On the Poncelet theorem
Solutions.

1 Poncelet theorem for $n = 3, 4$

1. Let the line joining I with vertex C of triangle ABC meet for the second time the circumcircle of ABC at point C' (fig.1).

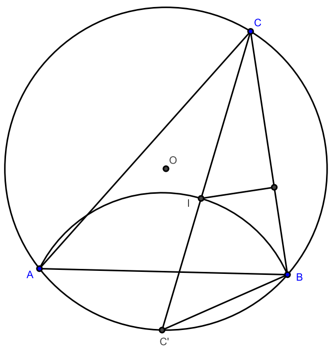

Fig.1

Since $C'A = C'B$, $\angle AIB = \pi - (\angle A + \angle B)/2 = (\pi + \angle C)/2$ and $\angle AC'B = \pi - \angle C$, we obtain that C' is the circumcenter of triangle AIB. Thus $IC' = C'B = 2R \sin \frac{\angle C}{2}$. On the other hand $IC = r/\sin \frac{\angle C}{2}$, therefore

$$R^2 - OI^2 = CI \cdot C'I = 2Rr.$$

2. Let a triangle ABC be given. Take the tangents from an arbitrary point C' of its circumcircle to its incircle and find their common points A', B' with the circumcircle distinct from C'. We have to prove that line $A'B'$ also touches the incircle.

Suppose the opposite. For example let $A'B'$ do not intersect the incircle of ABC. Increase angle $A'C'B'$ in such a way that line $C'I$ stays its bisector. Then the distances from I to lines $C'A'$ and $C'B'$ will increase, and the distance from I to $A'B'$ will decrease, thus sometimes a circle with center I and radius $r' > r$ will be the incircle of triangle $A'B'C'$. But by Euler formula the inradii of triangles ABC and $A'B'C'$ are equal — contradiction. The case when $A'B'$ intersect the incircle can be considered similarly.

3. By the **Feuerbach theorem**, the Euler circle passing through the midpoints A_0, B_0, C_0 of the sides of triangle ABC touches its incircle. From this the center of this circle coinciding with the midpoint of OH lie on the circle with center I and radius $R/2 - r$. Thus the trajectories of points M and H are homothetic to this circle with center O and coefficients $2/3$ and 2 respectively.

The trajectory of the Gergonne point is a circle coaxial with the circumcircle and the incircle. Synthetic proof of this fact is unknown.

The trajectory of the Lemoine point is an ellipse with the minor axis lying on OI. Synthetic proof of this fact is also unknown.

4. Let A'', B'', C'' be the second common points of the altitudes of $A'B'C'$ with the incircle of ABC. Then $A''A'$, $B''B'$, $C''C'$ are the bisectors of triangle $A''B''C''$, i.e. the orthocenter H' of triangle $A'B'C'$ coincides with the incenter of $A''B''C''$. Also it is easy to see that the sidelines of $A''B''C''$ are parallel to the corresponding sidelines of ABC, thus these triangles are homothetic. This homothety maps O and I to I to H' respectively, therefore H' lies on OI and $IH'/OI = r/R$. When the triangle rotates H' and the centroid of $A'B'C'$ dividing $H'I$ in ratio $2:1$ are immobile.

5. **Answer.** Let P' be the point inverse to P wrt the circumcircle. Consider a rotational homothety with center P', transforming P to I and find the image Q of I. The desired trajectory is a circle with center Q (fig.2). This can be proved using the complex numbers. Synthetic proof is unknown.

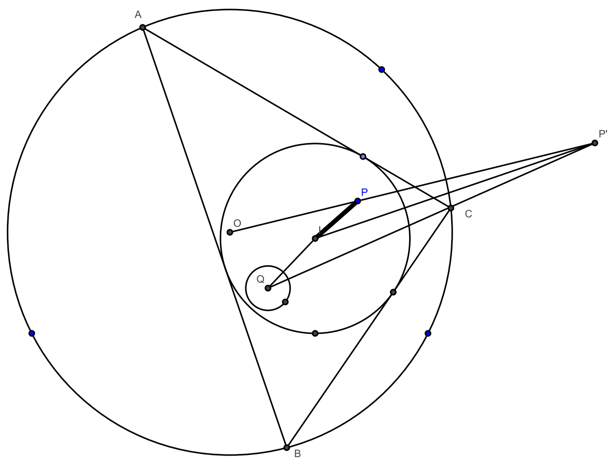

Fig.2

6. **Hint.** The Simson line bisects the segment between the corresponding point and the orthocenter H треугольника. The parallel line passing through H, meets for the second time the trajectory of H at a fixed point (fig.3).

