Halving graphs

I. Bogdanov, K. Kokhas

In this set of problems we deal with *geometrical graphs*, i.e. graphs drawn in the plane. We assume that no three vertices of a geometrical graph are collinear, and all the edges are depicted by segments; these segments may have common points different from vertices. The following two geometrical graphs are of the main interest.

Definition. 1) A *halving graph* $G(n)$ is a graph constructed as follows. Consider a set $\mathcal{S}$ of n points in the plane (where n is even), no three points being collinear; they form the set of vertices of our graph. A pair of vertices is connected by an edge exactly if the line passing through these points splits the whole set of vertices into two equal parts (that means that each open halfplane defined by this line contains $(n-2)/2$ points).

2) Consider also a more general case; let us define a *k-separating graph* $G_k(n)$. Let k and n be nonnegative integers with $n > 2k+2$ (now n is not necessarily even). Consider a set $\mathcal{S}$ of n points in the plane, no three of them being collinear. An **oriented** edge is drawn from A to B exactly if one of the two (open) halfplanes defined by the line AB contains exactly k points; this halfplane must be to the *right* of the line, when we go along the line from A to B. The set of vertices of the graph $G_k(n)$ consists of those points that have at least one edge (either ingoing or outgoing one). Sometimes we will also denote this graph by $G_k(\mathcal{S})$, when we wish to stress its dependence of the initial set $\mathcal{S}$.

1 Vertices

1.1. Prove that a halving graph contains no isolated vertices.

1.2. For a fixed value of n, determine all the values a degree of a vertex in a graph $G(n)$ can attain.

1.3. Does there exist a graph $G(50)$ containing 25 vertices of degree 1 and 25 vertices of degree 3?

1.4. Prove that each k-separated graph $G_k(n)$ contains at least $2k+3$ vertices.

1.5. a) Prove that a halving graph contains at most 3 vertices of degree $n-3$.
b) How many vertices of degree $n-3$ a halving graph may contain?

1.6. Does there exist a halving graph containing exactly 8 vertices and exactly 9 edges?

1.7. Prove that any halving graph $G(100)$ contains at most 60 vertices of degree 41.

2 Properties of graphs

2.1. Assume that a halving graph $G(2n)$ contains exactly n edges. Prove that each two of the segments representing these edges have a common point.

2.1$\frac{1}{2}$. A geometrical oriented graph is drawn in the plane. Prove that it cannot have both types $G_{10}(n)$ and $G_{15}(n')$.

2.2. Prove that no halving graph $G(2n)$ contains a Hamiltonian path (i.e. a path that passes through every vertex exactly once).

2.3. Let $n = 103$. Find all k for which a k-separated graph $G_k(n)$ is necessarily connected.

2.4. Prove that each connected component in a k-separated graph $G_k(n)$ contains an Eulerian path (i.e. a path that passes along every edge exactly once).

2.5. Assume that there exist two halving graphs on n vertices that contain k_1 edges and k_2 edges, respectively. Prove that for each m with $k_1 \leqslant m \leqslant k_2$ there exists a halving graph on n vertices having m edges.

2.6. a) Consider a k-separated graph $G_k(n)$ depicted on the plane; let $\mathcal{S}'$ be the set of its vertices. Prove that this graph coincides with the graph $G_{k'}(\mathcal{S}')$ for some k'.
b) Prove that each connected component of a graph $G_k(n)$ is also of the form $G_{k'}(n')$ for some k' and n'.

2.7. a) Prove that each (abstract) graph is a subgraph of some halving graph.
b) Prove that each (abstract) graph is an induced subgraph of some halving graph.

