Fermat Points and Euler Lines

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Solutions

1 The Fermat, Napoleon, and Apollonius points

1.1. *Solution 1.* By the sine theorem for $\triangle AT_aB$ and $\triangle AT_aC$,

$$\sin\angle BAT_a : \sin\angle ABT_a = BT_a : AT_a = CT_a : AT_a = \sin\angle CAT_a : \sin\angle ACT_a.$$

Therefore, $\sin\angle BAT_a : \sin\angle CAT_a = \sin(\angle B + \pi/3) : \sin(\angle C + \pi/3)$.

Analogously, $\sin\angle ACT_c : \sin\angle BCT_c = \sin(\angle A + \pi/3) : \sin(\angle B + \pi/3)$ and $\sin\angle CBT_b : \sin\angle ABT_b = \sin(\angle C + \pi/3) : \sin(\angle A + \pi/3)$.

By the sine Ceva theorem, the statement of the problem follows.

The second part of the problem is resolved analogously.

Solution 2. Since a 60° rotation about A maps $\triangle ABT_b$ onto $\triangle AT_cC$, the lines BT_b and CT_c and the circumcircles of $\triangle ABT_c$ and $\triangle CAT_b$ meet at a point. By reasoning analogously about B and C, we obtain that AT_a, BT_b, CT_c, and the circumcircles of $\triangle ABT_c$, $\triangle BCT_a$, and $\triangle CAT_b$ all meet at a point. (Figure 1)

The second part of the problem is resolved analogously.

1.2. The solution is analogous to the first solution of the previous problem.

1.3. *Solution 1.*

a) By the cosine theorem for $\triangle CN_aN_b$, we obtain

$$N_aN_b^2 = \frac{CA^2}{3} + \frac{CB^2}{3} - 2\frac{CA\cdot CB}{3}\cos(\angle C + \pi/3) = \frac{AB^2 + BC^2 + CA^2}{6} + \frac{2S_{ABC}}{\sqrt{3}}.$$

Since the right-hand side is symmetric, we obtain identical expressions for the two remaining sides of $\triangle N_aN_bN_c$.

The proof for $\triangle N'_aN'_bN'_c$ is analogous.

b) By the easily verified identities $A\vec{N}_b + B\vec{N}_c + C\vec{N}_a = \vec{0}$ and $A\vec{N}'_b + B\vec{N}'_c + C\vec{N}'_a = \vec{0}$.

c) By the expressions for the sides of $\triangle N_aN_bN_c$ and $\triangle N'_aN'_bN'_c$ obtained in a).

Solution 2.

a) Since AT_1 is a common chord of the circumcircles of AT_cBT_1 and AT_bCT_1, it is perpendicular to the line N_bN_c connecting the centers of those circles. Analogously, $BT_b \perp N_cN_a$ and $CT_c \perp N_aN_b$. Thus $\triangle N_aN_bN_c$ is equilateral, and $\triangle N'_aN'_bN'_c$ is handled analogously.

c) We have that $\triangle AN_bN_c \simeq \triangle T_1N_bN_c$, $\triangle BN_cN_a \simeq \triangle T_1N_cN_a$, and $\triangle CN_aN_b \simeq \triangle T_1N_aN_b$. Therefore,

$$S_{N_aN_bN_c} = \frac{1}{2}(S_{ABC} + S_{ABN_c} + S_{BCN_a} + S_{CAN_b}).$$

[1] Problems and solutions by N. Beluhov and A. Zaslavsky. Conference presentation by A. Zaslavsky, P. Kozhevnikov, D. Krekov, and O. Zaslavsky.

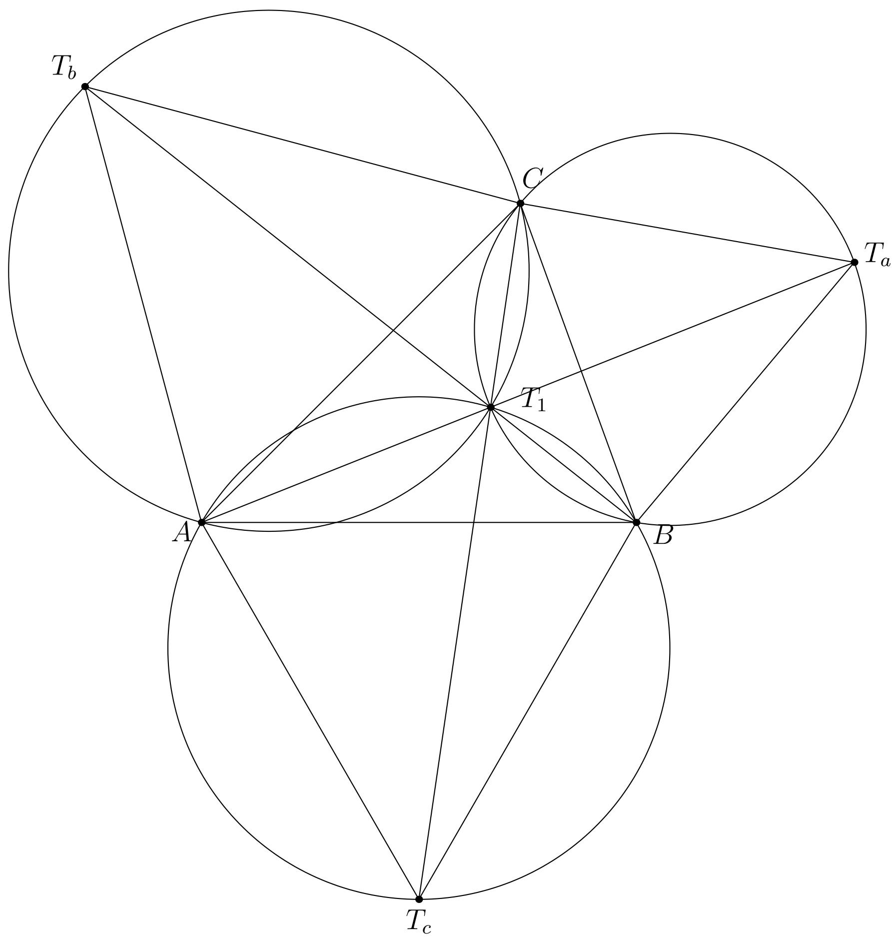

Figure 1

Analogously,

$$S_{N'_aN'_bN'_c} = \frac{1}{2}(-S_{ABC} + S_{ABN'_c} + S_{BCN'_a} + S_{CAN'_b}).$$

The claim follows.

