Fair cake division

Solutions after semifinal

Some general results

We start with problems 3.1–3.3, since their solutions allow to simplify the further text.

3.1. If $m \nmid n$, then it is impossible to split a cake into equal pieces which weigh $\frac{m}{n}$. So there will be some piece less than $\frac{m}{n}$. That means that some person will receive at least two pieces, and one of them weigh at most $\frac{m}{2n}$ by the pigeonhole principle.

If $f(m,n) = \frac{m}{2n}$, then every piece weighs $\frac{m}{2n}$ or $\frac{m}{n}$ (if its weight is between these numbers, then the person receiving it has all other pieces smaller than $\frac{m}{2n}$). We may split all the pieces of $\frac{m}{n}$ into two pieces which weigh $\frac{m}{2n}$ which may happen if and only if $m \mid 2n$ (while $m \nmid n$).

3.3. a) Consider any optimal distribution. If someone gets at least three pieces, then the minimal of them is at most $\frac{1}{3} \cdot \frac{m}{n}$; hence $f(m,n) \leq \frac{m}{3n}$.

3.2. a) Suppose that $f(m,n) > \frac{m}{n} - \frac{1}{2}$. For $\frac{m}{n} > \frac{3}{4}$ we have $\frac{m}{n} - \frac{1}{2} > \frac{m}{3n}$. So by 3.3a) every person has not more than two pieces. If there is a piece of weight $\frac{m}{n}$, then we cut it into two equal pieces and, and we still have the optimal decomposition of cakes because of 3.1. So the total of pieces in decomposition is $2n$. Since $2n < 3m$, then by the pigeonhole principle there exists a cake with not more than two pieces. A cake with one piece is impossible, so we have a cake with two pieces, one of which is at least $\frac{1}{2}$. A person who has this piece also gets one more piece, the weight of which is not more than $\frac{m}{n} - \frac{1}{2}$. A contradiction.

b), c) Let $k \geq 3$. Suppose that $\frac{2}{k+1} < \frac{m}{n} < 1$ but $f(m,n) > \frac{m}{n} - \frac{1}{k}$. Since $\frac{m}{n} - \frac{1}{k} > \frac{m}{3n}$ for $\frac{m}{n} > \frac{2}{k+1}$, then by 3.1 and 3.3a) we may assume that each person has got two pieces and so the total number of pieces is $2n$. Since $2n < (m+1)k$, then there is a cake with not more than k pieces, one of which by the pigeonhole principle is at least $\frac{1}{k}$. A person with this piece has one more piece with weight not more than $\frac{m}{n} - \frac{1}{k}$. A contradiction.

3.3. b) Consider a segment of length m. Divide it by red points into m equal segments, and by blue points into $3n$ equal segments (the endpoints will be multicolored). The obtained segments with the red endpoints represent the cakes.

Now we simultaneously erase all blue points adjacent to a red point which is not blue. Next, cut the cakes by the remaining blue points. We claim that we have obtained a desired division.

Since $m < n$, each cake contained at least three blue points, hence at least one of them is not erased. Next, each piece not adjacent to the border of a cake is of length $\frac{m}{3n}$. The same holds for the segments having a multicolored endpoint. All the other segments are grouped into pairs sharing a common red endpoint, and the total length of each pair is $3 \cdot \frac{m}{3n} = 1$. Hence we may give each such pair to one man. All the remaining pieces are $\frac{m}{3n}$ in length, so we can give to every remaining man three such pieces.

c) Suppose, to the contrary, that $\frac{m}{n} \in \left(\frac{2}{3}, \frac{3}{4}\right]$ but $f(m,n) > \frac{1}{3}$. By 3.1 and 3.3a), we may assume that each man gets exactly two pieces. Then the total number of pieces is $2n < 3m$, hence by the pigeonhole principle there exists a cake consisting of no more than two pieces. Hence one of these pieces is at least $\frac{1}{2}$. Finally, the man getting this piece should get some other piece of weight at most $\frac{m}{n} - \frac{1}{2}$ which does not exceed $\frac{m}{3n}$ since $\frac{m}{n} \leq \frac{3}{4}$. A contradiction.

Finally, we present the solution of 1.4.