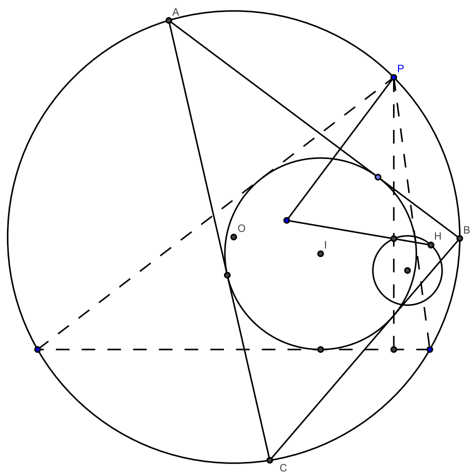

Fig.3

7.

a) Prove general assertion.

Let two circles one lying inside the other be given. From an arbitrary point X of the external circle draw the tangents to the internal one and inscribe to the obtained angle a circle touching the external circle. Then the locus of the centers of such circles is an ellipse having the center of the external circle as a focus.

Proof. Let O, I be the centers of the given circles, and let Y be a touching point. Then all chords XY meet OI at the same point — the homothety center H of the given circles. Since the center Z of the semi-inscribed circle is the common point of lines OY and IX, its trajectory is an ellipse. To prove that O is its focus consider a polar map wrt the circle with center O. It transforms I, H to a parallel lines i, h, also it transforms X, Y to the tangents x, y meeting at some point P of line h. Let U, V be the common point of x and y with i, and let Q be the vertex of parallelogram $PUQV$. Then Q lies on the line symmetric h about i. Since PQ is the fourth harmonic line to x, y, h, it meet the perpendicular from O to h at the fixed point — the pole of h. Using the homothety with the center in this point we obtain that all lines QU, QV (i.e. the polars of points Z) touches the same circle.

Alternative solution. It is sufficient to prove that all semiinscribed circles have common radical center L. Suppose this is true, and all semiinscribed circles have power equal to p wrt L. Then they all touch the image of Γ under inversion with center L and radius $\sqrt{|p|}$ followed by the symmetry in L, for $p < 0$.

Let us fix the outer circle Γ (let O be its center, and R be its radius), and the center I of the inner circle γ (let r be its radius, further r could vary). Let $Z, A, B, C, D \in \Gamma$ so that ZA, AB, BC, CD touch γ. Let ω_A, ω_B, ω_C be semiinscribed circles of triangles ZAB, ABC, BCD opposite to A, B, C, and touching Γ at T_A, T_B, T_C, respectively. We will prove that the radical center L of ω_A, ω_B, ω_C belongs to IO (we use the proof of Stoyan Boyev, he applied it for the case of three semiinscribed circles of a triangle). By the jump argument it is sufficient to conclude that all the circles from the family of semiinscribed circles have common radical center L.

Firstly, we will consider the homothety h taking the γ to Γ. According to the Monge's theorem we can easily find that lines AT_A, BT_B and CT_C are concurent and their intersection point T is the center of homothety h. Hence T, I and O are collinear points.

Let $Q_{AB} = AB \cap T_AT_B$, $Q_{BC} = BC \cap T_BT_C$. Note that $Q_{AB}Q_{BC}$ is the polar line of point T wrt Γ. Since $T \in IO$, we have $Q_{AB}Q_{BC} \perp IO$.

Let M_{AB}, M_{BC} be the midpoints of arcs AB, BC (take arcs opposite to I). Let P_{AB} is the common point of T_AT_B and the tangent to Γ at M_{AB}. P_{BC} defined similarly.

The radical axis $l(\omega_A, \omega_B)$ passes through M_{AB} and the common point of tangents to Γ through T_A and T_B. Hence P_{AB} is the pole of $l(\omega_A, \omega_B)$ wrt Γ. Similarly, and P_{BC} is the pole of the radical axis $l(\omega_B, \omega_C)$ wrt Γ. Thus $P_{AB}P_{BC}$ is the polar of the radical center L wrt Γ. To prove that $L \in IO$ it remains to prove that $P_AP_C \parallel Q_AQ_C$.

