A short proof of the Conway-Gordon-Sachs and Sachs Theorems

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Abstract

In this paper we present a short and apparently new proof of the Conway-Gordon-Sachs Theorem about the complete graph at 6 vertices embedded to $\mathbb{R}^3$ and the the Sachs Theorem about the the complete biparted graph at 8 vertices. We reduce this theorems to certain property of the complete graph at 5 vertices and the complete biparted graph at 6 vertices maped to a sphere or a plane.

Two triangles in the 3-dimensional space whose six vertices are in general position are *linked* if the outline of the first triangle intersects the interior of the second triangle exactly at one point.

Points in 3-dimensional space are in *general position* if no four of them are in one plane.

Rectilinear Conway-Gordon-Sachs Theorem. *Assume that six points in the* 3*-dimensional space are in general position. Then there exist two linked triangles with vertices at these points.*

Define a 2*-dimensional complex* as a set of triangles, segments and points in $\mathbb{R}^3$ that satisfies the following conditions:

- sides of any triangle from the complex are in the complex
- endpoints of any segment from the complex are in the complex

Two closed broken lines a and b without self-intersections in the 3-dimensional space are *linked* if there exist a 2-dimensional complex, denote it by A, embedded to $\mathbb{R}^3$, with a boundary a such that the number of intersection points of A with b is odd and vertices of the broken line b and the complex $A - a$ are in general position.

Denote by K_n the complete graph at n vertices. Denote by $K_{n,n}$ the complete biparted graph at $2n$ vertices.

Conway-Gordon-Sachs Theorem. *Assume that the graph K_6 is piecewise-linear embedded in the* 3*-dimensional space. Then in this graph exist two linked* 3*-length cycles.*

Remark. The statement of the theorem is meaningful because any 3-lenth cycle in this graph is a closed broken line.

Sachs Theorem. *Assume that the graph $K_{4,4}$ is piecewise-linear embedded in the* 3*-dimensional space. Then there exist two linked* 4*-length cycles in this grpah.*

Proof of the rectilinear Conway-Gordon-Sachs Theorem.

Let a, b be segments in the 3-dimensional space, S^2 be a sphere whose center is denoted by O. Let $f : \mathbb{R}^3 - \{O\} \to S^2$ be the central projection with the center O. A segment a is *higher* than a segment b, if

- $|f(a) \cap f(b)| = 1$, and
- O is closer to $f^{-1}(f(b)) \cap a$ than to $f^{-1}(f(a)) \cap b$.

Remark. The set $f^{-1}(f(b))$ is a 2-dimensional angle with the vertex O and sides joining O with the endpoints of b.

Lemma 1. *Assume that vertices of two triangles are in general position. Denote by $A_1A_2A_3$ the first triangle. Denote by S^2 a sphere with the center A_1 and radius so small that all the vertices of the triangles except A_1 are outside S^2. If the number of the sides of the second triangle that are lower than A_2A_3 is odd then these two triangles are linked.*

Remark. The condition that the vertices of the triangles except A_1 are outside the sphere could be avoided at the price of some complications both in the statement and the proof.

Proof of Lemma 1.

Denote by A_4, A_5, A_6 the vertices of the second triangle. Let $f : \mathbb{R}^3 - \{A_1\} \to S^2$ be the central projection with the center A_1. By the assertion of the lemma there exists a side, say A_4A_5, of triangle $A_4A_5A_6$ such that A_2A_3 is higher than A_4A_5. Then the point $f^{-1}(f(A_2A_3)) \cap A_4A_5$ is inside the 2-dimensional triangle $A_1A_2A_3$. Since $f(A_2A_3)$ is a ark of a circle on S^2 and $f(A_4A_5A_6)$ is a spherical triangle on S^2, $f(A_2A_3)$ intersects the projection of the outline of the triangle $A_4A_5A_6$ at most at 2 points. So there is a unique side A_4A_5 of the triangle $A_4A_5A_6$ that is lower than A_2A_3. Since the vertices of these two triangles are in general position the outlines of triangles $A_1A_2A_3$ and $A_4A_5A_6$ do not intersect. This implies that the outline of the triangle $A_4A_5A_6$ intersects the interior of the triangle $A_1A_2A_3$ at a unique point $f^{-1}(f(A_2A_3)) \cap A_4A_5$. So these two triangles are linked. QED

Continuation of the proof. Suppose that points $A_1, A_2, A_3, A_4, A_5, A_6$ are in general position in the 3-dimensional space. Consider the complete graph K_5 whose vertices are points A_2, A_3, A_4, A_5, A_6 and edges are segments joining pairs of these points. Consider a sphere S^2 with center A_1. Let this sphere be enough small to make points A_2, A_3, A_4, A_5, A_6 be outside the sphere. Consider the central projection $f : \mathbb{R}^3 - A_1 \to S^2$ with the center A_1. For ordered pair (e, e') of $e, e' \in K_5$ denote

$$e \circ e' := \begin{cases} 1, & \text{if } e \text{ is higher than } e' \\ 0, & \text{otherwise} \end{cases}.$$

For any edge $e \in K_5$ define its *linking number*

$$S_e := \sum_{e' \in (K_5 - e)} e \circ e'$$

Then

$$\sum_{e \in K_5} S_e \equiv \sum_{(e,e'), e,e' \in K_5} e \circ e' \equiv$$

$$\equiv \sum \{|f(e) \cap f(e')| : \{e, e'\} \text{ is a non-ordered pair of nonadjacent edges of } K_5\} \equiv 1 \mod 2.$$

Hence the linking number of some edge, say A_2A_3, is odd. Then Lemma implies that triangles $A_1A_2A_3$ and $A_4A_5A_6$ are linked.

