“Line avoidance” game – 1

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1 *Preliminary problems*

We consider games where two players Alica and Bob (aka First and Second, or simply, P1 and P2) move alternatively. In our games they color some objects (uncolored yet) with their own colors. For convenience, let Alice use an amber (or first) color and Bob use blue (or second) color. In all games Alice makes the first move. “And what about Bob?” you could ask. And Bob makes the second move.

Usually a move consists of coloring of one uncolored object (if the problem statement does not define another rule). What do they want to achieve by their colorings is formulated in the problem statement. But keep in mind that, generally speaking, it may happens that they will not reach their aims in coloring during the game. In this case a draw is declared. So a question “Who wins?” can be answered “Nobody!”.

1.1. Alica and Bob alternatively color the vertices of a dodecahedron. A player who colors two vertices of the same edge for the first time loses. Who wins if both players are playing with an optimal strategy?

1.2. Alice and Bob alternatively color vertices of some polyhedron. A player who colors all the vertices of some face for the first time wins. Who wins if both players are playing with an optimal strategy?

1.3. 99 points are marked on the circle. Alice and Bob alternatively color these points. A player who colors two neighbour points for the first time loses. Who wins if both players are playing with an optimal strategy?

1.4. There are n points marked on the circle. Alice and Bob, alternatively, draw chords by their own markers (connecting of neighbour points is allowed too). In a move they can draw several chords. A player who colors all sides of a triangle with vertices on the initial circle, loses. Who wins if both players are playing with an optimal strategy? You may investigate the question for sufficiently large n only.

2 *Boards with intersecting lines*

Let a finite number of points are marked on the plane (or elsewhere). We call these points nodes. Let several lines are passes through the nodes. You may think that a line is a «curve of an arbitrary shape» that passes through nodes. Let for any two lines the sets of nodes belonginig to the lines do not coincide. In fact, the lines are just the subsets of the set of nodes. We say that two lines intersect, if they pass through at least one common node. Geometrical arrangement of lines, including their intersections not in the nodes are inessential. Later we will use the configuration of lines and nodes as a board for a game and put some additional requirements for boards.

2.1. A board consists of n nodes. All the lines are pairwise intersecting.

a) Prove that the number of lines does not exceed 2^{n-1}.

b) Let the number of lines be equal to 2^{n-1}. Prove that if we add a node to any line we will obtain a line again.

2.2. A board consists of n nodes. All the lines are pairwise intersecting. Prove that it is possible to add several lines so that the number of lines is equal to 2^{n-1} and the whole set of lines is pairwise intersecting.

2.3. A board consists of n nodes and several lines. Let the minimal line contains k nodes. Prove that if any $k+1$ lines have a common node then all the lines have a common node.

2.4. Consider a cube $3\times 3\times 3$. Let its unit cubes be nodes of the board. We may consider them as a triples (x_1, x_2, x_3), $0 \leqslant x_1, x_2, x_3 \leqslant 2$. Let all the lines of the board contain exactly 3 nodes: a line containing nodes (x_1, x_2, x_3) и (y_1, y_2, y_3) contains also node $(-x_1-y_1, -x_2-y_2, -x_3-y_3)$ (calculations modulo 3). Who wins on this board?

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3 The general case of “Line avoidance” game

Consider a game board with nodes and lines as in the previous section. Alice and Bob alternatively color nodes. A player who colors all the nodes of some line for the first time, loses. If this situation does not happen until all the nodes on the board have been colored a draw is declared.

Remind that a permutation is a one-to-one mapping from a finite set onto itself. We will consider permutations of the set of the board nodes. We say that a permutation of the board nodes preserves lines if the image of the set of nodes of any line is a line, too. We will consider transitive boards only that means that for any two nodes u and v there exists a permutation σ of the board nodes preserving lines such that $\sigma(u) = v$.

Example. Sim (pencil game). n points are drawn on the circle. Two players take turns drawing chords by their own markers (it is allowed to connect neighbour points too). In contrast to the problem 1.4, drawing of one chord in a move is allowed only. A player who colors all sides of a triangle with the vertices belonging to the initial circle, loses.