1.4.

a) By the second solution of Problem 1.1.

b) Let X be any point in the plane. Let the $60°$ rotation about A mapping $\triangle ABT_b$ onto $\triangle AT_cC$ also map T_1 and X onto T' and Y. Then (Figure 2)

$$AX+BX+CX = CX+XY+YT_c \geq CT_c = CT_1+T_1T'+T'T_c = AT_1+BT_1+CT_1.$$

Equality is attained if and only if $X \equiv T_1$.

(That $XA+XB \geq XT_c$ also follows by Ptolemy's theorem for the quadrilateral ABT_cX.)

1.5. Let AT_1, BT_1, and CT_1 meet the circumcircle k of $\triangle ABC$ for the second time at A'', B'', and C''. Then $\triangle A''B''C''$ is similar to $\triangle A'B'C'$ and we are left to show that the Fermat lengths of $\triangle ABC$ and $\triangle A''B''C''$ are equal.

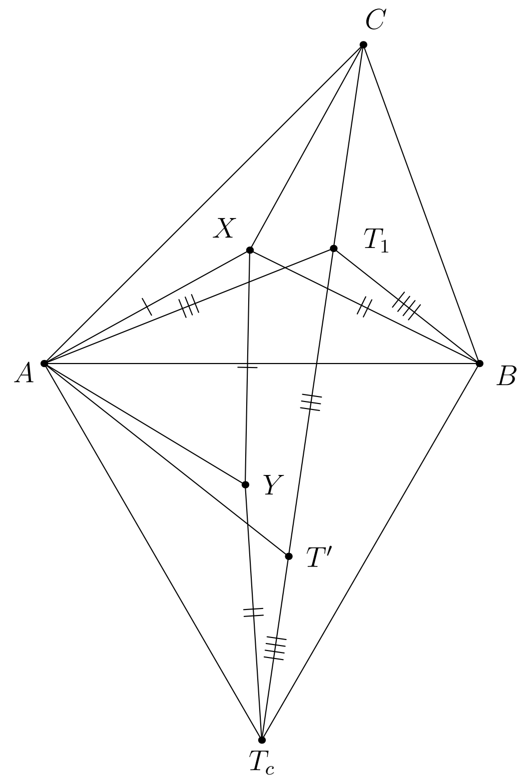

Figure 2

Solution 1. Let $\triangle B''C''T_a''$ be equilateral and external to $\triangle A''B''C''$. By the solution of the previous problem, the Fermat lengths of the two triangles equal AT_a and $A''T_a''$. (Figure 3)

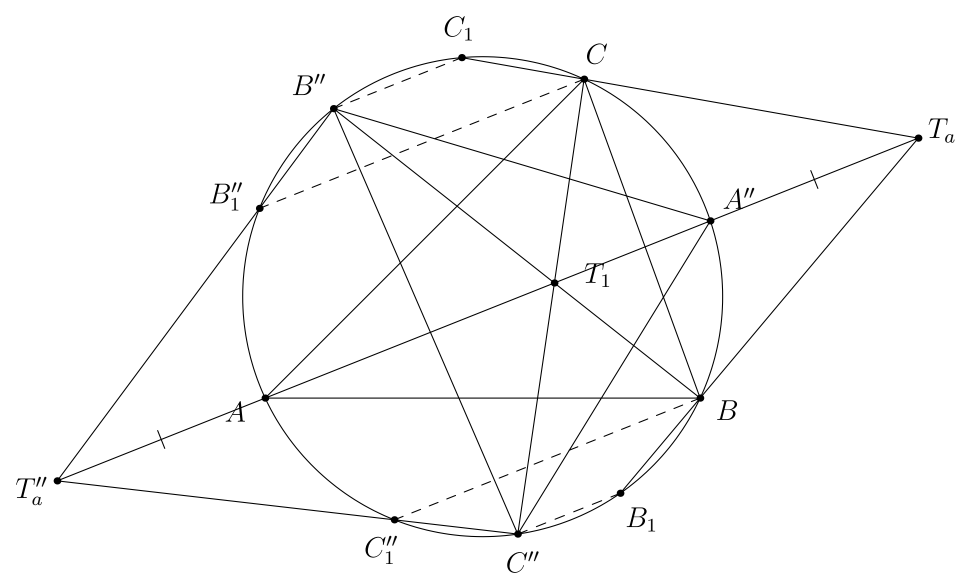

Figure 3

Let BT_a, CT_a, $B''T_a''$, and $C''T_a''$ meet k for the second time at B_1, C_1, B_1'', and

C_1''. By an angle chase, $BC_1''\|CB_1''\|B''C_1\|C''B_1$. Therefore, the figure is symmetric with respect to the perpendicular bisector $s_{AA''}$ of AA''. It follows that the segments AT_a and $A''T_a''$ are reflections of each other in $s_{AA''}$ and that their lengths are equal, as needed.

Solution 2. Let k' be the circle through T_1 concentric with k, and let k' meet the lines AA'', BB'', and CC'' for the second time at K_a, K_b, and K_c. We consider the configuration when the points A, T_1, K_a, A''; B, T_1, K_b, B''; and C, K_c, T_1, C'' lie in this order on the corresponding lines, and all other configurations are analogous.

By symmetry, $AT_1 = K_aA''$, $BT_1 = K_bB''$, and $CK_c = T_1C''$. We are left to prove that $T_1K_a+T_1K_b = T_1K_c$, and this holds by Ptolemy's theorem for $T_1K_aK_bK_c$.