The first equality follows from definition of S_e. The second equality holds because

- for any two edges $e, e' \in K_5$ $|f(e) \cap f(e')| \le 1$ because vertices of K_5 are in general position
- if edges $e, e' \in K_5$ are nonadjacent and $f(e) \cap f(e') \ne \emptyset$ then $e \circ e' + e' \circ e = 1$
- if edges $e, e' \in K_5$ are adjacent or $f(e) \cap f(e') = \emptyset$ then $e \circ e' + e' \circ e = 0$.

The third equality follows from Lemma 2.

Lemma 2. *For any general position linear map $f : K_5 \to S^2$ the number of self-intersections of $f(K_5)$ is odd.*

This Lemma is known, see e.g. [Sk, §1].QED

Proof of the Conway-Gordon-Sachs Theorem.

Consider a general position plane. Define what means that a segment a is *higher* than a segment b analogous to the definition in the linear case but replacing a 'sphere' with the 'general position plane' and the 'central projection' to the 'orthogonal projection to this plane'.

Lemma 3. *Consider two closed broken lines, denote them by A, B. Consider a general position plane. Assume that the number of ordered pairs (a, b) of sides $a, \in A$, $b \in B$ such that a is higher than b is odd. Then these two broken lines are linked.*

This Lemma is known, see [?].

Consider a general position plane π. Consider orthogonal projection $f : \mathbb{R}^3 \to \pi$.

For any ordered pair of broken lines (A, B) in the 3-dimensional space denote

$$A \circ B := \begin{cases} 1, & \text{if the number ordered pairs } (a, b) \text{ of sides } a \in A,\ b \in B \text{ such that } a \text{ is higher than } b \text{ is odd} \\ 0, & \text{otherwise} \end{cases}.$$

Denote by a one of the vertices of graph K_6. Denote by C_{ij} the cycle of edges of the graph $K_6 - \{a\}$ that does not contain the edge ij. Then the problem follows from

$$\sum_{bc \in K_6 - \{a\}} abc \circ C_{bc} \equiv \sum_{bc \in K_6 - \{a\}} (ab \circ C_{bc} + ac \circ C_{bc}) + \sum_{bc \in K_6 - \{a\}} bc \circ C_{bc} \equiv \sum_{bc \in K_6 - \{a\}} bc \circ C_{bc} \equiv$$

$$\equiv \sum \{|f(e) \cap f(e')| : \{e, e'\} \text{ is a non-ordered pair of nonadjacent edges of } K_6 - \{a\}\} \equiv 1 \mod 2,$$

Hence for some two cycles abc, C_{bc} of graph K_6 the number $abc \circ C_{bc}$ is equal to 1 and Lemma 2 implies that these cycles are linked. QED

Proof of the second equality.

Note that $ab \circ C_{bc} = \sum\limits_{e \in C_{bc}} ab \circ e$

For each $i \in K_6 - \{a\}$ and for each edge $e \in (K_6 - \{a\})$ there exist exactly two 3-length cycles in $K_6 - \{a\}$ containing this edge. So for each edge $ij \in K_6 - \{a, b\}$ the number $ab \circ ij$ appears twice in the sum $\sum\limits_{bc \in K_6 - \{a\}} (ab \circ C_{bc} + ac \circ C_{bc})$. Analogous for each edge $ij \in K_6 - \{a, c\}$ the number $ac \circ ij$ appears twice in this sum. Then this sum is even. QED

The proof of the third equality is the same to the *proof of the second equality* in the linear case.

The last equality follows from Lemma 4.

Lemma 4. *For any general position piecewise-linear map $f : K_5 \to \pi$ the number of self-intersections of $f(K_5)$ is odd.*

This lemma is the generalization of Lemma 2, see [Sk, §1].

Proof of the Sachs Theorem. Consider a general position plane π. Consider the orthogonal projection $f : \mathbb{R}^3 \to \pi$. Denote by a, b two vertices of graph $K_{4,4}$ from different parts. Denote by C_{ij} the cycle of edges of the graph $K_{4,4} - \{a, b\}$ nonadjacent to edge $ij \in K_{4,4} - \{a, b\}$. Denote by $xyzt$ a 4-length cycle of edges $xy, yz, zt, tx \in K_{4,4}$

$$\sum_{ij \in K_{4,4} - \{a,b\}} abij \circ C_{ij} \equiv \sum_{ij \in K_{4,4} - \{a,b\}} ab \circ C_{ij} + \sum_{ij \in K_{4,4} - \{a,b\}} (aj \circ C_{ij} + bi \circ C_{ij}) \equiv$$

$$\equiv \sum \{|f(e) \cap f(e')| : \{e, e'\} \text{ is a non-ordered pair of nonadjacent edges of } K_{4,4} - \{a, b\}\} \equiv 1 \mod 2$$

The second equality holds because for each $i \in K_{4,4} - \{a, b\}$ and for each edge of $K_{4,4} - \{a, b, i\}$ there exist four 4-length cycles containing this edge. So for each edge $kl \in K_{4,4} - \{a, b, j\}$ the number $aj \circ kl$ appears four times in the sum $\sum\limits_{ij \in K_{4,4} - \{a,b\}} (aj \circ C_{ij} + bi \circ C_{ij})$. And analogous for each edge $kl \in K_{4,4} - \{a, b, i\}$ the number $bi \circ kl$ appears four times in this sum. Hence this sum is even.

The proof of the second equality is the same to the *proof of the second equality* in the proof of the rectilinear Conway-Gordon-Sachs Theorem.

The last equality follows from Lemma 5.

Lemma 5. *For any general position piecewise-linear map $f : K_{3,3} \to \pi$ the number of self-intersections of $f(K_{3,3})$ is odd.*

This lemma is known, see [**?**].

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