$, and 0 to any other pair. The answer to Problem 2.1.5.V: $\delta(A, \sigma)$ is added to the intersection cocycle (componentwise and modulo 2).

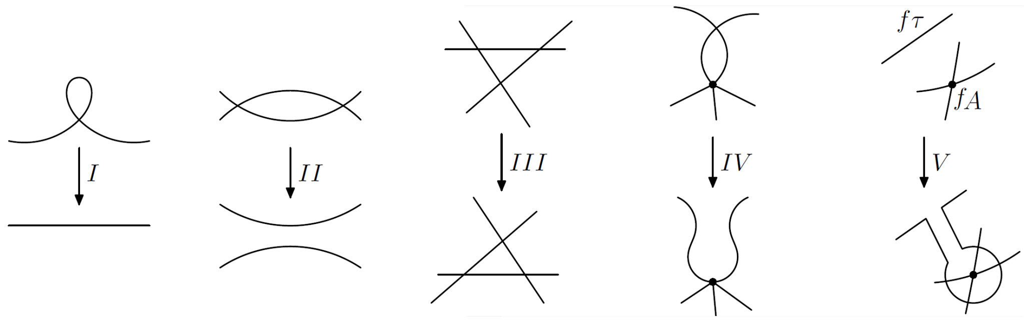

Figure 9: The Reidemeister moves for graphs in the plane.

2.1.6. (a) Find elementary coboundary of every pair (A, σ) for the graph K_4.

(b) The same for the graph K_5.

(c) Is the intersection cocycle of Problem 2.1.3.a an elementary coboundary? Is it the sum of elementary coboundaries?

(d) The same for the graph K_5, i.e., for Problem 2.1.3.b.

Two maps $\nu_1, \nu_2 : K^* \to \mathbb{Z}_2$ are called **cohomological** if

$$\nu_1 - \nu_2 = \delta(A_1, \sigma_1) + \ldots + \delta(A_k, \sigma_k)$$

for some vertices $A_1, \ldots, A_k$ and edges $\sigma_1, \ldots, \sigma_k$ (not necessarily distinct).

The answer to Problem 2.1.5 and the following Lemma 2.1.7 show that cohomology is the equivalence relation generated by changes of graph drawing.

2.1.7. (a) **Lemma.** The intersection cocycles of different plane drawings of the same graph are cohomological.

(b) A graph is $\mathbb{Z}_2$-planar if and only if the intersection cocycle of some (or equivalently, of any) general position map of this graph to the plane is cohomologous to the zero cocycle.

(In view of the Hanani-Tutte Theorem this gives a graph planarity criterion. By Proposition 2.1.2 this criterion is equivalent to 1.3.6.a.)

2.2 The topological Radon theorem for the plane

2.2.1. The topological Radon theorem for the plane. For any general position map $f : K_4 \to \mathbb{R}^2$ either

• the images of certain non-adjacent edges intersect, or

• the image of certain vertex belongs to the interior modulo 2 (see definition in §1.4) of the image of the cycle formed by those three edges that do not contain this vertex.

Proposition 2.2.1 follows from Propositions 2.2.2.ab, cf. assertion 1.1.2 and [Sk17'].

2.2.2. For any general position map $f : K_4 \to \mathbb{R}^2$ paint in red

• the intersections points of the images of non-adjacent edges, and

• those images of the vertices of K_4 that belong to the interior modulo 2 of the image of the cycle formed by three edges that do not contain this vertex.

(a) Mark the red points for a map f of your choice.

(b) The parity of the number of red points does not depend on f.

(c) The number of red points (i.e. of *Radon partitions*) is odd for any f.

Proposition 2.2.1 can be reformulated as follows, using the concept of continuous map of the circle to the plane (see Propositions 3.2.2 (4), (6)), instead of the notion of interior modulo 2.

2.2.3. (a) *The topological Radon theorem for the line.* For any continuous map of a triangle to the line the image of certain vertex belongs to the image of the opposite edge.

(b)* *The topological Radon theorem for the plane.* For any continuous map of a tetrahedron to the plane either

- the images of certain opposite edges intersect, or
- the image of certain vertex belongs to the image of the opposite facet.

(c)* If in (b) we replace 'continuous' by 'piecewise linear' (define this notion!), we obtain an equivalent assertion.

2.3 Toward Tverberg Theorem for the plane

In this subsection we shall show how to prove Tverberg Theorem (1.1.4.d and 1.1.5.c). This idea will actually work for the topological Tverberg Theorem (§3.2). You will prove assertion 1.1.4.d in the following stronger form.

2.3.1. * Any seven enumerated points $1, \ldots, 7$ in the plane can be decomposed (even in two ways) into three sets such that their convex hulls have a common point and none of this sets contains any of the sets $\{1, 6\}, \{3, 4\}, \{2, 5\}$.