1.4. Consider a set of pieces such that we may arrange it into n equal cakes with weight m, as well as into m equal cakes with weight n. Suppose that x is the maximal weight of the minimal piece of sets with given property. Then

$$nf(m,n) = x = mf(n,m),$$

and $f(m,n) = \frac{m}{n} f(n,m)$.

Some concrete values of f

Notation. In the examples we shall denote by $(i_1 \cdot a_1 + i_2 \cdot a_2 + \cdots + i_l \cdot a_l)$ the decomposition of a cake into $i_1 + i_2 + \cdots + i_l$ pieces, among which i_1 pieces weigh a_1, i_2 pieces weigh a_2, etc. The composition of a man's portion will be denoted in the same manner.

1.1. a) A particular case of 3.3c).
b) The bound follows from 3.2b). Example:

$$4 \times 210g = 2 \times (3 \cdot 70g) + 2 \times (3 \cdot 50g + 60g)$$

c) The bound follows from 3.2c) for $k = 12$. Example:

$$4 \times 3kg = 2 \times (12 \cdot 250g) + 2 \times (240g + 12 \cdot 230g) = 24 \times (230g + 250g) + (2 \cdot 240g) = 25 \times 480g.$$

1.2. a) **Answer.** $\frac{5}{21}$.
A particular case of 3.3c).
b) **Answer.** $\frac{5}{18}$.
The bound follows from 3.2a). Example:

$$7 \times 1 = 3 \times \left(2 \cdot \frac{1}{2}\right) + 2 \times \left(\frac{5}{18} + \frac{6}{18} + \frac{7}{18}\right) + 2 \times \left(2 \times \frac{5}{18} + \frac{8}{18}\right) =$$
$$= 6 \times \left(\frac{5}{18} + \frac{1}{2}\right) + \left(2 \cdot \frac{7}{18}\right) + 2 \times \left(\frac{6}{18} + \frac{8}{18}\right) = 9 \times \frac{7}{9}.$$

1.3. a) **Answer.** $\frac{1}{3}$.
We may assume that every person has two pieces, so there is a cake with three (or more) pieces, one of which is not more than $\frac{1}{3}$, and $f(8,9) \leq \frac{1}{3}$.
Example:

$$8 \times 1 = 2 \times \left(3 \cdot \frac{1}{3}\right) + 6 \times \left(\frac{4}{9} + \frac{5}{9}\right) = 6 \times \left(\frac{1}{3} + \frac{5}{9}\right) + 3 \times \left(2 \cdot \frac{4}{9}\right) = 9 \times \frac{8}{9}.$$

b) **Answer.** $\frac{2}{7}$.
The bound $f(11,14) \geq \frac{2}{7}$ follows from 3.2a). Example:

$$11 \times 1 = 5 \times \left(2 \cdot \frac{1}{2}\right) + 4 \times \left(2 \cdot \frac{2}{7} + \frac{3}{7}\right) + 2 \times \left(\frac{2}{7} + 2 \cdot \frac{5}{14}\right) =$$
$$= 10 \times \left(\frac{1}{2} + \frac{2}{7}\right) + 4 \times \left(\frac{3}{7} + \frac{5}{14}\right) = 14 \times \frac{11}{14}.$$

c) **Answer.** $\frac{5}{17}$.

As usual, we may assume that each person has got two pieces. Then there are 34 pieces in total. Note that each cake contains two or three pieces: a cake could not contain only one piece, and if it contains at least 4 pieces, then one of them is not more than $\frac{1}{4} < \frac{5}{17}$. So we have 6 *fat* cakes with three pieces and 8 *usual* cakes with two pieces. There are 18 pieces in fat cakes and 16 pieces in usual ones. So there is a person getting both his pieces from the fat cakes (if it is the same cake, we choose a second fat cake arbitrarily). The rest four pieces of these two cakes weigh in total $2 - \frac{14}{17} = \frac{20}{17}$. So one of them is not more than $\frac{5}{17}$.

Example:

$$14 \times 1 = 8 \times \left(\frac{9}{17} + \frac{8}{17}\right) + 4 \times \left(2 \cdot \frac{6}{17} + \frac{5}{17}\right) + 2 \times \left(\frac{7}{17} + 2 \cdot \frac{5}{17}\right) =$$
$$= 8 \times \left(\frac{5}{17} + \frac{9}{17}\right) + 8 \times \left(\frac{6}{17} + \frac{8}{17}\right) + \left(2 \cdot \frac{7}{17}\right) = 17 \times \frac{14}{17}.$$

2.1. a) **Answer.** $\frac{1}{3}$.