1.6.

a) The point N_c' lies on the circumcircle k_c of $\triangle ABT_c$, with $N_c'T_c$ being a diameter of k_c. Since T_1 also lies on k_c, $\angle N_c'T_1C = \pi/2$. On the other hand, $M_CM : M_CC = M_CN_c : M_CT_c = 1 : 3$, so $N_cM\|CT_c$. Therefore, N_cM is the perpendicular bisector of $N_c'T_1$ and $MN_c' = MT_1$. (Figure 4)

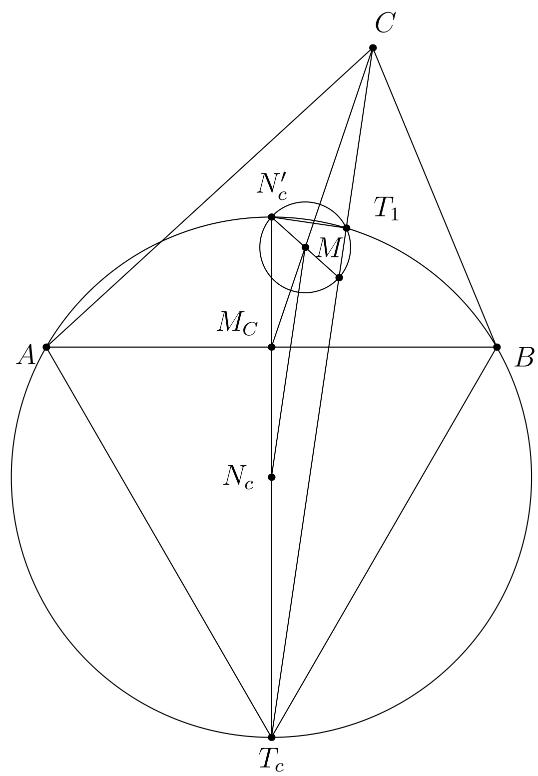

Figure 4

b) Since M is the circumcenter of $\triangle N_a'N_b'N_c'$ and $\angle N_c'T_1C = \pi/2$, the second intersection point of the circumcircle of $\triangle N_a'N_b'N_c'$ and the line CT_c is the reflection of N_c' in M. (Figure 4) Since the line CM divides the segment $N_c'T_c$ in ratio $1:3$, the claim follows by Menelaus' theorem.

1.7.

a) See the second solution to problem 1.3 a).

b) By the cosine theorem for $\triangle AT_bN_c$ and $\triangle BT_aN_c$, we have that $N_cT_a^2 - N_cT_b^2 = CT_a^2 - CT_b^2$ and the claim follows.

1.8.

a) The reflections of T_1^* in the sides of $\triangle ABC$ form a triangle δ such that the lines T_1A, T_1B, and T_1C are the perpendicular bisectors of its sides. (Figure 5) Therefore, all of δ's angles equal 60°.

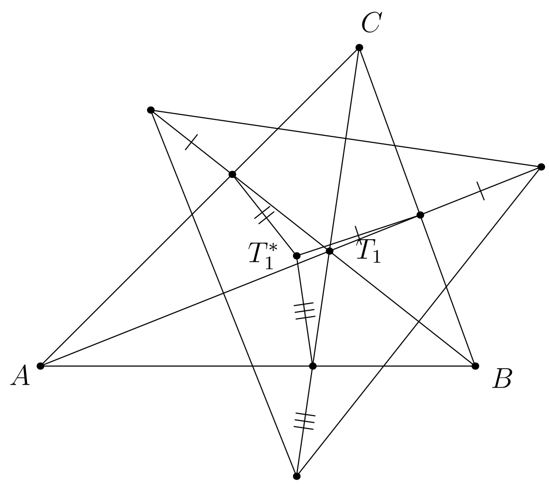

Figure 5

b) Let X and Y be the projections of T_1^* onto AC and BC. Then the quadrilateral CXT_1^*Y is inscribed in a circle on diameter CT_1^*. Therefore, $XY = CT_1^* \cdot \sin \angle C$ and the claim follows by a).

1.9. The lines AT_1, BT_1, and CT_1 contain the reflections of T_1^* in the sides of $\triangle ABC$. (Figure 5) Therefore, all three billiard balls eventually arrive at T_1^*, and at the same moment. (A corollary of this problem was on the 28th Tournament of Towns, Spring, 10-11.7: The reflections of AT_1, BT_1, and CT_1 in the opposite sides of $\triangle ABC$ are concurrent.)

1.10. The locus of the points X such that $AX : BX = AC : BC$ is a circle Ω_c perpendicular to the circumcircle k of $\triangle ABC$. Since T_1^* and T_2^* are the intersection points of Ω_a, Ω_b, and Ω_c, they are mutually inverse with respect to k.

1.11. The centers S_a, S_b, and S_c of Ω_a, Ω_b, and Ω_c are the intersections of the sides of $\triangle ABC$ with the tangents to k at the opposite vertices. Since the line $S_aS_bS_c$ is the polar of L with respect to k, it is perpendicular to OL. Since the line $T_1^*T_2^*$ contains O and is perpendicular to $S_aS_bS_c$, it coincides with OL.

The two remaining parts of the problem are special cases of Problem 3.3.

1.12. A special case of Problem 3.3.

1.13. A special case of Problem 3.3.

1.14. A special case of the three pairs of isogonal conjugates theorem (see [1]).

1.15. By Problems 3.1 and 3.2.

2 Euler lines and Steiner ellipses

In all solutions for this section, we only consider the configuration when T_1 lies inside $\triangle ABC$ and the quadrilateral $ABCT_2$ is convex. All other configurations are handled analogously.

2.1. We show that the Euler line of $\triangle BCT_1$ passes through M. All other lines in the problem are handled analogously.

Let M_a be the medicenter of $\triangle BCT_1$. Since N_a is the circumcenter of $\triangle BCT_1$, the Euler line of $\triangle BCT_1$ is M_aN_a. (Figure 6)

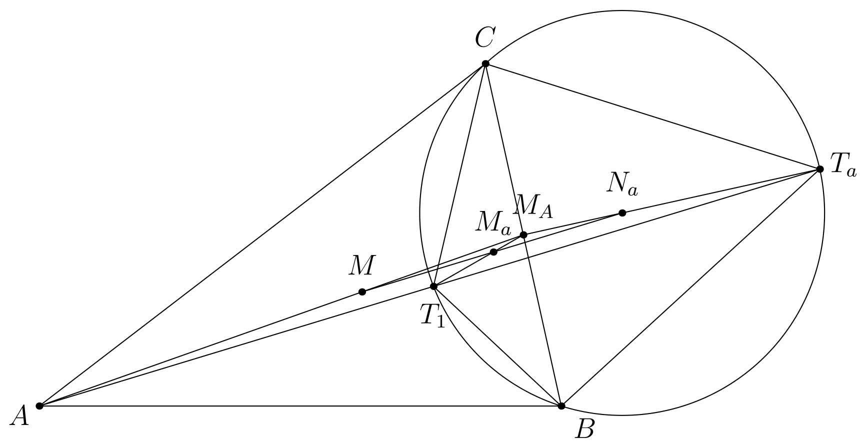

Figure 6

Since the points A, T_1, and T_a are collinear and the points M, M_a, and N_a divide the segments M_AA, M_AT_1, and M_AT_a in the same ratio $1:2$, they are collinear as well.