For a subset $X \subset [7] := \{1, 2, 3, 4, 5, 6, 7\}$ an ordered partition $X = R_1 \sqcup R_2 \sqcup R_3$ into 3 non-empty subset is **rainbow** if $7 \in R_3$ and no set R_1, R_2, R_3 contains two numbers whose sum is 7 (i.e. contains any of the subsets $\{1, 6\}, \{3, 4\}, \{2, 5\}$).

2.3.2. How many rainbow partitions of [7] are there?

2.3.3. For each $j \in [6]$ and each rainbow partition S of $[7] - \{j\}$ there are exactly two rainbow partitions of [7] extending S.

(Comment. For example, a rainbow partition $(\{1\}, \{6, 4, 5\}, \{3, 7\})$ of $[7] - \{2\}$ extends to two rainbow partitions $(\{1, 2\}, \{6, 4, 5\}, \{3, 7\})$ and $(\{1\}, \{6, 4, 5\}, \{3, 2, 7\})$ of [7], but $(\{1\}, \{2, 4, 5, 6\}, \{3, 7\})$ would not be allowed because $2 + 5 = 7$.)

Denote by $\langle X \rangle$ the convex hull of a finite subset $X \subset \mathbb{R}^2$.

The *triple van Kampen number modulo 2* for an ordered set $f = (1, \ldots, 7)$ of general position points in the (oriented) plane is

$$v(f) \ := \ \sum_R |\, \langle R_1 \rangle \cap \langle R_2 \rangle \cap \langle R_3 \rangle \,| \mod 2 \ \in \ \mathbb{Z}_2,$$

where the summation is over all rainbow partitions $R = (R_1, R_2, R_3)$ of [7].

2.3.4. (a) Find $v(f)$ for some f of your choice.

(b) The residue $v(f)$ is zero for each f.

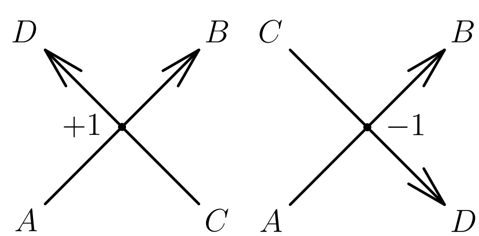

Figure 10: The sign of intersection point

Define **the sign** of intersection point of oriented segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ as the number $+1$ if ABC is oriented clockwise and the number -1 otherwise.

2.3.5. Suppose we have 14 points in general position in the plane: 7 red points and 7 yellow points. Electric current flows through every red segment. The sum of the currents flowing to any red point equals the sum of the currents issuing out of the point. The current also flows through the yellow segments conforming to the same Kirchhoff's law. Let us orient each red or yellow segment accordingly to the direction of the current passing through it. Assign to each intersection point of oriented red and yellow segments the product of currents passing through these segments and the sign of the intersection point. Are there points and currents such that the sum of all assigned products (that is *the flow of the red current through the yellow one*) would be equal to 42A^2?

Hint: for $3+3$ points the flux of the red current through the yellow one equals to zero.

Lemma 2.3.6 (Triviality)**.** *For each two general position closed polygonal lines in the plane the sum of signs of their intersection points is zero.*

2.3.7. There is a unique map csgn : Ra $\to \{+1,-1\}$ from the set Ra of all rainbow partitions of $[7]$ to $\{+1,-1\}$ such that

• $\operatorname{csgn}(\{1,2,3\},\{4,5,6\},\{7\}) = +1$.

• For every $j \in [6]$ and every rainbow partition S of $[7]-\{j\}$ take the two rainbow partitions R, T of $[7]$ extending S (see assertion 2.3.3). Then $\operatorname{csgn} R + \operatorname{csgn} T = 0$.

For every partition $R = (R_1, R_2, R_3)$ of the ordered set $(1,\dots,7)$ of general position points in the (oriented) plane define the **geometric sign** $\operatorname{gsgn} R$ as follows:

• $\operatorname{gsgn} R = +1$, if up to an even permutation of (R_1, R_2, R_3)

∘ either $R_1 = \{a,b,c\}$, $R_2 = \{d,e,f\}$, $a<b<c$, $d<e<f$, and the plane triangles abc, def have the same orientation;

∘ or $R_1 = \{a,b,c\}$, $R_2 = \{d,e\}$, $R_3 = \{f,g\}$, $a<b<c$, $d<e$, $f<g$ and the plane triangles abc, deg have the opposite orientation;

• $\operatorname{gsgn} R = -1$ otherwise.

The *triple van Kampen number* for an ordered set $f = (1,\dots,7)$ of general position points in the (oriented) plane is

$$V(f) := \sum_R \operatorname{csgn} R \cdot \operatorname{gsgn} R \cdot |\, \langle R_1\rangle \cap \langle R_2\rangle \cap \langle R_3\rangle \,|,$$

where the summation is over all rainbow partitions $R = (R_1, R_2, R_3)$ of f.