Let X_{AB}, X_{BC} be touching points of ω_B with AB and BC. Note that T_B, XAB, M_{AB} are collinear, T_B, XBC, M_{BC} are collinear, and $\frac{T_BX_{BC}}{T_BM_{BC}} = \frac{T_BX_{AB}}{T_BM_{AB}}$ from homothety with center T_B taking ω_B to Γ. We have $\frac{T_BQ_{BC}}{T_BP_{BC}} = \frac{T_BX_{BC}}{T_BM_{BC}} = \frac{T_BX_{AB}}{T_BM_{AB}} = \frac{T_BQ_{AB}}{T_BP_{AB}}$, and this completes the proof.

Considering a particular case with symmetry in OI, we obtain $\vec{OL} = \frac{2Rr}{R^2-d^2-r^2}\vec{OI}$, where $d = OI$.

b) From previous result we obtain that all semi-inscribed circles touche the circle with center O and another circle with the center in the remaining focus of the ellipse.

c) **Hint.** The locus of the centers of such circles is an ellipse with foci O and I.

8. **Hint.** The trypolar passes through the Lemoine point L and meets for the second time the trajectory of L at a fixed point (fig.4).

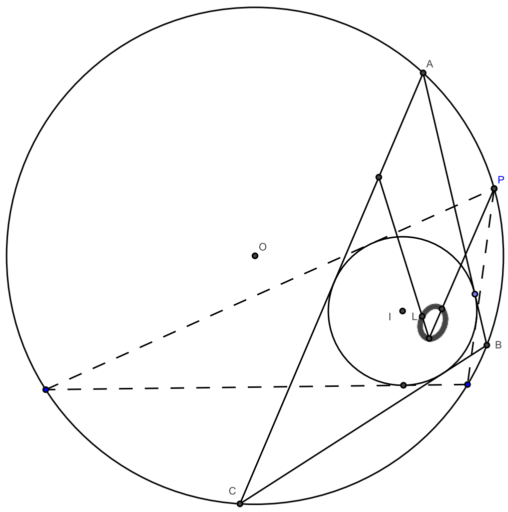

Fig.4

9. Let R, r be the radii of the given circles, let O be the center of the external circle, and O' be the circumcenter of triangle ABI. We have that O and O' lie on the perpendicular bisector to segment AB.

Use the cosines theorem to triangles $AO'O$ and $OO'I$:

$$R^2 = O'A^2 + O'O^2 - 2O'A \cdot O'O \cos\angle AO'O$$

$$OI^2 = O'I^2 + O'O^2 - 2O'I \cdot O'O \cos\angle IO'O.$$

Substracting the second equality from the first one we obtain:

$$R^2 - OI^2 = 2O'O(O'A\cos\angle AO'O - O'I\cos\angle IO'O = 2O'O \cdot r.$$

Therefore the desired locus is the circle with center O and radius $(R^2 - OI^2)/2r$.

Alternative solution. Let us fix the outer circle Γ (let O be its center, and R be its radius), and the center I of the inner circle γ (let r be its radius, further r could vary). Let $A, B, C \in \Gamma$ so that AB and BC touch γ. Let S and R be circumcenters of triangles AIB, BIC, respactively. We have $OS \perp AB$, $OR \perp BC$, and $SR \perp IB$ (since SR is the perpendicular bisector of IB). Since $\angle(AB, BI) = \angle(IB, BC)$, we have $\angle(OS, SR) = \angle(SR, RO)$, hence $OS = OR$.

Let $X, Y \in \Gamma$, and XY touches γ so that $\angle(\vec{IX}, \vec{IY}) < \pi$. Define the map $g_r : \Gamma \to \Gamma$ such that $g_r(X) = Y$. Let $S_r(X)$ be the circumcenter of triangle $XIg_r(X)$, and $OS_r(X) = \rho_r(X)$. Note that $\rho_r(X)$ is continous over (X, r).

We have proved that $\rho_r(X) = \rho_r(g_r(X))$ (we made a jump). From this jump we will derive that $\rho_r(X)$ is independent of X.

Let us call r *regular* if the orbit $\{X, g_r(X), g_r^2(X), \ldots\}$ is dence in Γ. For each regular r we obtain that $\rho_r(X)$ is independent of X. If r_0 is not regular (there are closed Ponselet trajectories), then take a limit $\rho_{r_0}(X) = \lim\limits_{r \to r_0} \rho_r(X)$ over regular r. Thus we get that $\rho_r(X)$ is independent of X for all r.