In order to interpret Sim as a transitive game we denote the initial points by A_1, A_2, …, A_n and consider a board with nodes N_{ij}, $1 \leqslant i < j \leqslant n$ (the node N_{ij} corresponds to the chord A_iA_j). For each set of three elements $\{i, j, k\}$ (where $i < j < k$) define a line that consists of three nodes N_{ij}, N_{ik}, N_{jk}. We obtain a transitive board. The game on this board is equivalent to Sim. A chord drawing turns to a node coloring and a triangle turns to a line!

At first glance it looks as the first player has a disadvantage in “Line avoidance” game on a transitive board. Since all the points on the board are “the same” due to transitivity, the earlier coloring starts the earlier line constructing happens. Also the first player colors more points than the second one on the board with odd number of points (if the game continues till the end), besides that the second player has a choice in the last move and the first player does not.

The main question. For which n does a transitive board with n nodes exist such that Alice wins?

3.1. Let all the lines on the board consist of two nodes only. Prove that Alice does not win.

3.2. Consider a “greedy” game in which a player in a move can color any non empty set of nodes by his choice. Prove that there does not exist a board for which Alice can win greedy game.

3.3. Does a board with 6 nodes exist for which Alice wins?

3.4. Let n be an odd composite number. Prove that there exists a board with n nodes such that Alice wins on it.

3.5. Prove that for $n = 12k + 6$ there exists a board such that all its lines contain exactly 3 nodes and Alice wins.

3.6. Prove that for any number r there exist an arbitrary large number n and a board with n nodes such that each line contains exactly r nodes and Alice wins on this board.

3.7. a) Prove that for $n = 2b$ where b is odd, there exists a board such that Alice wins on it.

b) Prove that for $n = 2^a b$ where b is odd, $a > 1$, there exists a board such that Alice wins. You may suggest solutions for partial cases of this question.

3.8. Prove that Alice can not win on any board that contains exactly 2^n nodes.

3.9. Open question. For which prime numbers n there exists a board with n nodes for which Alice wins?

For example, what can you say about small prime numbers: $n = 3$, 5, 7, 11, 13?

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4 Special boards

We assume that all the boards under consideration are transitive.

Denote by A a set of numbers n for which a transitive board with n nodes exists such that any its two lines are intersecting and the number of lines equals 2^{n-1}. What integers n belong to the set A is an open question.

Remind that a permutation is a one-to-one mapping of the set $[n] = \{1, 2, \ldots, n\}$ onto itself. A permutation τ is called a product of permutations ρ and σ if $\tau(k) = \rho(\sigma(k))$ for all $k \in [n]$ i.e. if τ is a composition of the mappings ρ and σ. Multiplication of permutations is non commutative in the general case. A set G of permutations is called a group if it closed under taking the operation of permutation product. In this case G necessarily contains the identical permutation e and for any $\sigma \in G$ the set G contains permutation $\bar{\sigma}$, which is inverse to σ, i.e. $\bar{\sigma}\sigma = \sigma\bar{\sigma} = e$.

We say that a permutation is a flip if it partitions the points of the set $[n]$ onto pairs and maps each point to the point of its pair. We call a permutation a flop if the length of any of its cycles equals a power of 2. Let B be a set of numbers n for which there exists a group of permutations of the set $[n]$ that does not contain a flop and such that this group acts transitively on the set $[n]$.

4.1. a) If a flip preserving lines exists for the given board then Alice can not win on that board.

b) In the toys shop they sale a board with n nodes where n is even, all the lines are pairwise intersecting and their number is 2^{n-1}. Prove that there exists a board with n nodes such that Alice wins on it.

4.2. Let a transitive group G of permutations of the set $[2k]$ contain a permutation τ that maps some k-element set X to its complement $[2k] \setminus X$. Prove that group G contains a flop.

4.3. Let a board contain n nodes and all the lines be pairwise intersecting. Let there is no a flop that preserves lines on this board. Prove that we can add several lines so that the number of lines is 2^{n-1}, the board remains to be transitive and there is no a flop preserving lines.

4.4. Let $n = 2^a \cdot b$ where $b > 1$ is odd. Prove that there exists a board with n nodes such that any flip does not preserve the lines.

4.5. Prove that $A = B$.

4.6. Prove that the set A is closed under multiplication i.e. $k, n \in A$ implies that $kn \in A$.

4.7. Prove that if $b > 1$ is an odd number then $2b \in A$.

4.8. Prove that if $b > 3$ is an odd number then $4b \in A$.

Solutions

1 *Preliminary problems*

1.1. Answer: the first player wins.

If all the vertices of dodecahedron are colored in two colors then each face of the dodecahedron contains 2 neighbor vertices colored in the same color. So a draw is impossible in this game. The first player does not lose when uses central symmetry strategy.