2.3.8. * Cf. [BMZ09, MTW10].

(a) Find $V(f)$ for some f of your choice.

(b) Is the residue $V(f) \mod 3$ independent of f?

(Hint. Analogously to Problem 2.3.4.a.)

(c) Is it correct that $V(f) \not\equiv 0 \mod 6$ for each f?

(Hint. Follows from (a), (b) and Problem 2.3.4.c.)

If (b) and (c) do not hold, then redefine $V(f)$ in an analogous way so that the analogues of (b) and (c) for the new $V(f)$ would be correct.

(5) State and prove analogues of parts (a,b,c) for decompositions of 13 points into 5 sets.

(p) The same for decompositions of $3p-2$ points into p sets, for each prime p.

(4) The same for decompositions of 10 points into 4 sets.

(pk) The same for decompositions of $3\cdot 2^k - 2$ points into 2^k sets, for each integer k.

(6) The same for decompositions of 16 points into 6 sets.

Invariants of graph drawings in the plane

presented by A. Enne, A. Ryabichev, A. Skopenkov and T. Zaitsev

Hints and solutions for some problems after the semifinal

2.1.1. (a) Note that if the graph K' is obtained from a graph K by subdivision of an edge, then a piecewise-linear map $K' \to \mathbb{R}^2$ corresponds to the piecewise-linear map $K \to \mathbb{R}^2$. Therefore, the $\mathbb{Z}_2$-planarity of the graph K_5 with an edge subdivided implies the $\mathbb{Z}_2$-planarity of the graph K_5. However, by Lemma 1.5.4 we have that the graph K_5 is not $\mathbb{Z}_2$-planar.[9]

(b) Analogously to (a), Lemma 1.5.4 and Statement 1.5.5 imply that any graph homeomorphic to K_5 or $K_{3,3}$ is not $\mathbb{Z}_2$-planar. Using Kuratowski Theorem 1.3.3, we obtain the required assertion.

2.1.2. Put the images of the vertices on the unit circle preserving the order and let the images of the edges be the chords.

2.1.3. Order the points round the quadrilateral (resp. pentagon) perimeter and let the edges be ordered lexicographically. Then the answer is the following:

$$
\text{(a)} \begin{pmatrix}
- & - & - & - & - & 0 \\
- & - & - & - & 1 & - \\
- & - & - & 0 & - & - \\
- & - & 0 & - & - & - \\
- & 1 & - & - & - & - \\
0 & - & - & - & - & -
\end{pmatrix}, \quad
\text{(b)} \begin{pmatrix}
- & - & - & - & - & - & - & 0 & 0 & 0 \\
- & - & - & - & - & 1 & 1 & - & - & 0 \\
- & - & - & - & 0 & - & 1 & - & 1 & - \\
- & - & - & - & 0 & 0 & - & 0 & - & - \\
- & - & 0 & 0 & - & - & - & - & - & 0 \\
- & 1 & - & 0 & - & - & - & - & 1 & - \\
- & 1 & 1 & - & - & - & - & 0 & - & - \\
0 & - & - & 0 & - & - & 0 & - & - & - \\
0 & - & 1 & - & - & 1 & - & - & - & - \\
0 & 0 & - & - & 0 & - & - & - & - & -
\end{pmatrix}.
$$

2.1.4. (a) For $(x, y) \in \partial(\sigma \times \tau)$ either x or y is a vertex of the graph K. So we have $f(x) = f(y)$ only if the image of a vertex and the image of some point of an edge coincide. This is impossible when f is a general position map.

(b)[10] Take the piecewise-linear map $\gamma : [0; 1] \to \mathbb{R}^2$ corresponding to $f\sigma$. If $\gamma(t) \notin f\tau$, then the number of rotations for the 'piece' $\gamma|_{[0;t]}$ of the edge σ from 0 to t and the edge τ is well defined. And if $\gamma(t) \in f\tau$, then these numbers of rotations for restrictions $\gamma|_{[0;t-\varepsilon]}$ and $\gamma|_{[0;t+\varepsilon]}$ differ by ± 1 (depending on the sign of the intersection point, see Fig. 10). This argument becomes rigorous after defining 'the number of rotations' formally.