Considering a particular case with symmetry in OI, one can easily obtain $\rho_r(X) = \frac{R^2-d^2}{2r}$, where $d = OI$.

10. Let O, D be the centers of the given circles, and let R, r be their radii; let A', B', C' be the midpoints of arcs BC, CA, AB, and I be the incenter of ABC. Then I is the orthocenter of triangle $A'B'C'$, thus $\vec{OI} = \vec{OA'} + \vec{OB'} + \vec{OC'}$. Therefore vector $C'I$ lies on CD, and its length is equal to $2R\sin\angle OA'B' = 2Rr/CD$. On the other hand $C'D = (R^2 - OD^2)/CD$, i.e the ratio $C'I/C'D$ do not depend on C. Thus DI/DC'' also do not depend on C, i.e. I lies on the circle homothetic to the given external circle with center D.

Alternative solution. Let us fix the outer circle Γ (let O be its center, and R be its radius), and the center I of the inner circle γ (let r be its radius, further r could vary). Let $A, B, C \in \Gamma$ so that AB, AC, BD touch γ. Let AI and BI intersect Γ for the second time at A' and B', respectively. Let S and R be incenters triangles ABC, ABD, respectively. We have $B'R/B'I = B'A/B'I = \lambda = A'B/A'I = A'S/A'I$. Hence S and R lie on the image of Γ under homothety with center I and ratio $1-\lambda$.

Considering special case one can calculate λ.

We have made a jump. From this solution follows similarly to Problem 9.

11. Let O be the center of the circle containing point C, and O' be the center of the remaining circle. Since $OO' = \sqrt{3}$, we obtain that $A'B'$ touches the second circle at some point C'. Therefore $\angle A'O'A = \angle AO'C' + \frac{1}{2}\angle C'O'B = 2\angle ABC' + \angle C'AB = \angle CB'A' + \frac{1}{2}\angle CA'B'$, $\angle O'A'O = \angle O'A'B' + \angle B'A'O = \frac{\pi}{2} - \angle C'O'A' + \frac{\pi}{2} - \angle BCA = \pi - \angle BCA - \frac{1}{2}\angle CA'B' = \angle CB'A' + \frac{1}{2}\angle CA'B'$. Since $O'A = OA'$, we obtain that $AO'A'O$ is an isosceles trapezoid, and $AA' = OO' = \sqrt{3}$.

12. Let C be the fourth vertex of rectangle $PACB$. Since $OP^2 + OC^2 = OA^2 + OB^2$, C lies on the circle with center O. Thus the midpoint of AB lies on the circle having the center at the midpoint of OP, and the inverse common point of the tangents also lies on a circle.

13. **Hint.** Prove that the segments joining the touching points of the incircle with the opposite sides are perpendicular and use the previous problem.

14. **Answer.**

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}.$$

15. **Answer.**

$$\frac{R+OP}{R-OP} = \frac{(R+d)^2}{(R-d)^2}.$$

16. Using the formula of the previous problem we obtain that P is the limit point of the pencil containing the incircle and the circumcircle. Thus for any point X of the circumcircle the ratio of distance XP and the length of the tangent from X to the incircle is the same. From this fact using that the lines joining the opposite touching points pass through P we obtain the assertion of the problem.

17. Point M is the midpoint of the segment between the midpoints of the diagonals. Since the midpoints of the diagonals and the incenter are collinear it is easy to see that the trajectory of M is a circle.

18.

a) Let U, V be the midpoints of the diagonals. Since U, V lie on the circle with diameter OP, and line UV passes through I, we have

$$\mathrm{tg}\angle UPO\,\mathrm{tg}\angle VPO = \frac{UO \cdot VO}{UP \cdot VP} = \frac{S_{OUV}}{S_{PUV}} = \frac{OI}{IP} = \sqrt{\frac{R^2}{R^2 - OP^2}}.$$

b) **Hint.** The lengths of the diagonals are $2\sqrt{R^2 - OP^2 \sin^2 \angle UPO}$ and $2\sqrt{R^2 - OP^2 \sin^2 \angle VPO}$. From this and previous equality we obtain that the product of the diagonals is equal to $4R\sqrt{R^2 - OP^2}$.