2.2. Let M_a and O_a be the medicenter and circumcenter of $\triangle AT_1T_2$. Since N_bN_c is the perpendicular bisector of AT_1 and $N_b'N_c'$ is the perpendicular bisector of AT_2, the lines N_bN_c and $N_b'N_c'$ meet at O_a. (Figure 7)

Let x be the acute angle between the lines MO_a and $N_b'N_c'$.

Since M is the common center of $\triangle N_aN_bN_c$ and $\triangle N_a'N_b'N_c'$, $\angle MN_cN_b = \angle MN_b'N_c' = 30^\circ$. Consequently, the quadrilateral $MO_aN_cN_b'$ is cyclic and

$$\angle N_bN_cN_b' = \angle O_aN_cN_b' = 180^\circ - \angle N_b'MO_a = \angle MO_aN_b' + \angle O_aN_b'M = x + 30^\circ.$$

Notice that $\triangle AN_bT_b$, $\triangle AN_cB$, and $\triangle AN_b'C$ are similar and identically oriented. It follows that similitude of center A maps $\triangle N_bN_cN_b'$ onto $\triangle T_bBC$. Thus

$$\angle T_1BC = \angle T_bBC = \angle N_bN_cN_b' = x + 30^\circ.$$

Let T' be the reflection of T_1 in M_A. Then $\angle BT'C = \angle BT_1C = 120^\circ$. Consequently, the quadrilateral $BT'CT_2$ is cyclic and

$$\angle BT_2T' = \angle BCT' = \angle T_1BC = x + 30^\circ.$$

Since the lines BT_2 and $N_a'N_c'$ are perpendicular, the lines BT_2 and $N_b'N_c'$ make a 30° angle. Thus from $\angle BT_2T' = x + 30^\circ$ it follows that $MO_a \| T'T_2$.

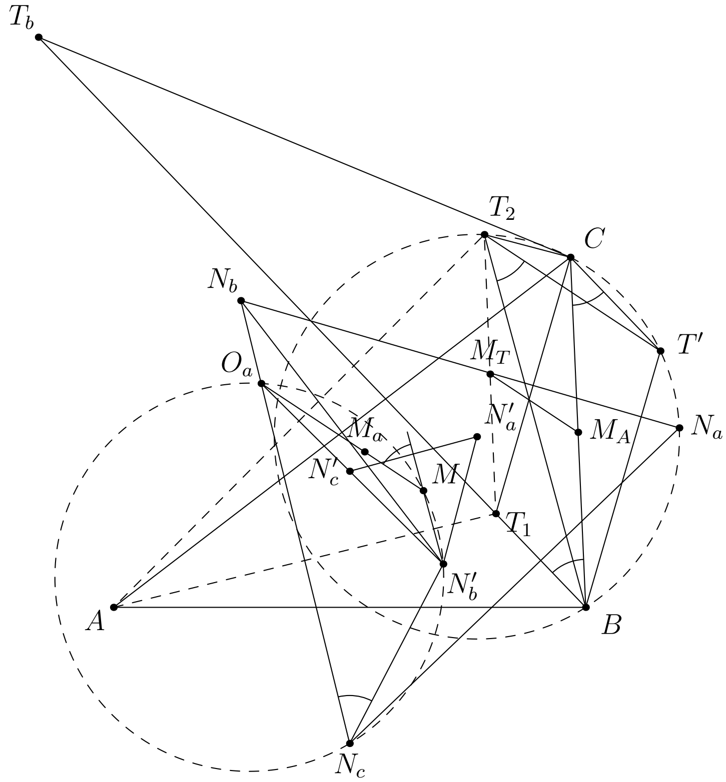

Figure 7

Since M_A and M_T are midpoints of T_1T' and T_1T_2, $M_AM_T \| T'T_2$.

Since M and M_a divide the segments AM_A and AM_T in the same ratio $2:1$, $MM_a \| M_AM_T$.

Consequently, $MO_a \| T'T_2 \| M_AM_T \| MM_a$. It follows that the points M, M_a, and O_a are collinear, M lies on l_A, and $l_A \| M_AM_T$.

2.3. *Solution 1.* By Problem 2.1 and the solution to Problem 2.2.

Solution 2. By Problem 4.4 d) applied to $\triangle ABT_1$ and the point T_2, l_A, l_B, and the Euler line of $\triangle ABT_2$ are concurrent. Analogously, l_A, l_B, and the Euler line of $\triangle ABT_1$ are concurrent. By Problem 2.1, it follows that l_A and l_B meet at M.

2.4. There exist two distinct points T_3 satisfying the conditions of the problem, corresponding to the two possible orientations of $\triangle T_1T_2T_3$, namely $T_3 \equiv P$ and $T_3 \equiv Q$. Ordinarily, orientation does not matter much in geometry problems. However, in this problem the two cases for T_3 need to be handled very differently.

Case 1. $T_3 \equiv P$.

Let A' be the isogonal conjugate of A in $\triangle T_1T_2P$ and let A'' be the reflection of A' in the line T_1T_2. (Figure 8)

Then $\angle A''T_1T_2 = \angle T_2T_1A' = 180^\circ - \angle PT_1A =$ (since $\angle AT_1C = 120^\circ$ and $\angle PT_1T_2 = 60^\circ$) $= \angle CT_1T_2$.

It follows that the points T_1, A'', and C are collinear. Analogously, the points T_2, A'', and C are collinear. Thus $A'' \equiv C$.