2.1.5. Answer to I-IV: the intersection cocycle does not change.

2.1.6. For example, the coboundaries $\delta\big(3, (12)\big)$ for K_4 and K_5 (with vertices and edges ordered as in the proof of Problem 2.1.3) are the following partial matrices:

[9] This also may be deduced from Problems 2.1.6.d and 2.1.7.a

[10] Statements 2.1.4.b and 3.1.2.e can also be proved analogously to the proof that $v(f) = o(f)$ for $r = 2$ in [Sk16, §3.4].

$$
\text{(a)} \begin{pmatrix}
- & - & - & - & - & 1 \\
- & - & - & - & 0 & - \\
- & - & - & 0 & - & - \\
- & - & 0 & - & - & - \\
- & 0 & - & - & - & - \\
1 & - & - & - & - & -
\end{pmatrix}, \quad \text{(b)} \begin{pmatrix}
- & - & - & - & - & - & - & 0 & 1 & 1 \\
- & - & - & - & - & 0 & 0 & - & - & 0 \\
- & - & - & - & 0 & - & 0 & - & 0 & - \\
- & - & - & - & 0 & 0 & - & 0 & - & - \\
- & - & 0 & 0 & - & - & - & - & - & 0 \\
- & 0 & - & 0 & - & - & - & - & 0 & - \\
- & 0 & 0 & - & - & - & - & 0 & - & - \\
1 & - & - & 0 & - & - & 0 & - & - & - \\
1 & - & 0 & - & - & 0 & - & - & - & - \\
0 & 0 & - & - & 0 & - & - & - & - & -
\end{pmatrix}.
$$

In other words, $\delta_{K_4}\big(3,(12)\big) = \{((12),(34))\}$ $\delta_{K_5}\big(3,(12)\big) = \{((12),(34)),((12),(35))\}$.

(c) Yes, it equals $\delta\big(1,(24)\big)$ (it also equals $\delta\big(2,(13)\big)$, $\delta\big(3,(24)\big)$ and $\delta\big(4,(13)\big)$).

(d) No, it is not: addition of an elementary coboundary does not change the parity of the number of units above the diagonal, while initially this number equals five.

2.1.7. (a) Suppose a graph K and two general position maps $f,f' : K \to \mathbb{R}^2$ are given. Suppose these maps are in general position relatively to each other, cf. the proof of the linear analogue of Lemma 1.5.4.

The proof of the particular case when the maps f and f' differ only on the interior of one edge σ. Take a point O in the plane. For every vertex B of the graph K, if B is not an end of the edge σ, join fB to O by a polygonal line b in general position to the cycle $\widehat{\sigma} := f\sigma \cup f'\sigma$. Then by the Parity Lemma 1.4.4 for every edge B_1B_2 nonadjacent to σ we have

$$0 \underset{2}{\equiv} |\widehat{\sigma} \cap (b_1 \cup b_2 \cup f(B_1B_2))| \underset{2}{\equiv} |\widehat{e} \cap b_1| + |\widehat{\sigma} \cap b_2| + |\widehat{\sigma} \cap f(B_1B_2)| \quad \Rightarrow$$

$$\Rightarrow \quad |\widehat{\sigma} \cap f(B_1B_2)| \underset{2}{\equiv} |\widehat{\sigma} \cap b_1| + |\widehat{\sigma} \cap b_2|.$$

Denote by $B_1,\dots,B_k$ all the vertices for which the polygonal lines $b_1,\dots,b_k$ intersect the cycle $\widehat{\sigma}$ at an odd number of points. (The set $B_1,\dots,B_k$ depends on the choice of the point O, but the following equality holds for every choice.) Then the difference of the intersection cocycle of f and f' equals

$$\delta(B_1,\sigma) + \ldots + \delta(B_k,\sigma) = \sum_{B \notin \sigma} |\widehat{\sigma} \cap b| \cdot \delta(B,\sigma).$$

(b) Use (a) and Fig. 9.V.

2.2.1. By assertion 2.2.2 we have that there exists at least one red point, as it is desired.

2.2.2. (a) Take a linear map of the graph K_4 to the plane such that the images of the vertices are the vertices of the regular triangle and its center. Then there is only one red point, namely the center.

(b) Let $f, f' : K_4 \to \mathbb{R}^2$ be maps in general position. We may assume that they are in general position *relatively to each other*, cf. proof of linear analogue of Lemma 1.5.4. The reduction of the general case to the particular one when maps f and f' differ from each other only on the interior of one edge σ is similar to Lemma 1.5.4. Let us prove the particular case (cf. Lemma 1.5.4). Consider a point O in the plane. For each of the two vertices B of graph K_4 that are not the ends of edge σ connect $f(B)$ and O to a polygonal chain b, which is in general position to the cycle $\widehat{\sigma} := f\sigma \cup f'\sigma$. Then for edge B_1B_2 non-adjacent with σ we get

$$0 \underset{2}{\equiv} |\widehat{\sigma} \cap (b_1 \cup b_2 \cup f(B_1B_2))| \underset{2}{\equiv} |\widehat{\sigma} \cap b_1| + |\widehat{\sigma} \cap b_2| + |\widehat{\sigma} \cap f(B_1B_2)| \underset{2}{\equiv} (v_1+v_2)(f) - (v_1+v_2)(f').$$

(c) Follows from (a) and (b).

2.2.3.(a) Obviously, the image of some vertex lies between the images of two other vertices. Then use Intermediate value theorem.

Assertion (b) follows from (c) and piecewise-linear case topological Radon theorem for the plane. The piecewise-linear case follows from 2.2.2.