2 An algebraic view on the Poncelet theorem

19.

a) Write an equation of line A_0A_1 and use that the distance from point $(d, 0)$ to this line is equal to r. We obtain

$$((R+d)^2 - r^2)t_0^2t_1^2 - r^2(t_0^2 + t_1^2) + 2(R^2 - d^2)t_0t_1 + ((R-d)^2 - r^2) = 0$$

b) **Hint.** Denote the above polynomial as P_1. Excluding t_1 from the system $P_1(t_0, t_1) = 0$, $P_1(t_1, t_2) = 0$, we obtain the polynomial from t_0, t_2, having degree 4 on each variable. Dividing it to $(t_0 - t_2)^2$, we obtain the desired relation.

c) The desired assertion can be obtained by induction similarly as the previous result.

20. Let starting from some point t_0 we return to it through n steps. Then we have: $P_n(t_0, t_n) = 0$, $t_0 = t_n$. Substituting the second equation to the first one we obtain the equation of degree 4 such that t_0 is its root. It is evident that $t_1, t_2, \ldots$ are also its roots. Since the number of roots of this equation is greater than its degree we obtain that it is an identity when $n \geq 5$. That proves the Poncelet theorem.

21. **Hint.** Let chords A_0A_1 and A_1A_2 touche two circles of the given pencil. Writing the corresponding equations and excluding t_1 we obtain a polynomial from t_0, t_2 having degree 4 on each variable. This polynomial can be decomposed to two multipliers each of them corresponding to some circle of the pencil. Therefore A_0A_2 touches one of these two circle depending on the choice of the tangents. Now we can reason similarly to the previous problem.

22. Denote $x_i = r_i/(R+d_i)$, $y_i = r_i/(R-d_i)$, where r_1, r_2 are the radii of the circles touching the sides and the diagonals respectively, and d_1, d_2 are the distances from their centers to the circumcenter. Considering the symmetric polygons we obtain

$$x_1 = (x_2^2 + y_2^2 - 1)/(1 - x_2^2 + y_2^2), \quad y_1 = (x_2^2 + y_2^2 - 1)/(1 + x_2^2 - y_2^2) \qquad (1).$$

resolving this system wrt x_2, y_2 we obtain:

$$x_2 = \sqrt{(x_1(1+y_1)/(x_1+y_1))}, \quad y_2 = \sqrt{(y_1(1+x_1)/(x_1+y_1))} \qquad (2)$$

23.

a) For $n = 3$ we have the Euler formula: $1/r = 1/(R+d) + 1/(R-d)$ i.e. $x + y = 1$ or $x = \sin^2 t$, $y = \cos^2 t$, $0 < t < \pi/4$. For $n = 6$ from (2) we obtain $x = \sin t \cdot \sqrt{1 + \cos^2 t}$, $y = \cos t \cdot \sqrt{1 + \sin^2 t}$.

b) $x = \sqrt{\frac{\sin t \cdot (1+\cos t)}{(\sin t + \cos t)}}$, $y = \sqrt{\frac{\cos t \cdot (1+\sin t)}{(\sin t + \cos t)}}$.

c) Define x_i, y_i and reason similarly to the conclusion of (1) we obtain:

$$x_2 = (x_1^2 + y_1^2 - 1)/(1 - x_1^2 + y_1^2), \quad y_2 = (x_1^2 + y_1^2 - 1)/(1 + x_1^2 - y_1^2)$$

$$x_1 = (1 - x_2^2 - y_2^2)/(1 - x_2^2 + y_2^2), \quad y_1 = (1 - x_2^2 - y_2^2)/(1 + x_2^2 - y_2^2)$$

Resolving the second system wrt x_2, y_2 and comparing two expressions for x_2 we obtain (index 1 is ejected)

$$x(1-y)(1 - x^2 + y^2)^2 - (x+y)(x^2 + y^2 - 1)^2 = 0$$

It is easy to see that this equation is identity when $x = -1$, $y = 0$, $x + y = 1$. Dividing to corresponding multipliers we obtain:

$$(x + y - 1)(x + y + 1)^2 = 4xy(x + y)$$

24. When we change R to d and vice versa defined above x do not change, and y change the sign. Since (x, y) satisfy the closing condition for the n-th step, we obtain that $(x, -y)$ satisfy the closing condition for the $2n$-th step, i.e. the broken line close through k steps where k is some divisor of $2n$. Since this assertion is symmetric k may be equal only to $n/2$, n or $2n$.

25.

a) Let the external circle is rationally parametrized and $t_0, t_1, t_{-1}, \ldots$ are the parameters corresponding to the vertices of the polygon. Using the Vieta theorem from the relation between t_0 and t_i find $t_i + t_{-i}$ and $t_i * t_{-i}$. Now express through t_0 the sums of the coordinates of the corresponding vertices and summing for all i, find the coordinates of the centroid. Let they be $x = P(t_0)/Q(t_0)$, $y = R(t_0)/S(t_0)$.