Reflection in the internal bisector of $\angle T_1PT_2$ followed by reflection in T_1T_2 maps

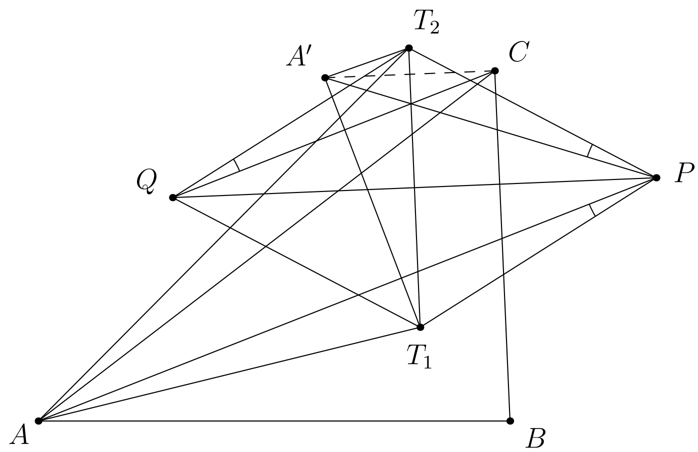

Figure 8

the line AP onto the line CQ, so $AP \| CQ$. Analogously, $BP \| AQ$ and $CP \| BQ$.

Therefore, when $T_3 \equiv P$, the three lines in the problem meet at Q.

Case 2. $T_3 \equiv Q$.

Let X be the intersection point of AP and BQ, and let $\triangle RYB$ and $\triangle ZCP$ be translation copies of $\triangle AQX$. (Figure 9)

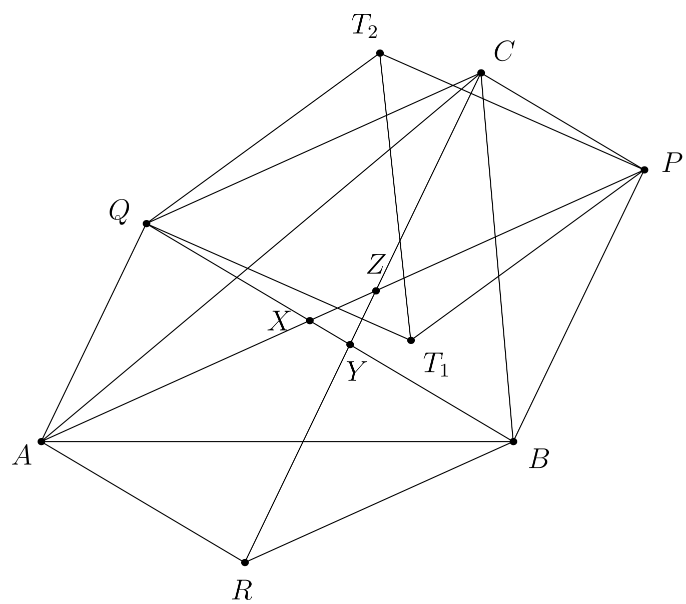

Figure 9

Then the points C, Y, Z, and R are collinear, $AR \| BQ \| CP$, $AP \| BR \| CQ$, and $AQ \| BP \| CR$.

Therefore, when $T_3 \equiv Q$, the three lines in the problem meet at R.

2.5. *Solution 1.* By Problem 2.2, l_A is parallel to $M_A M_T$. By Problem 2.4, CP and BQ are parallel. Since M_T is the midpoint of PQ, this means that $M_A M_T$ is the midline of the two parallel lines CP and BQ, and is thus parallel to both of them.

Solution 2. By Problem 4.1 b), P lies on the Neuberg cubic for $\triangle AT_1T_2$. Since the isogonal conjugate of P with respect to that triangle is C, by Problem 4.2 l_A is parallel to CP.

2.6. Let X be such that the line through A parallel to BX, the line through B parallel to CX, and the line through C parallel to AX are concurrent at Y.

Let α be an affine transformation mapping $\triangle ABC$ to an equilateral $\triangle A'B'C'$. Let $\alpha(X) = X'$ and $\alpha(Y) = Y'$. Then the line through A' parallel to $B'X'$, the line through B' parallel to $C'X'$, and the line through C' parallel to $A'X'$ are concurrent at Y'.

Consider a rotation ρ concentric with $\triangle A'B'C'$ such that ρ maps A' onto B', B' onto C', and C' onto A'.

Let $\rho(Y') = Z'$. Then each of $\angle X'A'Z'$, $\angle X'B'Z'$, and $\angle X'C'Z'$ equals either $60°$ or $120°$. Consequently, the points X', Z', A', B', and C' are concyclic.

It follows that X' lies on the circumcircle of $\triangle A'B'C'$. Therefore, X lies on the circumscribed Steiner ellipse of $\triangle ABC$.

2.7. By Problems 2.4 and 2.6.

2.8. By Problem 2.5 and Case 2 of the solution to Problem 2.4.

2.9–2.11. Define the affine transformation α and $\triangle A'B'C'$ as in the solution to Problem 2.6. Let $\alpha(\triangle PQR) = \triangle P'Q'R'$.

As in the solution to Problem 2.6, $\triangle A'B'C'$ and $\triangle P'Q'R'$ are concentric and equal equilateral triangles. Thus they have the same area and a common medicenter. Since both of those properties are preserved by affine transformations, so do $\triangle ABC$ and $\triangle PQR$.

Since the midpoint of $P'Q'$ lies on the common incircle e' of $\triangle A'B'C'$ and $\triangle P'Q'R'$, the midpoint of PQ lies on the common inscribed Steiner ellipse e of $\triangle ABC$ and $\triangle PQR$.

Since $P'Q'$ is tangent to e' at its midpoint and tangency is preserved by affine transformations, PQ is tangent to e at M_T. Since T_1T_2 is perpendicular to PQ at M_T, it is a normal to e.

3 The Kiepert hyperbola

3.1.

a) Similarly to Problems 1.1 and 1.2.

b) We are going to need the following easily verified claim (see [1] and [3] for details).

Lemma. Let A and B be two points in the plane, and let f be a transformation mapping every line through A onto a line $f(l)$ through B in such a way that cross-ratios are preserved. Then the locus of the intersection points of l and $f(l)$ as l varies is a conic through A and B. (When $f(AB) = AB$, that conic degenerates into the union of the line AB and one more straight line.)

The transformation f defined by $f(AA') = BB'$ satisfies the conditions of the Lemma. Therefore, the required locus is a conic through A and B.