1.2. Answer: two possible results of the game are either a draw or the first player wins

For a tetrahedron the result is a draw. For octahedron it is easy to see that the first player wins. For any polyhedron the second player can not win because the first player can steal his winning strategy.

1.3. Answer: the second player wins.

Enumerate points along the circle. Let the first player color point 1 at his first move. Then the second player colors point 98 in his first move and after that uses symmetrical (with respect to a symmetry $k \leftrightarrow 99 - k$) strategy.

1.4. Answer: the second player wins for $n \geqslant 6$.

The first player can not win due to problem 3.2. A draw is impossible for $n \geqslant 6$ because Ramsey number $R(3,3) = 6$.

2 *Boards with intersecting lines*

2.1. a) A set and its complement can not both be lines. And just in case, if the number of lines is 2^{n-1}, then in any pair of the complementary sets exactly one is a line.

b) Let X be a set of the board nodes, G be an arbitrary subset of X. If F is a line, $F \subset G$, then $F \cap (X \setminus G) = \varnothing$. Therefore $X \setminus G$ is not a line. Then G is a line by p. a).

2.2. [1, lemma 2.2]. Let X be a set of the board nodes and $\mathcal{L}$ be a set of the lines. We know that any two lines intersect. The set of lines that we want to construct should satisfy 2.1 b). Consider all the sets of nodes that contain line, let

$$\mathcal{L}_1 = \{B \subset X : A \subset B \text{ for some line } A \in \mathcal{L}\}.$$

Particularly, $\mathcal{L} \subset \mathcal{L}_1$. If there exists a set B that intersects any line and in the same moment does not contain any line (it means that its complement $\bar{B}$ intersects any line and does not contain any line, too) then it's worth to assign one of the sets B, $\bar{B}$ to be a "new" line. Choose one of the two sets that contains more nodes (in the case of equality of the numbers of nodes we can choose any of them). Let $\mathcal{L}_2$ be a family of sets that we choose by the described way.

Check that $\mathcal{L}_1 \cup \mathcal{L}_2$ is a family of pairwise intersecting lines that contains exactly 2^{n-1} elements. It's clear by the construction that any two lines in this family are intersecting.

Let Y be an arbitrary subset of X and $\bar{Y}$ be its complement. If Y contains a line then $Y \in \mathcal{L}_1$; else if Y does not intersect any line then $\bar{Y} \in \mathcal{L}_1$; in all other cases Y or $\bar{Y}$ is contained in $\mathcal{L}_2$.

2.3. It's the proposition of Ph. Zumstein. If $L = \{x_1, x_2, \ldots, x_k\}$ is a minimal line and no one of its points is a common point of the intersection of all sets then for any i, $1 \leqslant i \leqslant k$, there exists a line L_i for which $x_i \notin L_i$. But then the intersection $L \cap L_1 \cap \ldots \cap L_k$ is empty.

2.4. Answer: Alice wins.

We make some notes about construction of the game board. First of all any two distinct cubes indeed define a line consisting of 3 cubes (in the sense that the third cube found by the receipt of problem statement does not coincide with the two initial ones). Further, the rule describing lines is very symmetrical: if construction from the problem statement gives the cube C by the cubes A and B then the same construction gives the cube B by the cubes A and C and the cube A by the cubes B and C. It is not hard to check also that each cube belongs to 13 different lines.

Note that all cubes on this board except the cube $(0,0,0)$ can be partitioned into pairs by the mapping $x \mapsto -x$. Let Alice color the cube $(0,0,0)$ at her first move. After that let she color at each move the cube corresponding to the cube colored by her opponent at the previous move. Alice can not lose using this strategy.

