On the Poncelet theorem

G.Chelnokov, E.Diomidov, V.Kalashnikov, P.Kozhevnikov, A.Zaslavsky

The simplest formulation of the Poncelet theorem is next.

Poncelet theorem. Let two circles be given, and one of them lies inside the second one. A tangent from an arbitrary point A_0 of the external circle Ω to the internal circle ω meets Ω for the second time at point A_1. Similarly from point A_1 construct point A_2 etc. Then if $A_0 = A_n$ for some point A_0, this ill be true for any other point of Ω.

Not formally an inscribed and circumscribed polygon[1] can be "rotated" between two circles (its form can change during this rotation). We will call such "rotating" polygon *a Poncelet polygon*.

This project purposes to prove the Poncelet theorem and to examine some properties of Poncelet polygons. Also we consider the generalizations of Poncelet theorem and some similar theorems.

1 Poncelet theorem for $n = 3, 4$

1. Let O, I be the circumcenter and the incenter of the triangle, and R, r be the radii of the circumcircle and the incircle respectively. Prove **the Euler formula**

$$OI^2 = R^2 - 2Rr.$$

2. Prove the Poncelet theorem for $n = 3$.

A set of remarkable points or centers is associated with any triangle. When we "rotate" a triangle between its circumcircle and incircle, these points move on some curves. In the next problems we have to find the corresponding trajectories.

3. Find the trajectory of

a) the centroid M;

b) the orthocenter H;

c*) the Gergonne point G (the common point of the segments between the vertices of the triangle and the touching points of the opposite sides with the incircle)

d*) the Lemoine point L, isogonally conjugated to M

of the Poncelet triangle?

4. Let A', B', C' be the touching points of the incircle with the sides. Find the trajectory of the centroid M_0 of triangle $A'B'C'$.

5*. Let a Poncelet triangle and a fixed point P be given. Find the trajectory of the point isogonally conjugated to P.

6*. Let X be a fixed point of Ω. Prove that its Simson line passes through a fixed point Y (and line ℓ, passing through Y and perpendicular to XY, touches ω).

7. A circle touching sides AC, BC and the circumcircle of triangle ABC is called *semi-inscribed* circle of this triangle.

a) Find the trajectory of the center of the semi-inscribed circle.

b) Prove that the semi-inscribed circle of the Poncelet triangle touches some circle distinct from Ω.

c) Prove the same assertion for the circle, passing through two vertices of the Poncelet triangle and touching ω.

[1] we will use this term in place of closed broken line, which may be self-intersecting

8.* Let triangle ABC and point X be given. Lines AX, BX, CX meet BC, CA, AB respectively at points A', B', C'. Then the common points of lines $A'B'$ and AB, $B'C'$ and BC, $C'A'$ and CA are collinear on the line which is called *a trypolar* of X wrt ABC.

a) Prove that the trypolar of a fixed point X of Ω wrt the Poncelet triangle passes through a fixed point Y.

b) Find a locus of points $Y(X)$.

9. A circle with center I lies inside an other circle. Find the locus of the circumcenters of triangles IAB, where AB is an arbitrary chord of the external circle touching the internal one.

10. Let two circles be given and one of them lies inside the second one. Find the locus of the incenters of triangles ABC, where AC and BC are two chords of the external circle touching the internal one.

11. Two circles with radii 1 meet at two point, and the distance between these points also is equal to 1. C is an arbitrary point on one of these circles, the tangents CA and CB to the second circle meet the first circle for the second time at points B' and A'. Find the distance AA'.

12. Let a circle and a point P inside it be given. Two perpendicular rays with origin P meet the circle at points A and B.

a) Find the locus of the midpoints of segments AB.

b) Find the locus of the common points of the tangents to the circle at points A and B.

13. Prove the Poncelet theorem for $n = 4$.

14. Let two circles with centers O, I and radii R, r satisfy the Poncelet theorem for $n = 4$. Find the relation between R, r and $d = OI$.

15.

a) Prove that the diagonals of the Poncelet quadrilateral meet on the same point P, lying on OI.

b) Find the relation between OP, R and d.