When $\triangle AB'C$ and $\triangle BA'C$ are constructed internally and their base angles equal $\angle C$, the lines AA' and BB' coincide with AC and BC, and we obtain that C also lies on this conic.

Since the required locus also contains M and H (corresponding to setting the base angles of the three isosceles triangles to $0°$ and to letting them tend to $90°$), it is a rectangular hyperbola.

Note. Since five points uniquely determine a conic, the locus of the intersection of AA' and CC' is the same hyperbola. This gives us a second proof of part a).

3.2. Isogonal conjugation with respect to a triangle maps a circumscribed conic onto a straight line. Since the Kiepert hyperbola contains M and H, its isogonal conjugate is the line OL.

Since the points T_1^* and T_2^* are mutually inverse with respect to the circumcircle, isogonal conjugation maps them onto opposite points on the Kiepert hyperbola. Therefore, M_T is the center of the Kiepert hyperbola and thus lies on the Euler circle. By Problem 2.9, we obtain also that the point R lies on the circumcircle of $\triangle ABC$.

Let $A'(\varphi)$, $B'(\varphi)$, and $C'(\varphi)$ be the third vertices of the isosceles triangles corresponding to base angle φ, so that $AA'(\varphi)$, $BB'(\varphi)$, and $CC'(\varphi)$ meet at $X(\varphi)$.

3.3. *Solution 1.*

a) When the Kiepert hyperbola is projected from A onto the perpendicular bisector of BC, the points $X(\varphi)$ and $X(-\varphi)$ are mapped onto the points $A'(\varphi)$ and $A'(-\varphi)$ symmetric with respect to BC. Therefore, the transformation from the Kiepert hyperbola onto itself mapping $X(\varphi)$ onto $X(-\varphi)$ preserves cross-ratios.

The transformation mapping X onto the second intersection point of the Kiepert hyperbola with the line XL also preserves cross-ratios. Therefore, to establish the claim it suffices to find three values of φ such that the two transformations coincide.

When $\varphi = \angle A$, $X(-\varphi)$ coincides with A and $X(\varphi)$ lies on the symmedian through A. Analogously, the two transformations coincide when $\varphi = \angle B$ and $\varphi = \angle C$.

b), c) Similarly to part a).

Solution 2. a), b) By Problem 3.4. c) Similarly to the first solution of Problem 3.4. Use the fact that AE contains $A'(90° - \angle A)$ and analogously for BE and CE.

The special cases $\varphi = \pi/3$ and $\varphi = \pi/6$ of this problem yield Problems 1.11–1.13.

3.4. *Solution 1.* (N. Beluhov) We use the following lemma.

Lemma. Let $\varphi_1 + \varphi_2 = \angle A$. Then the lines $AA'(\varphi_1)$ and $AA'(\varphi_2)$ are symmetric with respect to the bisectors of $\angle A$.

Proof. Let L_A and L_A' be the midpoints of the arcs BC not containing A and BAC of the circumcircle k of $\triangle ABC$. (Figure 10) Then BL_A and BL_A' are the bisectors of $\angle A'(\varphi_1)BA'(\varphi_2)$ in $\triangle BA'(\varphi_1)A'(\varphi_2)$. It follows that the circumcircle k of $\triangle BL_AL_A'$ is the Apollonius circle for B in $\triangle BA'(\varphi_1)A'(\varphi_2)$. Therefore, AL_A and AL_A' are the bisectors of $\angle A'(\varphi_1)AA'(\varphi_2)$.

Let the line $X(\varphi_1)X(\varphi_2)$ meet AB at Y.

For $X^*(\varphi_3)$ to lie on $X(\varphi_1)X(\varphi_2)$, it is necessary and sufficient that $AX^*(\varphi_3)$ and $BX^*(\varphi_3)$ meet the line $X(\varphi_1)X(\varphi_2)$ at the same point. This holds if and only if the cross-ratio of the four lines AY, $AX(\varphi_1)$, $AX(\varphi_2)$, and $AX^*(\varphi_3)$ equals the cross-ratio of the four lines BY, $BX(\varphi_1)$, $BX(\varphi_2)$, and $BX^*(\varphi_3)$.

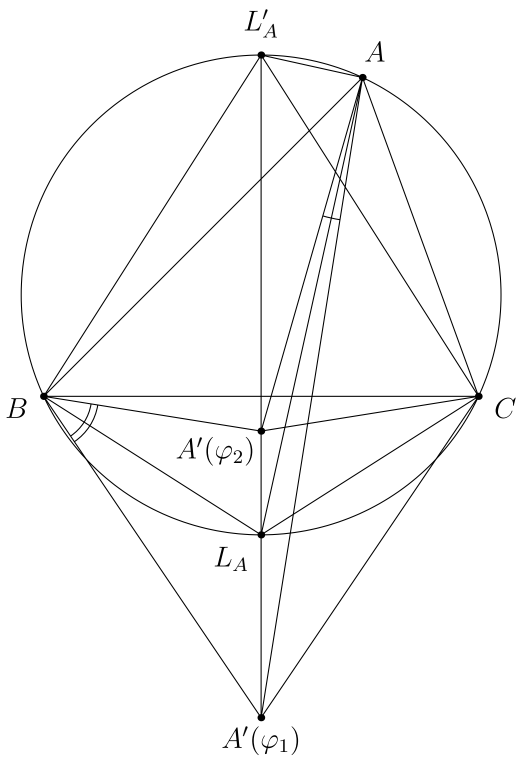

Figure 10

The lines AY, $AX(\varphi_1)$, $AX(\varphi_2)$, and $AX^*(\varphi_3)$ meet the perpendicular bisector of AC at $A'(-\angle A)$, $A'(\varphi_1)$, $A'(\varphi_2)$, and, by the Lemma, $A'(\angle B-\varphi_3) \equiv A'(\angle B+\varphi_1+\varphi_2)$. It follows that the angles between successive lines equal $\angle A+\varphi_1$, $\varphi_2-\varphi_1$, and $\angle B+\varphi_1$.

Analogously, the angles between successive lines in the sequence BY, $BX(\varphi_1)$, $BX(\varphi_2)$, and $BX^*(\varphi_3)$ are the same, only ordered in reverse.