2.3.2. The answer is $6^3 - 2 \cdot 2^3 = 200$. Consider analogous partitions where the sets can be empty. For such partitions each of the pairs $\{1,2\}, \{3,4\}, \{5,6\}$ can be distributed in 6 ways and element 7 always lies in R_3. Therefore, the number of such partitions equals 6^3.

Then count the considered partitions that are not rainbow. In each of them one of the sets R_1, R_2 is empty, for each of the pairs $\{1,2\}, \{3,4\}, \{5,6\}$ there are two ways to distribute it. So the number of excess partitions equals $2 \cdot 2^3$.

2.3.3. The element j can be added to two of three sets of partition S since one of the sets of partition contains the "partner" $7-j$ of j that cannot be contained in the same set as j.

2.3.4. (b) Assign to a rainbow partition R_1, R_2, R_3 the rainbow partition R_2, R_1, R_3 (this correspondence is a bijection). We can observe that the number of nonzero summands in the formula is even, so $v(f) = 0$.

Also the Problem 2.3.4 can be solved analogously to Problems 1.2.1.a, 1.5.4, 2.2.2.b, cf. Problems 2.1.7.a and [Sk17'].

2.3.5. *Proof for analogue with 3+3 points.* The current is constant for each loop, and the signs of intersection points alternate, therefore the sum equals zero.

Answer. There is no such point and currents, because the sum of all assigned products always equals zero.

Solution. Denote by *red current* (resp. yellow) assignment of currents for red (resp. yellow) segments conforming to the Kirchhoff's law. Note that if we consider two red currents and one yellow current, then the flow of the sum of the red currents through the yellow one equals the sum of the flows. Analogously for one red current and two yellow currents the sum of the flows equals the flow of the sum (that is, the flow is *biadditive*).

Add a point C to the yellow points and a point D to the red points so that all 16 points are in general position, as in Problem 1.4.1. Assume that the currents on the segments CA_i and DB_j equal zero. For each segment A_iA_j consider the current flowing through the triangle CA_iA_j equal the initial yellow current through A_iA_j (and equal zero out of the triangle CA_iA_j). Then the sum of these $\binom{7}{2}$ currents equals the initial yellow current. Analogously decompose the red current into the sum of $\binom{7}{2}$ currents flowing through triangles DB_kB_l. The required statement follows from property of biadditivity and analogue for 3+3 points.

2.3.6. The proof is analogous to the proof of the Parity Lemma 1.4.4.

3 Additional problems for advanced teams

3.1 Intersections with signs and for line drawings

In this section we define the van Kampen obstruction to $\mathbb{Z}$-*planarity* of a graph (Problems 3.1.1-3.1.4) and to $\mathbb{Z}_2$-*embeddability of a graph into the line* (Problem 3.1.5). These generalizations of constructions from Section 2.1 we formally will not use further. However, it is useful to obtain these simple generalizations before moving to the more complicated generalisation in Problems 3.2.4, 3.2.5.

A graph is called **$\mathbb{Z}$-planar** if there exists a general position map of this graph to the plane such that the sum of the signs of intersection points of f-images of any two non-adjacent edges is zero for some (or equivalently any) orientations of its edges [Sk16, Fig. 4].

3.1.1. (a) Draw two (not closed) oriented general position polygonal lines, intersecting each other in an even number of points, sum of whose signs is not equal to zero.

(b) A graph is planar if and only if it is $\mathbb{Z}$-planar.

Orient the edges of a graph K. Suppose $f : K \to \mathbb{R}^2$ is a general position map. Assign to each ordered pair of non-adjacent edges (σ, τ) the sum of the signs of the intersection points of its f-images $f\sigma \cdot f\tau$. Denote by $\widetilde{K}$ the set of all ordered pairs of non-adjacent edges of the graph K. The obtained map $\widetilde{K} \to \mathbb{Z}$ is called **integral intersection cocycle**. In this section we drop the word 'integral'.

3.1.2. (a) Orient the edges of the graph K_5. Find the intersection cocycle for a linear general position map $K_5 \to \mathbb{R}^2$ such that the images of the vertices form a convex pentagon.

(b) $f\sigma \cdot f\tau = -f\tau \cdot f\sigma$.

(c) How does the intersection cocycle change if we change the orientation of an edge?

(d) How does the intersection cocycle change if we change the orientation of the plane, that is if we compose f with a rotational symmetry?

(e)* In Problem 2.1.4.b the number $f\sigma \cdot f\tau$ equals the number of rotations of the vector while one goes around $\partial(\sigma \times \tau)$.

3.1.3. (I-V) How does the intersection cocycle change under the Reidemeister moves in Fig. 9.I-V? (The vertex A is not an end of the edge τ in Fig. 9.V.)