16. Prove that the lines joining the touching points of the opposite sides of the Poncelet quadrilateral with the incircle are the bisetors of the angles formed by its diagonals.

17. Find the trajectory of the centroid M of the Poncelet quadrilateral.

18.* Prove that

a) the product of the tangents of the angles between line OI and the diagonals;

b) the product of the lengths of the diagonals

of the Poncelet quadrilateral is constant.

2 An algebraic view on the Poncelet theorem

When $n > 4$ the Poncelet theorem also can be proved synthetically. By the examination of the properties of the Poncelet polygons using only geometric methods is difficult. The methods of algebraic geometry are more effective. Firstly prove the Poncelet theorem using these methods.

Consider center O of Ω as an origin of a coordinates system and line OI as an abscissa axis. Let R, r — be the radii of the circles, and d be the distance between its centers, i.e. I has the coordinates $(d, 0)$. Define the coordinates of the points of Ω as $x = R(1-t^2)/(1+t^2), y = R \cdot 2t/(1+t^2)$, this concordance between the points of Ω and the significances of t will be one-one, if point $(-R, 0)$ correspond to $t = \infty$. Such defining of a curve is called its rational parametrization. Let $t_0, t_1, \dots, t_{n-1}$ be the significances of t, corresponding to the vertices of the polygon.

19.

a) Find the relation between t_0 and t_1.

b) Find the relation between t_0 and t_2.

c*) Prove that t_0 and t_n satisfy to the relation $P_n(t_0, t_n) = 0$, where $P_n(x, y)$ is some symmetric polynomial, having degree 2 on each variable.

20. Prove the Poncelet theorem.

The general Poncelet theorem. Let circles $\omega_1, \dots, \omega_n$ lie inside circle Ω, and let all these circles be coaxial, i.e they have a common radical axis. If there exists a polygon $A_1 \dots A_n$ inscribed into Ω and such that A_1A_2 touches ω_1, A_2A_3 touches ω_2,...,A_nA_1 touches ω_n, then there exists an infinite set of such polygons.

21.

a) Prove the general Poncelet theorem.

b) Prove "the generalest" Poncelet theorem, in which the coaxial circles are replaced by the conics passing through four fixed points.

$>$From the general Poncelet theorem we obtain that if $A_1 \dots A_n$ is the Poncelet polygon inscribed into circle Ω and circumscribed around circle ω, then its diagonals A_iA_{i+k} for any fixed k touche the same circle coaxial with Ω and ω.

22. Let R and r be the radii of the circumcircle and the incircle of the Poncelet polygon , and d be the distance between its centers. Find the radius of the circle touching the diagonals A_iA_{i+2} and the distance from the center of this circle to the circumcenter.

23. Find the relations between R, r and d for the Poncelet

a) hexagon;

b) octagon;

c) pentagon.

24. (S.Markelov) Let R, r and d be the radii of the circumcircle and the incircle of the Poncelet n-gon and the distance between its centers. Prove that d, r and R are also the radii of the circumcircle an the incircle and th distance between its centers for some Poncelet polygon, having n, $2n$, or $n/2$ sides.

25. Find the trajectory

a) the centroid of the vertices;

b) the centroid of the touching points of the incircle with the sides

of the Poncelet polygon.

26. Let an incenter, a circumcenter and a centroid of an inscribed and circumscribed n-gon be given. Can this n-gon be restored by a compass and a ruler if

a) $n = 3$?

b) $n = 4$?

27.

a) Let $t_1, \dots, t_n$ be the values of the parameter corresponding to the vertices $A_1, \dots, A_n$ of a Poncelet n-gon; let $\sigma_1 = t_1 + \dots + t_n$, $\sigma_2 = t_1t_2 + t_1t_3 + \dots + t_{n-1}t_n$,...,$\sigma_n = t_1 \cdots t_n$ be the Vieta polynomials from $t_1, \dots, t_n$. Prove that all even Vieta polynomials are constant and all odd polynomials are proportional to σ_1.

b) Let $d_1, \dots, d_n$ be the lengths of the tangents from the vertices $A_1, \dots, A_n$ of a Poncelet n-gon to its incircle; let $\sigma_1 = d_1 + \dots + d_n$, $\sigma_2 = d_1d_2 + d_1d_3 + \dots + d_{n-1}d_n$,...,$\sigma_n = d_1 \cdots d_n$ be the Vieta polynomials from $d_1, \dots, d_n$. Prove that all even Vieta polynomials are constant and all odd polynomials are proportional to σ_1.