Solution 2. Let $\varphi_1 + \varphi_2 = 2\varphi_0$. Reasoning as in the previous problem, we obtain that the quadrilateral $X(\varphi_1)X(\varphi_0)X(\varphi_2)X(\pi/2 + \varphi_0)$ is harmonic, i.e., the line $X(\varphi_1)X(\varphi_2)$ contains the pole of the line $X(\varphi_0)X(\pi/2 + \varphi_0)$.

However, for all φ_0 the line $X(\varphi_0)X(\pi/2+\varphi_0)$ contains E. It follows that when φ_1 and φ_2 vary so that $\varphi_1 + \varphi_2$ remains equal to $2\varphi_0$, all lines $X(\varphi_1)X(\varphi_2)$ meet at a point X on the polar of E, which by the previous problem coincides with the line OL.

Since the Kiepert hyperbola and the line OL are isogonal conjugates, X is the isogonal conjugate of some point $X(\varphi_3)$ on the Kiepert hyperbola. To work out the relation between φ_1 and φ_2 on the one hand, and φ_3 on the other hand, we are going to use the two pairs of isogonal conjugates theorem.

Let $X(\varphi_1)$ and $X(\varphi_2)$ be two points on the Kiepert hyperbola and let $X^*(\varphi_1)$ and $X^*(\varphi_2)$ be their isogonal conjugates on the line OL. By the two pairs of isogonal conjugates theorem, the lines $X(\varphi_1)X^*(\varphi_2)$ and $X^*(\varphi_1)X(\varphi_2)$ meet in a point on the Kiepert hyperbola.

The value of φ corresponding to that point equals, on the one hand, $f(\varphi_1) - \varphi_2$, and, on the other hand, $f(\varphi_2) - \varphi_1$, where f is some (as of yet unknown) function.

It follows that $f(\varphi_1)+\varphi_1 = f(\varphi_2)+\varphi_2 = \text{const}$. Substituting, say, $\varphi_2 = 0$, we obtain $f(\varphi) = -\varphi$, completing the solution.

3.5.

a) Since the perpendiculars from the vertices of $\triangle A'B'C'$ onto the sides of $\triangle ABC$ are concurrent (being the perpendicular bisectors of the sides of $\triangle ABC$), the perpendiculars from the vertices of $\triangle ABC$ onto the sides of $\triangle A'B'C'$ are concurrent as well.

b) We use the following lemma.

Lemma. The lines $A'(\varphi)B'(\varphi)$ and $CC'(\pi/2-\varphi)$ are perpendicular.

Indeed, it follows from the Lemma that the orthologic center of $\triangle ABC$ and $\triangle A'B'C'$ is the point $X(\pi/2-\varphi)$.

Proof 1. Let A'' and B'' be the reflections of C in $A'(\varphi)$ and $B'(\varphi)$. Then $\angle CAB'' = \angle CBA'' = \pi/2$ and $A''B'' \| A'(\varphi)B'(\varphi)$. (Figure 11)

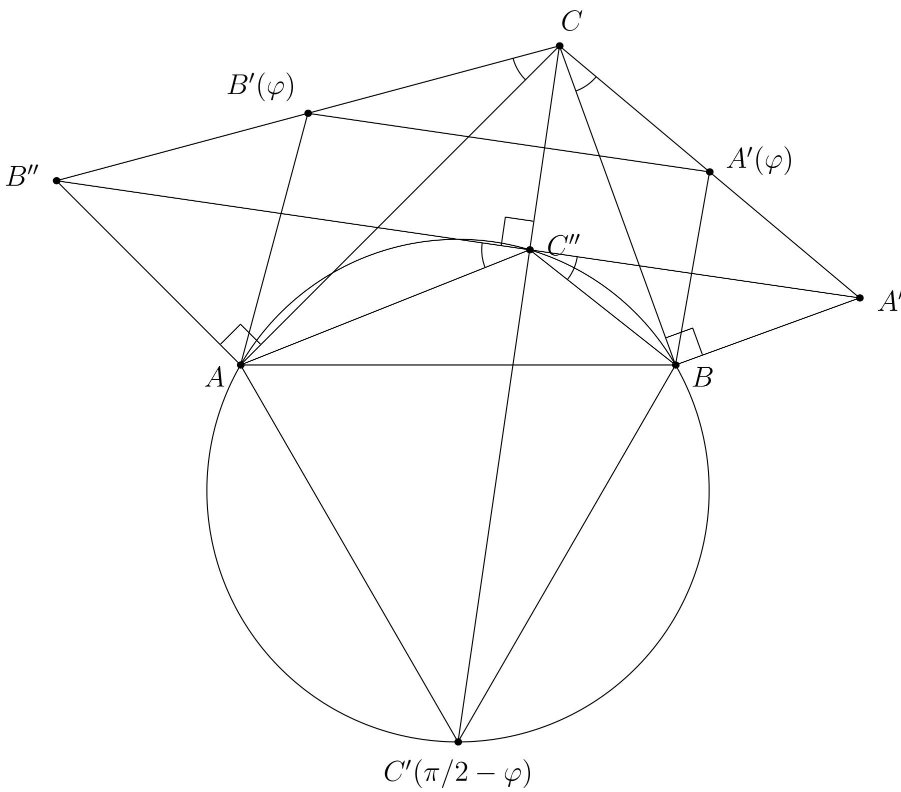

Figure 11

Let C'' be the projection of C onto the line $A''B''$. Since the quadrilateral $CC''BA''$ is inscribed in a circle of diameter CA'', we obtain that $\angle BC''A'' = \angle BCA'' = \varphi$. Analogously, $\angle AC''B'' = \varphi$.

It follows that the quadrilateral $AC'(\pi/2-\varphi)BC''$ is cyclic. Thus $\angle AC''C'(\pi/2-\varphi) = \angle ABC'(\pi/2-\varphi) = \pi/2-\varphi$ and C, C'', and $C'(\pi/2-\varphi)$ are collinear, as needed.