3 *Convex chains and windmills*

We will group all the edges of a halving graph into several *convex chains*. For this, let us first rotate a graph so that none of its edges is vertical, and draw a vertical line (not passing through the vertices) that has equal number of vertices on both sides of it. Now, we draw a vertical line ℓ through the leftmost vertex V_1. Rotate this line clockwise around V_1 until it passes through some edge, say V_1V_2. Next, we continue rotation of the line ℓ around the vertex V_2 clockwise until it passes through a next edge, say V_2V_2, and so on. If the line becomes vertical, we terminate the process. Now we say that a polyline $V_1V_2V_3\dots$ is a *convex chain*. After that we start a new process taking as a starting point the leftmost vertex that contains no edge lying in the chains yet constructed.

As a result, after there are no more unused edges, all the edges will be partitioned into several convex chains. Notice that this partition depends on the direction initially chosen as vertical.

3.1. Prove that the process above partitions the edges of a halving graph into exactly $n/2$ convex chains, each chain starting in the left halfplane and ending in the right halfplane; moreover, no two chains have a common edges.

3.2. Prove that the sum of degrees of any two vertices of a halving graph does not exceed n.

In the next series of problems we deal with a finite set $\mathcal{S}$ of points in the plane, no three of which are collinear.

Definition. A *windmill* is the following process. Choose a line ℓ that passes through a single point $T \in \mathcal{S}$. This line rotates clockwise about the pivot T until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, U, takes over as a new pivot, and the line ℓ now rotates clockwise about U, until it next meets a point of $\mathcal{S}$, and so on.

3.3. Prove that one can choose a point T and a starting line ℓ so that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

3.4. Prove that for each set $\mathcal{S}$ there exists a windmill such that the windmill line sweeps all the points of the plane.

3.5. Prove that for each set $\mathcal{S}$ there exists a point in it such that each windmill starting from this point passes through all the points of $\mathcal{S}$.

3.6. Consider a finite set of points in the plane, no three being collinear, and a line a that passes through no points of the set. Let us colour all the points on one side of a in red, and all the other marked points in blue; assume that there are K red and M blue points. Prove that for each $k < K$ and $m < M$ there exists a line b such that one of halfplanes defined by b contains exactly k red and m blue points.

4 *Extremal problems*

4.1. a) Does there exist a positive integer n and a halving graph with n vertices and $2013n$ edges?
b) Does there exist a positive integer n and a graph $G_{10}(n)$ with n vertices and $2013n$ edges?

4.2. Find the maximal number of edges in a path in a halving graph $G(n)$.

4.3. Find the maximal number of edges in a cycle in a halving graph $G(n)$.

4.4. Prove that a clique of size k can be a subgraph of a halving graph that has
a) $O(k^3)$ vertices; b) $O(k^2)$ vertices.

4.5. Prove that the statement b) of the previous problem is asymptotically exact; more exactly, prove that if a halving graph with n vertices contains a clique of size k, then $n \geqslant \lfloor k^2/2 \rfloor$.

4.6. Denote by $e_{k,n}$ the maximum of number of edges in a k-separated graph $G_k(n)$. Prove that

$$e_{2n,2k+1} \geqslant 2e_{n,k} + n.$$

4.7. Prove that a graph $G_k(n)$ for $k < (n-2)/2$ contains at most $4\sqrt{(k+1)(n-k-1)n}$ edges.

5 *Circular sequences*

In this section we suggest some approach to problem 4.7; in this approach, we rephrase our problem in different terms. Thus, problems in this section look quite different from the other ones.

Definition. 1) A *circular n-sequence* is a sequence of $\binom{n}{2}+1$ permutations of the set $\{1, 2, \dots, n\}$ satisfying the following properties:

a) for every two consecutive permutations, one of them may be obtained from the other by swapping two neighboring numbers (such a swapping hereafter is called a *flip*, more specifically a *k-flip* if it swaps the numbers on kth and $(k+1)$st positions);

b) the last permutation of a sequence is obtained from the first one by reversing the order of all elements;
c) every two numbers from the set $\{1, 2, \ldots, n\}$ participate in exactly one flip of the sequence.