Suppose A is a vertex which is not an end of an edge σ. **An elementary coboundary** of the pair (A, σ) is the map $\delta(A, \sigma) : \widetilde{K} \to \mathbb{Z}$ which assigns

- 1 to any pair (σ, τ) with τ directed out of A and any pair (τ, σ) with τ directed into A,
- -1 to any pair (σ, τ) with τ directed into A and any pair (τ, σ) with τ directed out of A,
- 0 to any other pair.

The answer to Problem 3.1.3.V: $\delta(A, \sigma)$ is added to the intersection cocycle.

Two maps $N_1, N_2 : \widetilde{K} \to \mathbb{Z}$ are called **(integrally) cohomologous**, if

$$N_1 - N_2 = m_1\delta(A_1, \sigma_1) + \ldots + m_k\delta(A_k, \sigma_k)$$

for some vertices $A_1, \ldots, A_k$, edges $\sigma_1, \ldots, \sigma_k$ and integer numbers $m_1, \ldots, m_k$ (not necessarily distinct).

3.1.4. (a), (b) State and prove integral analogous of problems 2.1.7.ab.

(c) Doubled intersection cocycle of any general position map of a graph in the plane is null-cohomologous.

3.1.5. (a) For any general position map $f : K \to \mathbb{R}$ of a graph K (define this notion!) and a pair of non-adjacent edges $\{AB, CD\}$ of the graph K

$$|f(A) \cap f(CD)| + |f(B) \cap f(CD)| + |f(AB) \cap f(C)| + |f(AB) \cap f(D)| \underset{2}{\equiv} 0.$$

(b)* Take a general position map $f : K \to \mathbb{R}$ of a graph K. The map assigning the number $|f(A) \cap f(BC)|$ to any pair A, BC consisting of a vertex A and an edge BC such that $A \neq B, C$ is called *intersection cocycle*. Define analogues of at least two of the Reidemeister moves in Fig. 9 for maps of a graphs to the line. How does the intersection cocycle change for the analogous moves?

(c)* Define graph $K^{*(1)}$ as follows. The vertices of $K^{*(1)}$ are unordered pairs $\{A, B\}$ of different vertices of a graph K. For each pair A, BC consisting of a vertex A and an edge BC of K such that $A \neq B, C$ connect vertex $\{A, B\}$ to $\{A, C\}$ with an edge in graph $K^{*(1)}$. Denote this edge by $A \times BC = BC \times A$.

Find $K^{*(1)}$, if K is a cycle with 3 vertices, a triod, K_4.

(d)* For a vertex B of a graph G define *an elementary coboundary* $\delta_G B$ as a map from the set of the edges of G to the set $\{0, 1\}$ which assigns 1 to the edges that contain this vertex, and 0 to any other edge. (In other words, $\delta_G B$ corresponds to the set of edges containing B.) Dispositions ω_1 and ω_2 of zeroes and units on edges of graph $K^{*(1)}$ are called *cohomologous* if $\omega_1 - \omega_2$ is the sum of some elementary coboundaries $\delta_{K^{*(1)}}\{A, B\}$. Is it correct that there is a general position map $f : K \to \mathbb{R}$ of a graph K such that $f(A) \notin f(\sigma)$ for each vertex A and edge $\sigma \not\ni A$, if and only if the intersection cocycle of some general position map $f' : K \to \mathbb{R}$ is zero?

(e)* *A cocycle* is a map from the edges of the graph $K^{*(1)}$ to the set $\{0, 1\}$ such that the sum of images of edges $A \times CD$, $B \times CD$, $C \times AB$, $D \times AB$ is even for any non-adjacent edges AB, CD of graph K (cf. part (a)). For each cocycle ν assign to any unordered pair $\{AB, CD\}$ of disjoint edges of graph K the sum of two numbers on 'opposite' edges $A \times CD$ and $B \times CD$ of 'square' $AB \times CD$. That is, define a map $\mathrm{Sq}^1 \nu : K^* \to \mathbb{Z}_2$ by the formula

$$\mathrm{Sq}^1\nu\{AB, CD\} := \nu(A \times CD) + \nu(B \times CD) = \nu(AB \times C) + \nu(AB \times D).$$

Prove that $\mathrm{Sq}^1(\mu + \nu) = \mathrm{Sq}^1 \mu + \mathrm{Sq}^1 \nu$.

(f)* Express $\mathrm{Sq}^1(\mu\nu)$ in terms of $\mu, \nu, \mathrm{Sq}^1 \mu, \mathrm{Sq}^1 \nu$. (Here we consider componentwise sum and product.)

(g)* Prove that $\delta_{K^{*(1)}}\{A, B\}$ is a cocycle.

(h)* Prove that $\mathrm{Sq}^1 \delta_{K^{*(1)}}\{A, B\} = \sum\limits_{\sigma \ni B} \delta(A, \sigma) =: \delta(A \times \delta_K B)$.