It remains to prove that the final position of the game can not be a draw. Assume that we have a draw in the end of the game (no matter what strategy the players used). 14 cubes in this position are colored by Alice. Each of $\binom{14}{2} = 7 \cdot 13$ pairs of these cubes defines a line that contains a third cube C. Since we have a draw here this cube is colored by Bob. So for each $7 \cdot 13$ pairs we detect a cube colored by Bob. But Bob has colored 13 cubes totally, therefore we detect one of his cubes at least 7 times. Thus, we find 7 lines (A_1, B_1, C), (A_2, B_2, C), $\ldots$, (A_7, B_7, C), that contain Bob's cube C. All the 14 cubes A_i, B_i here are colored by Alice and no two of them coincide. Hence these lines contain all Alice's cubes and then other 6 lines that pass trough C consist of Bob's cubes. A contradiction.

3 The general case of "Line avoidance" game

3.1. [2, theorem 3]. If a winning strategy for the first player exists, it can start by coloring of an arbitrary node. Denote by S_z the strategy that starts from the coloring of node z.

Let Alice use a winning strategy S_x. We will prove that Bob can steal Alice's strategy. Let $\{x, y\}$ be an arbitrary line that contains node x. Then Bob can ignore the fact that the node x has been already colored and plays as the first player according to the strategy S_y. Since Alice has colored node x she will never color it again. As to Bob his strategy S_y never requires to color node x because he loses in that case. Therefore Bob can always move according to the strategy S_y. So he will not lose.

3.2. [2, theorem 4]. Let a winning strategy for Alice exist, assume that the first move of Alice according to this strategy is a coloring of a set of nodes S.

We will prove that for each Alice's first move Bob has an answer that allows him do not lose. Let Alice color a set U at the first move. If U is the whole board then the game is already finished and Bob has not lost. In the opposite case there exists node x in the board that has not been colored yet. The game is transitive, so there exists a winning strategy F for the first player that begins from coloring of a set S' containing x. Let Bob color node x and those nodes of S' that are not colored yet. Let he play after that according to the strategy F.

There are 2 obstacles in this way.

1) In fact some nodes in S' were possibly colored by Alice. It is not a disadvantage for Bob, and does not prevent him to use strategy F due to avoidance nature of the game.

2) Several nodes outside the set S' (namely, the nodes in $U \setminus S'$) have been colored by Alice before Bob's first move. Then consider Alice's second move, let she color set V. Now Bob can think that the game begins from coloring of S' by him at the first move and continues with coloring of $U \setminus S'$ and V by Alice at the second move.

3.3. A n s w e r: yes. Example from [2]. Arrange 6 nodes in 2 rows, 3 nodes in a row, let lines be the sets containing 3 points of one of two kinds: either this set contains exactly one of two nodes from each column and even number of nodes in the upper row or it contains two points from one column and one point from the next column in cyclic order. It is now verified directly that this board is transitive and the first player wins.

3.4. [2, theorem 6]. Let $n = pq$. We assume that q non intersecting circles are drawn on the board, each circle contains p nodes. Let $p' = (p+1)/2$, $q' = (q+1)/2$. Choose an arbitrary q' circles and choose p' nodes in each of these circles. Consider a family of all sets W consisting of $p'q'$ nodes that can be obtained by this way for all choices. Let lines be sets of $p'q'$ nodes that do not belong to W. Transitivity of this board is provided by permutations that permute all nodes in each circle only or permute the whole circles only and by compositions of these permutations.

It is clear that any two sets of W are intersecting. So if we want to prove that Alice wins it is sufficient to check that she can construct a set from W after her $p'q'$ moves (then Bob on his next move will construct a line). We call a circle active if Alice has colored in it at least one node but less than p' nodes. We call a circle full if Alice has colored p' nodes in it. The winning strategy for Alice is to apply the first rule from the list below that could be applied.

1) If the last move of Bob was in active circle then Alice chooses any node in this active circle.

2) If less than q' circles are active or full then Alice chooses any node in empty circle.

3) Alice chooses any node in any active circle.

The first rule guarantees that Alice always has colored more nodes than Bob in any active circle. The second rule guarantees that more than one half of non empty circles are active or full after Alice's move.

According to this strategy after $p'q'$ Alice will color exactly p' nodes in exactly q' circles.

3.5. [2, theorem 24]. Let a board consist of $2k + 1$ copies of the 6-node board from solution 3.3 (and lines are sets of 3 nodes, belonging to the same copy and forming a line in it in a sense of solution 3.3). The transitivity of this board is clear.

Now let Alice play on the first board as in the solution 3.3. All other boards she splits onto pairs and plays symmetrically on them.