Lemma 1. $S = Q$ and $degQ = 2n$. In fact, the denominator of a rational function is defined by its poles. The centroid is an infinite point iff one of the vertices of the n-gon coincide with one if two infinute points of the circle. Thus the roots of Q and S coincide with these $2n$ points.

Now choose parametrize the circle in such a way that the centroid does not tend to infinity when t_0 tends to infinity. Then the degrees of P and R are not greater than the degree of Q. Now count the common points of our curve with an arbitrary line. Substituting the rational functions with degrees $2n$ to the equation of the line we obtain an equation with degree $2n$ wrt t_0, it has $2n$ roots but each point of the curve corresponds to n significances of the parameters, thus there exists two points on the curve.

So our curve meets any line at 2 points, thus its degree is 2. Now consider its infinite points. If $n - 1$ points of the circle tend to the finite point, and one point tends to the infinite point, then the enroid tends to the same infinite point. Therefore our conic meet the infinite line at 2 points lying on any circle, i.e. it is also a circle.

b) Suppose that the centroid is not a fixed point. Then its trajectory intersect the infinite line. This can happen only if one of touching points is infinite. But the cicumcircle and the incircle meet at the infinite points. Therefore if for example the touching point of side A_1A_2 with the incircle is infinite then some endpoint of this side for example A_1 also is infinite. Thus the touching point of side A_1A_n is infinite. When A_1 tends to an infinite point two touching points move on the opposite directions and the trajectory of the centroid is defined by the midpoint of the corresponding segment.

Now use that the midpoint of the segment between two touching points is inverse to the common point of the corresponding tangents. It is easy to see that when A_1 tends to infinity, the inverse point tends to some finite point. Thus the centroid does not intersect the infinite line, therefore it is a fixed point.

26.

a) No. Let t_a, t_b, t_c be the significances of the parameter corresponding to the vertices A, B, C of the triangle. Expressing the coordinates of the centroid through t_a we obtain an equation wrt t_a. Since t_b, t_c are also the roots of this equation, its degree is equal to 3. But a cubic equation can not be solving by a compass and a ruler.

b) Yes. The centroid M lies on the circle having I and the midpoint of OP as opposite points. Thus the perpendicular from M to line MI meets OI at the midpoint of OP, i.e. we can find the common point P of the diagonals and calculate the radii of the circumcircle anf the incircle. Now line MI meets the circle with diameter OP at the midpoints of the diagonals, which allows to restore the vertices of the quadrilateral.

27.

a) The relation between t_0 and t_1 is symmetric and has degree 2 wrt each variable. Thus we can express all Vieta polynomials through t_0. All these functions have the same poles, therefore each of them is a linear function of σ_1. Since the odd polynomials are odd functions, and the even polynomials are even functions we obtain the desired assertion.

28. It is easy to prove next

Lemma. Let three coaxial circles be given. From an arbitrary point C on one of them take the tangents CA, CB to two remaining circles. Let point D divide segment AB in some fixed ratio. Then the locus of points D is the circle coaxial with the given circles.

Now the desired assertion can be obtained by induction.

29.

30. It is evident that the relation between d_1, d_2, d_3 is symmetric and has degree 2 on each variable. Thus we can write it as $P(\sigma_1, \sigma_2, \sigma_3) = 0$, where P is a polynomial with degree 2, and σ_1, σ_2, σ_3 are the Vieta polynomials from d_1, d_2, d_3. If one of conics coincide with he external one then two remaining conics coincide i.e if $d_3 = 0$ then $P = (d_1 - d_2)^2$, therefore $P = \sigma_3(a\sigma_3 + b\sigma_2 + c\sigma_1 + d) + \sigma_1^2 - 4\sigma_2$. Substituting $d_1 = d_2 = d_3 = t$, we obtain an equation wrt t with degree 6. Two of its roots are equal to zero. Find 4 remaining roots excluding r from the system: $t^2 = 1 - 2r$ (Euler formula) $l^2 - 1 = (l - t)^2 - r^2$ (the condition of coaxility), where l is the abscissa of the common point of the center line and the radical axis.

As result we obtain

$$P = (\sigma_3 - \sigma_1)^2 + 8l\sigma_3 - 4\sigma_2$$

References

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