Proof 2. Let A'' be the reflection of A in the line $CB'(\varphi)$. Then $\triangle CA'(\varphi)B$ and $\triangle CB'(\varphi)A''$ are similar and identically oriented. Therefore, so are $\triangle CA'(\varphi)B'(\varphi)$ and $\triangle CBA''$ and the lines $A'(\varphi)B'(\varphi)$ and BA'' make an angle of φ. (Figure 12)

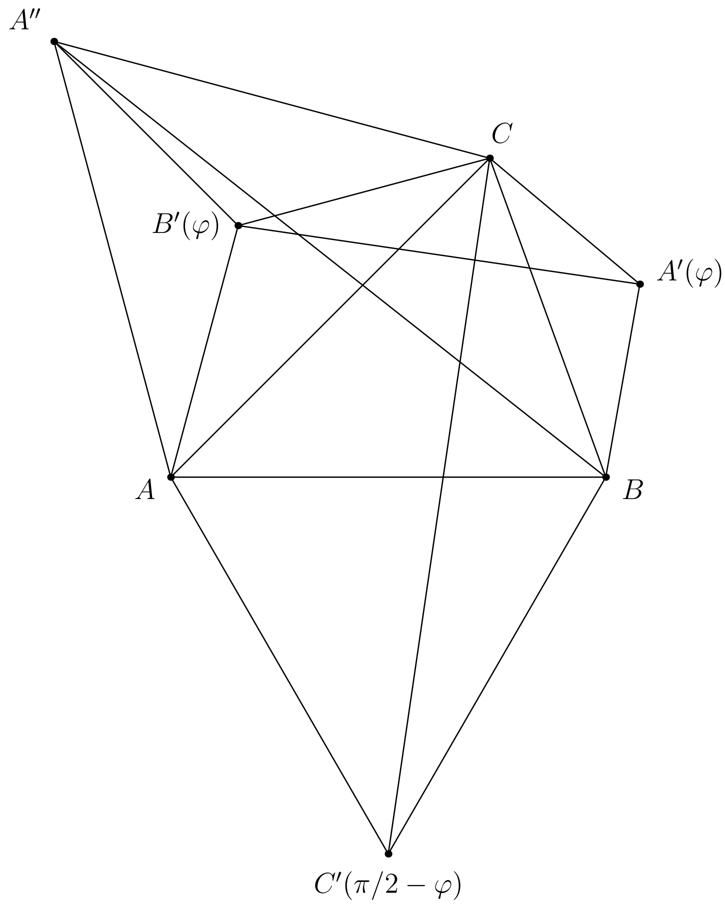

Figure 12

Analogously, $\triangle AC'(\pi/2-\varphi)B$ and $\triangle ACA''$ are similar and identically oriented. Therefore, so are $\triangle ACC'(\pi/2-\varphi)$ and $\triangle AA''B$ and the lines $CC'(\pi/2-\varphi)$ and $A''B$ make an angle of $\pi/2-\varphi$.

4 The Neuberg cubic

4.1.

a) All three lines meet at one of the two Napoleon points of $\triangle X_uX_vX_w$.

b) (From [2]) *Solution 1.* The two triangles formed by the medicenters and circumcenters of $\triangle XUV$, $\triangle XVW$, and $\triangle XWU$ are orthologic, and both orthologic centers coincide with the center of $\triangle UVW$. However, two orthologic triangles with coinciding orthologic centers are always perspective.

Solution 2. Hint. All three lines meet at one of the two Napoleon points of the triangle formed by the circumcenters of $\triangle XUV$, $\triangle XVW$, and $\triangle XWU$.

4.2. See [4].

4.3. (From [2]) We give an elementary proof that the Euler lines of $\triangle ABX$, $\triangle BCX$, and $\triangle CAX$ are concurrent when $X \equiv I$, I_a, I_b, I_c, T_1^*, and T_2^*. We use the following lemma.

Lemma. Let X be a point not on the circumcircle k of $\triangle ABC$, and let AX, BX, and CX meet k for the second time at A', B', and C'. Then the Euler lines of $\triangle ABX$, $\triangle BCX$, and $\triangle CAX$ are concurrent if and only if the Euler lines of $\triangle A'B'X$, $\triangle B'C'X$, and $\triangle C'A'X$ are concurrent.

Proof. Let O_a be the circumcenter of $\triangle BCX$, and define O_b, O_c, O'_a, O'_b, and O'_c analogously.

Suppose that the Euler lines of $\triangle A'B'X$, $\triangle B'C'X$, and $\triangle C'A'X$ are concurrent at Y'. Let Y'' be the isogonal conjugate of Y' in $\triangle O'_aO'_bO'_c$ and let Y be such that the quadrilaterals $O_aO_bO_cY$ and $O'_aO'_bO'_cY''$ are similar. Then the Euler lines of $\triangle ABX$, $\triangle BCX$, and $\triangle CAX$ are concurrent at Y.

When X coincides with I, I_a, I_b, or I_c, X is the orthocenter of $\triangle A'B'C'$ and the claim follows by the Lemma.

When X coincides with T_1^* or T_2^*, $\triangle A'B'C'$ is equilateral and the claim follows by the Lemma and Problem 4.1 b).

4.4.

a) Since the Neuberg cubic is its own isogonal conjugate, it contains the (imaginary) intersection points of the line at infinity and the circumcircle. Every conic through this pair of points is a circle.

b) The Neuberg cubic contains the internal bisector of $\angle C$. Furthermore, it contains A, B and their reflections in the opposite sides.

c) By Problem 4.1 b), the Neuberg cubic contains T_a, T_b, and T'_c. Since it also contains C, it meets the external bisector of $\angle C$ at four distinct points and therefore contains it. Since A, B, O, and H all lie on the reflection of the circumcircle in the line AB, the circle part of the Neuberg cubic coincides with this circle.

d) Similarly to part c).

e) Suppose that the Neuberg cubic consists of a straight line l and a circle s. Since s cannot contain all vertices of $\triangle ABC$, without loss of generality l contains C. Since the Neuberg cubic is its own isogonal conjugate, the isogonal conjugate l' of l also belongs to the Neuberg cubic. Thus $l \equiv l'$. By Problem 4.2, l is then parallel to the Euler line.

f) (From [2]) We are going to use e). Let L_C be the midpoint of the arc ACB of the circumcircle k of $\triangle ABC$. Suppose that $AC \neq BC$ and the Euler line is parallel to CL_C. Then CL_COH is a parallelogram, so CH equals the circumradius of $\triangle ABC$ and $\angle C = 60°$.

4.5. See [4].

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