28*. Let two circle be given one of them lying inside the other. Consider a broken lines $A_1A_2 \dots A_{n+1}$, with the vertices lying on the external circle, and the links touching the internal one. Find the locus of the centroids of the touching points.

29*. Define the Simson line of point X wrt a cyclic n-gon using the induction as the line containing the projections of X to the Simson lines of X wrt $(n-1)$-gons, obtaining by deleting of each vertex. Prove that the Simson line of a fixed point of Ω wrt the Poncelet polygon passes through a fixed point.

30. Let triangle ABC is inscribed into circle Ω with radius 1, and let lines AB, BC, CA touches circles ω_1, ω_2, ω_3, in such a way that all these circles are coaxial. Find the relation between the distances d_1, d_2, d_3 from the centers of ω_1, ω_2, ω_3 to the center ofΩ.

3 The other closing theorems

The Poncelet theorem is the example of a closing theorem. Take some other examples of such theorems.

The Steiner porism. Let two circles be given: α and lying inside it β. Consider a chain of circles $\omega_1, \omega_2, \dots$, touching α internally, touching β externally, and such that ω_{i+1} touches ω_i. If for some circle ω_1 circle ω_n touches ω_1, then this is true for any ω_1.

The zigzag theorem. Let two circles α and β be given. Take an arbitrary point A_0 on α and find such point B_0 on β that $A_0B_0 = 1$. Now find point A_1 on α distinct from A_0 and such that $A_1B_0 = 1$ etc. If A_n coincides with A_0, then this is true for any other point A_0.

Note that the zigzag theorem is correct even for two circles not lying in the same plane.

The Emch theorem. Let three circles be given: α, β lying inside α and γ lying inside β. Consider a chain of circles $\omega_1, \omega_2, \dots$, touching α internally, touching γ externally, and such that ω_{i+1} and ω_i meet at the point lying on β. If for some ω_1 circle ω_n touches ω_1, then this is true for any ω_1.

The Brocard broken line theorem. Let a circle ω, a point P inside it and an angle ϕ be given. For an arbitrary point X_0 of ω construct such point X_1, that $\angle PX_0X_1 = \phi$. Similarly for point X_1 construct X_2 etc. If for some X_0 $X_n = X_0$, then this is true for any X_0.

The Protasov theorem. Let S_0, S_1, S_2 be three spheres with non-collinear centers. Consider a family Σ of spheres touching S_1 and S_2 (the spheres of Σ touche each of spheres S_1, S_2 by the same way — internally or externally) and perpendicular to S_0. Let ω — be a circle in the space, which do not lie on any sphere from Σ and do not pass through the common points of many than two spheres of Σ. For an arbitrary point X_0 on ω take a sphere $s_1 \in \Sigma$ passing through it and find the second common point X_1 of s_1 and ω. Take a sphere $s_2 \in \Sigma$ distinct from s_1 and passing through X_1 and find its second common point X_2 with ω etc. If for some X_0 $X_n = X_0$, then this is true for any X_0.

31. Prove these theorems algebraically.

An arbitrary circle on the plane can be given by an equation $x^2 + y^2 + ax + by + c = 0$. Correspond to such circle a point in the space with coordinates (a, b, c).

32. Is this a one-one correspondence?

33. Which pairs of points correspond to touching circles?

34. Which assertions correspond to the Steiner and Emch theorems?

35. Obtain

a) the Emch theorem and the Brocard broken line theorem from the Poncelet theorem;

b) the Poncelet theorem, the zigzag theorem and the Steiner porism from the Emch theorem.

36. Find the Protasov theorem

37. Obtain the zigzag theorem, the Poncelet theorem, the Emch theorem and the Steiner porism from the Protasov theorem.