3.6. [2, corollary 25]. We modify the example of the solution 3.5. Let $k > 2r$ and k be as big as we will need. Consider the 6-node board from solution 3.3, let L be the set of lines on this board as in solution 3.3, each line contains 3 nodes. Consider a big board consisting of $n = 2k + 1$ copies of 6-node board. Define a set of lines in it (each lines contains r nodes) as

$$L' = \{\ell' \text{ is a set of } r \text{ nodes} : \ell' \supset \ell \ \text{ for some } \ \ell \in L\}$$

Alise's strategy looks like that in the problem 3.5: Alice plays the game from the problem 3.3 on the first board, and plays symmetrically on others. Then at some moment of time Bob constructs a 3-node line $\ell \in L$ (and Alice has constructed no lines from L yet). If r pairs of moves have been made till this moment, Bob loses because line ℓ plus all other his moves form a line from the set L'. If less than r pairs of moves have been made, then Alice begins just to wait making each move on a new board.

3.7. a) [2, proposition 10]. Let $n = 2b$, $b' = (b - 1)/2$. We may think that the board is a square cylinder $b \times 2$, for which we fix the (cyclic) direction of its long side and the nodes are the cells of the cylinder. Denote nodes by pairs (x, ϵ), where x is a residue modulo b, $\epsilon = 0$ or 1.

We call a set of b cells winning if

— either it contains one cell in each column and the total number of cells in the upper row is odd;

— or it contains the unique "full" column (i.e. both cells of this column belong to the set) and the "empty" column (without cells of the set) is situated at most b' cells later in the chosen direction.

Denote by W the set of all winning sets. It is clear that a winning set contains one half of all cells of the board and that the complement of the winning set is not a winning set. Therefore any 2 sets in W intersect. We call a set a line if it is the complement of a winning set. The obtained board is transitive because the rotations of the cylinder along the long side and along the short side preserve lines.

Check that Alice wins on this board. For a set A that contains at most one cell in each column denote by $\bar{A}$ a set that contains the second cells of those columns which contain cell of A. In each moment denote positions of a game by pair (A, B) where A is the set that has been colored by Alice and B is the set that has been colored by Bob.

Observe that for positions of the form $(A \cup (x, \epsilon), \bar{A} \cup (z, \delta))$, where $1 \leqslant z - x \leqslant b'$, Alice has a simple forced victory: she just should color cell $(x, 1 - \epsilon)$ and then continue a game by coloring one cell in each column except column z. Due to the restrictions on x and z Alice will color a winning set of the second type at the end of the game (and hence Bob will color a line).

Now describe the strategy of Alice. If there are no simple forced victory, Alice should leave to Bob after her move a position of the form $(A \cup (x, \epsilon), \bar{A})$, where set A contains at most one cell in each column, and (x, ϵ) is an arbitrary additional cell. Look at this strategy in details. Assume that Bob colors cell (z, δ) in his next move. Then the following cases are possible.

1) $1 \leqslant z - x \leqslant b'$. Then Alice has a forced victory.

2) $(z, \delta) \neq (x, 1 - \epsilon)$. Then Alice colors cell $(z, 1 - \delta)$ and obtains a position in agreement with her strategy.

3) $(z, \delta) = (x, 1-\epsilon)$. Here we have after Bob's move a position of the form $(B, \bar{B})$, in which all columns are either full (both cells are colored) or empty. Then choose the first empty column x'. If $x' < b'$ then Alice colors cell $(x', 0)$ and obtains a position in agreement with the strategy. (We believe also that this case corresponds to the initial position of the game and the first move of Alice was $(0, 0)$.)

If $x' \geqslant b'$, then Alice should play carefully on order to construct a winning set of the first type, for this she should trace the parity. In the very first occurrence of this case all the columns from 0 to b' have been already colored and the column b' is empty because Alice has not made a move in this column yet and if Bob has colored a cell there than he provides the forced victory for Alice. Starting from that moment Alice colors one cell in the first empty column as before. Bob should color the second cell in the same column because otherwise he presents Alice the forced victory. On the last move Alice should chooses the cell that provides desired parity.

b) See [2, theorem 13]. The construction is too big. Looking forward to find shorter one.

3.8. [2, theorem 9]. Non elementary proof uses serious group theory technique.

3.9. [2]. For $n = 3$, 5, 7 this board does not exist. For $n = 11$, 13 examples are found by computer.