2) A *double circular n-sequence* is a sequence of $2\binom{n}{2}+1$ permutations of the same set, such that the first $\binom{n}{2}+1$ its elements form a circular n-sequence, and the last elements form the "reflection" of this sequence. I.e., the $(\binom{n}{2}+i)$th permutation is obtained from the ith one by reversing the order of all elements; thus, if the ith and $(i+1)$st permutations differ by a k-flip, then their reflections differ by a $(n-k)$-flip, and these flips swap the same numbers.

Some circular sequences may be constructed in the following geometrical way. In the plane, consider a set $\mathcal{S}$ of n points in a general position (we assume also that the lines connecting these points are pairwise non-parallel). Let us take some line and project these points onto this line; we will obtain some permutation of our points on the line. Now, we start rotating the line around some fixed center; the order of the points will sometimes change by a flip, so we will receive some sequence of the permutations. When a line rotates at 180° (360°), we will get a (double) circular sequence.

3) Let $\mathcal{T}$ be some (double) circular sequence. A set $\mathcal{P} \subset \{1, 2, \ldots, n\}$ is a *halfplane* (with respect to $\mathcal{T}$) if there exists a permutation σ in $\mathcal{T}$ such that the elements of $\mathcal{P}$ (in some order) form the leftmost piece of σ. A halfplane consisting of k elements is also called a *k-set*. Denote the number of k-sets in a double circular sequence $\mathcal{T}$ by $s_k(\mathcal{T})$.

5.1. Consider a double circular sequence $\mathcal{T}$. Prove that the number of its k-flips coincides with the number of its k-sets.

5.2. For every double circular sequence $\mathcal{T}$ prove that $\sum_{k=1}^{n-1} s_k(\mathcal{T}) = n(n-1)$.

5.3. Let $S_k(n)$ be the maximal possible sum of the form $\sum_{i=1}^{k} s_i(\mathcal{T})$, where $\mathcal{T}$ is a double circular n-sequence. Prove that $S_k(n) = kn$ for all $1 \leqslant k < n/2$.

5.4. Assume that a double circular n-sequence $\mathcal{T}$ is constructed from a set of points S. Prove that the number of edges in a graph $G_k(S)$ is equal to $s_{k+1}(\mathcal{T})$ (for every k, $1 \leqslant k \leqslant (n-2)/2$).

5.5. Determine whether every circular sequence arises from some set of points in the plane.

5.6. Consider any circular n-sequence $\mathcal{T}$. Let P be its first permutation, and let $P = XYZ$ with $|Y| = y > 0$ (the substrings X and Z may happen to be empty). Prove that for every k with $1 \leqslant k \leqslant n-1$, the number of k-flips in $\mathcal{T}$ involving at least one element of Y does not exceed $\binom{y}{2} + 2k$.

5.7. Prove that there exists a constant C (depending on none of n and k) such that for every double circular n-sequence and for every $1 \leqslant k \leqslant n/2$ the inequality $s_k(\mathcal{T}) \leqslant Cn\sqrt{k}$ holds.

6 *Intersections and the maximal number of edges*

Definition. We say that a pair of distinct edges in a geometrical graph is an *intersection* if these edges have a common point (different from a common vertex). In this section, we investigate the number of intersections.

6.1. Let V be the set of vertices of some halving graph with $|V| = n$ (surely, n is even). Denote by d_v the degree of a vertex $v \in V$. Let X be the number of intersections in our graph. Prove that

$$X + \sum_{v \in V} \binom{(d_v+1)/2}{2} = \binom{n/2}{2}.$$

6.2. Assume that the numbers n and e satisfy the conditions $10^6 n < e < 10^{-6} n^2$. Prove that there exists a (geometrical) graph on n vertices having e edges and at most $10^6 e^3/n^2$ intersections.

6.3. Prove that there exists a positive constant c such that every geometrical graph on n vertices with $e > 100n$ edges has at least $c \cdot e^3/n^2$ intersections.

6.4. Prove that the number of edges in a halving graph on n vertices does not exceed $Cn^{4/3}$, where C is some absolute constant (not depending on n).