3.2 The topological Tverberg conjecture for the plane

The winding number of a closed oriented polygonal line $A_1 \dots A_n$ around a given point O that does not belong to the polygonal line is a sum

$$\deg_{A_1 \dots A_n} O := (\angle A_1 O A_2 + \angle A_2 O A_3 + \ldots + \angle A_{n-1} O A_n + \angle A_n O A_1)/2\pi$$

of the oriented angles divided by 2π.

3.2.1. (a) Find a winding number of the regular pentagon (with arbitrary orientation) around its center and around the intersection point of two lines passing throw non-adjacent sides.

(b) Find the winding number for each (arbitrarily oriented) polygonal line from figure 4 and a point of your choice (in any finite domain).

3.2.2. * (3) For every map $f : K_7 \to \mathbb{R}^2$ there exists an enumeration of the vertices by the numbers $1, \dots, 7$ such that either

• the winding number of the cycle 567 around some intersection point of the edges 12 and 34 is nonzero, or

• the winding number of each of the cycles 567 and 234 around the point 1 is nonzero.

(4) **The topological Tverberg theorem for the plane.** If r is a power of a prime, then for every map $f : K_{3r-2} \to \mathbb{R}^2$ there exists an enumeration of the vertices by the numbers $1, \dots, 3r-2$ such that either

• the winding number of each of the cycles $3t-1, 3t, 3t+1$, $t = 2, 3, \dots, r-1$, around some intersection point of the edges 12 and 34 is nonzero, or

• the winding number of each of the cycles $3t-1, 3t, 3t+1$, $t = 1, 2, 3, \dots, r-1$, around the point 1 is nonzero.

(The cases $r = 5$, arbitrary prime r, $r = 4$, $r = 2^k$ are counted as separate problems.)

(6) **The topological Tverberg conjecture for the plane.** Is the analogue of assertion (4) correct, if r is not a power of a prime?

Hints to problem 3.2.2 are in the following problem, cf. problem 2.3.8. By [Sc05, SZ] Theorem 3.2.2 is equivalent to the following result: *If r is a power of a prime, then for every map of the $(3r-1)$-simplex to the plane there exist r pairwise disjoint faces, images of that have a common point.* Problem 3.2.2 has the analogous reformulation.

3.2.3. * (a) For every general position map $f : K_7 \to \mathbb{R}^2$

$$\sum_{\substack{|R_1|=1,\\ |R_2|=|R_3|=3}} \operatorname{csgn} R \deg_{fR_2}(fR_1) \deg_{fR_3}(fR_1) \;+ \sum_{\substack{|R_1|=|R_2|=2,\ |R_3|=3,\\ X\in fR_1\cap fR_2}} \operatorname{csgn} R \operatorname{sgn} X \deg_{fR_3} X \underset{6}{\equiv} 2.$$

Here the summation is over all the rainbow partitions $R = (R_1, R_2, R_3)$ with specified properties, and fR_i is the image of a vertex R_i, edge R_i or an oriented cycle $R_i = abc$, $a < b < c$, under f.

(b) State and prove the analogue of (a) for partitions of the vertices of K_{3r-2} into r sets. (The cases $r = 5$, arbitrary prime r, $r = 4$, $r = 2^k$, r the power of prime and $r = 6$ are counted as separate problems.)

3.2.4. * Suppose $f : K_7 \to \mathbb{R}^2$ is a general position map. Define *threefold intersection cocycle* and *cohomologous cocycles* analogously to §3.1 and Problems 2.3.8, 3.2.3. Do the same for arbitrary graph instead K_7 and for r-fold intersections.

3.2.5. * (3) Tripled threefold intersection cocycle of any general position map $K_7 \to \mathbb{R}^2$ is null-cohomologous.

(4) The 4-fold intersection cocycle of any general position map $K_{10} \to \mathbb{R}^2$ multiplied by 8 is null-cohomologous.

(r) If integer r is not a power of a prime p, then the r-fold intersection cocycle of any general position map $K_{3r-2} \to \mathbb{R}^2$ multiplied by $r!/\sum_{k=1}^{\infty}\left\lfloor \frac{r}{p^k}\right\rfloor$ is null-cohomologous.

(0) **Özaydin theorem.** If r is not a prime number, then the r-fold intersection cocycle of any general position map $K_{3r-2} \to \mathbb{R}^2$ is null-cohomologous. (The same is correct for every graph.)

References

[AMSW] *S. Avvakumov, I. Mabillard, A. Skopenkov and U. Wagner.* Eliminating Higher-Multiplicity Intersections, III. Codimension 2, arxiv:1511.03501.

[BE82] ** V.G. Boltyansky and V.A. Efremovich.* Intuitive Combinatorial Topology. Springer. Russian version: Moscow, Nauka, 1982, `http://ilib.mccme.ru/djvu/geometry/boltiansky-nagl-topo.htm`.