4 Special boards

4.1. a) [2, lemma 8]. Flip allows to define symmetric strategy.

b) [2, proposition 7]. Buy the board in the shop and remove all its lines except those that have $n/2$ nodes. Let antiline be the complement of a line. Then the complement of each antiline is a line. Let Alice play a game "Construct an antiline". She can win this game by stealing a strategy if necessary. Victory in this game is a victory in Line avoidance game also.

4.2. All the cycles of the permutation τ have even length, let their lengths equal $2^{a_1}b_1$, $2^{a_2}b_2$, $\ldots$, $2^{a_\ell}b_\ell$ where all the numbers b_i are odd. Let $b = b_1 b_2 \ldots b_\ell$. Then τ^b is a flop.

4.3. [1, proposition 2.3]. Not paying attention to how weird it seems, the text below is the problem solution suitable for odd n also, nevertheless its sense in this case seems so doubtful at each moment of time.

Let G be a permutation group preserving lines. By the result of the previous problem, we know that if a permutation $\tau \in G$ permutes some set with its complement then some power of the permutation τ is a flop but it's prohibited . Thus the orbits of the group G on the family of sets of size $n/2$ are split into "mutually complement" pairs: if $\{F_1, F_2, \ldots, F_k\}$ is an orbit of the set F_1 then $\{\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_k\}$ is an orbit of the set $\bar{F}_1$.

Then we can use the construction of the problem 2.2. It's evident that the group G maps the lines of the set $\mathcal{L}_1$ onto lines of the same set. As to lines from the set $\mathcal{L}_2$ the test checking whether a set belongs to $\mathcal{L}_2$ (the sets B and $\bar{B}$ have to intersect with each line of $\mathcal{L}$) is invariant with respect to the action of the group G (i. e. for $\tau \in G$ the sets τB and $\tau \bar{B}$ intersect with any line too). So the remark above about complement pairs of orbits allows to choose which of the sets B, $\bar{B}$ should be included in $\mathcal{L}_2$ by G-invariant way.

4.4. [1, theorem 3.2]. It is sufficient to construct a group of permutations that acts transitively on the set of nodes and does not contain flops. Let nodes be a lattice points of the rectangle $2^a \times b$. Denote by s_i, $i = 1, \ldots, b$, a cyclic shift of the nodes on the i-th horizontal, and by t a permutation that cyclically permutes all the horizontals (in each vertical row the nodes are permuted by the same way). Then the set of permutations of the form

$$s_1^{k_1} s_2^{k_2} \ldots s_b^{k_b} t^k, \quad \text{where } \ k_1 + k_2 + \ldots + k_b \,\vdots\, 2^a \qquad (*)$$

is a group (this set is closed under compositions). Since b is odd a flip in that group must be of the form $s_1^{k_1} s_2^{k_2} \ldots s_b^{k_b} t^k$ where $k = 0$ and the exponents k_i could be equal to 0 or 2^{a-1} only. Since b is odd at least one of exponents k_i has to be equal to zero.

4.5. Verify the implication $(k \in B) \Rightarrow (k \in A)$. If G is a group of permutations of the set $[n]$ that does not contain a flop take an empty set of lines as beginning set. Then by the assertion of the problem 4.3 it can be enlarged to intersecting family of 2^{n-1} lines.

To prove implication $(k \in A) \Rightarrow (k \in A)$ note that the group of permutations acting on the transitive board with 2^{n-1} lines can not exchange any set with its complement. Then this group does not contain flops as we see in the solution of problem 4.3.

4.6. We will better check this property for the set B. If a group G_1 acts on a board X_1 with k nodes and does not contain flops and a group G_2 acts on a board X_2 with n nodes and does not contain flops also then the group $G_1 \times G_2$ acts on the board $X_1 \times X_2$ with kn nodes and, trivially, does not contain flops.

4.7. In the solution 4.4 we construct a transitive permutation group on the board with $2^a b$ nodes, that does not contain a flip. For $a = 1$ this group can not contain a flop. Indeed, an element of the form $(*)$ can not be a flop because for $k \neq 0$ its order is divisible by b, and for $k = 0$ this is an element of the second order, but a flop of the second order is a flip that is impossible.

4.8. The construction of the solution 4.4 can be generalized. See details in [1, theorem 2.4].

References

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