[BM04] *Boyer, J. M. and Myrvold, W. J.* On the cutting edge: simplified $O(n)$ planarity by edge addition, Journal of Graph Algorithms and Applications, 8:3 (2004) 241–273.

[BMZ09] P. V. M. Blagojevič, B. Matschke, G. M. Ziegler, Optimal bounds for the colored Tverberg problem, J. European Math. Soc., arXiv:0910.4987.

[BZ] P. V. M. Blagojevič and G. M. Ziegler, Beyond the Borsuk-Ulam theorem: The topological Tverberg story, arXiv: 1605.07321.

[CLR] ** T. H. Cormen, C. E.Leiserson, R. L.Rivest, C. Stein.* Introduction to Algorithms, MIT Press, 2009.

[Fo04] ** R. Fokkink.* A forgotten mathematician, Eur. Math. Soc. Newsletter 52 (2004) 9–14, `http://www.ems-ph.org/journals/newsletter/pdf/2004-06-52.pdf`.

[Fr15] *F. Frick*, Counterexamples to the topological Tverberg conjecture, arXiv:1502.00947.

[Gr10] M. Gromov, *Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry*, Geometric and Functional Analysis 20 (2010), no. 2, 416–526.

[HT74] *J. Hopcroft and R. E. Tarjan,* Efficient planarity testing, Journal of the Association for Computing Machinery, 21:4 (1974) 549–568.

[Ma03] *J. Matoušek*, Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry, Springer, 2003.

[MTW10] *J. Matoušek, M. Tancer, U. Wagner.* A geometric proof of the colored Tverberg theorem, arXiv:1008.5275.

[MW15] *I. Mabillard and U. Wagner.* Eliminating Higher-Multiplicity Intersections, I. A Whitney Trick for Tverberg-Type Problems. arXiv:1508.02349.

[Oz] *M. Özaydin,* Equivariant maps for the symmetric group, unpublished, `http://minds.wisconsin.edu/handle/1793/63829`.

[Pr95] ** V. V. Prasolov.* Intuitive topology. Amer. Math. Soc., Providence, R.I., 1995. Russian version: Moscow, MCCME, 1995, `http://www.mccme.ru/prasolov`.

[Pr06] ** V. V. Prasolov.* Elements of Combinatorial and Differential Topology, 2006, GSM 74, AMS, Providence, RI.

[Sa91] *K. S. Sarkaria.* A one-dimensional Whitney trick and Kuratowski's graph planarity criterion, Israel J. Math. 1991. 73. P. 79-89.

[Sc05] *T. Schöneborn,* On the Topological Tverberg Theorem, arXiv:math/0405393.

[Si16] S. Simon, Average-Value Tverberg Partitions via Finite Fourier Analysis, Israel J. Math., 216 (2016), 891-904, Arxiv:1501.04612.

[Sk'] ** A. Skopenkov.* Algebraic Topology From Algorithmic Viewpoint, draft of a book, in Russian, `http://www.mccme.ru/circles/oim/algor.pdf`.

[Sk15] ** A. Skopenkov.* Algebraic Topology From Geometric Viewpoint (in Russian), MCCME, Moscow, 2015. `http://www.mccme.ru/circles/oim/obstruct.pdf`.

[Sk16] ** A. Skopenkov,* A user's guide to the topological Tverberg Conjecture, arXiv:1605.05141.

[Sk17'] * A. Skopenkov, On van Kampen-Flores, Conway-Gordon-Sachs and Radon theorems, arxiv:1704.00300.

[SS] ** A. Skopenkov.* Realizability of hypergraphs and Ramsey link theory, arxiv:1402.0658.

[Sk05] ** A. Skopenkov.* On the Kuratowski graph planarity criterion, Mat. Prosveschenie, 9 (2005), 116-128. arXiv:0802.3820.

[Sk03] *M. Skopenkov.* Embedding products of graphs into Euclidean spaces, Fund. Math. 2003. 179. P. 191-198.

[ST07] ** A. Skopenkov and A. Telishev*, Once again on the Kuratowski graph planarity criterion, Mat. Prosveschenie, 11 (2007), 159-160. arXiv:0802.3820.

[SZ] *T. Schöneborn and G. Ziegler*, The Topological Tverberg Theorem and Winding Numbers, `http://page.math.tu-berlin.de/~schoenei/papers/The_Topological_Tverberg_Theorem_and_Winding_Numbers.pdf`.

[Ta] * Handbook of Graph Drawing and Visualization. Ed. R. Tamassia, CRC Press. `https://cs.brown.edu/~rt/gdhandbook/`.

[Vi02] ** E. B. Vinberg.* A Course in Algebra. Graduate Studies in Mathematics, vol. 56. 2003.

[vK32] *E. R. van Kampen*, Komplexe in euklidische Räumen, Abh. Math. Sem. Hamburg, 9 (1932) 72–78; Berichtigung dazu, 152–153.

Books, reviews and popular articles in this list are marked by the stars.