Denote $s = f(3k-1, 3k)$. Firstly we prove that $s \le \frac{1}{3}$. Assume the contrary. Then $s > \frac{1}{3}$, hence by 3.1 and 3.3a) each man gets two pieces, and as usual we may assume that he gets exactly to pieces. Hence there exists a cake split into at least three parts, and the minimal of them is at most $\frac{1}{3}$. A contradiction.

To construct an example, let us multiply all the weights by $3k$. We have

$$(3k-1) \times 3k = 2 \times \big(3 \cdot k\big) +$$
$$+ 3 \times \big((2k-1) + (k+1)\big) + 3 \times \big((2k-2) + (k+2)\big) + \cdots + 3 \times \big((k+1) + (2k-1)\big) =$$
$$= 3 \times \big(k + (2k-1)\big) + 3 \times \big((k+1) + (2k-2)\big) + \cdots + 3 \times \big((2k-1) + k = (3k+2) \times (6k+2).$$

b) **Answer.** $\frac{2k+1}{2(3k+2)}$.

Denote $s = f(3k+1, 3k+2)$, $t = \frac{2k+1}{2(3k+2)}$. Firstly, let us prove that $s \le t$. Assume the contrary. As in the previous problem, by $s > t \ge \frac{3k+1}{3(3k+2)}$ we may assume that each man gets exactly two pieces. If some cake is divided into at least four parts then the least of them is at most $\frac{1}{4} \le \frac{2k+1}{2(3k+2)}$ which is impossible. Hence each cake is split into two or three pieces, and it is easy to see that there are exactly two cakes split into three parts.

Now consider the following graph. Its vertices are the cakes, and to each man corresponds an edge connecting two cakes the pieces of this man are taken from. This graph has two *distinguished* vertices of degree 3 and all other vertices of degree 2, hence it consists of three paths connecting distinguished vertices (some of them may be circuits). The length of one of these paths is at least $k+1$; consider this path $v_0, v_1, \ldots, v_{k+1}$.

Denote the portion of the man corresponding to the edge (v_i, v_{i+1}) as (p_i, q_i) so that p_i is taken from cake v_i, and q_i is taken from v_{i+1}. Then $p_i + q_i = \frac{3k+1}{3k+2}$ for $i = 0, 1, \ldots, k$, while $q_i + p_{i+1} = 1$ for $i = 1, 2, \ldots, k$. Hence $p_{i+1} - p_i = \frac{1}{3k+2}$, and hence $p_k \ge \frac{k}{3k+2} + p_0 \ge \frac{k}{3k+2} + t = \frac{4k+1}{2(3k+2)}$. Finally, we get $q_k = \frac{3k+1}{3k+2} - p_k \le t$. A contradiction.

This proof also provides a hint of how to construct an example: there should be two paths of length $k+1$ and one path of length k. To construct such an example, let us multiply all the weights by $2(3k+2)$. We have

$$(3k+1) \times (6k+4) = 2 \times \big(2 \cdot (2k+1) + (2k+2)\big) +$$
$$+ 2 \times \big((4k+1) + (2k+3)\big) + 2 \times \big((4k-1) + (2k+5)\big) + \cdots + 2 \times \big((2k+3) + (4k+1)\big) +$$
$$+ \big(4k + (2k+4)\big) + \big((4k-2) + (2k+6)\big) + \cdots + \big((2k+4) + 4k\big) =$$
$$= 2 \times \big((2k+1) + (4k+1)\big) + 2 \times \big((2k+3) + (4k-1)\big) + \cdots + 2 \times \big((4k+1) + (2k+1)\big) +$$
$$+ \big((2k+2) + 4k\big) + \big((2k+4) + (4k-2)\big) + \cdots + \big(4k + (2k+2)\big) = (3k+2) \times (6k+2).$$

c) **Answer.** $\frac{k}{3k+1}$.

Denote $s = f(3k, 3k+1)$, $t = \frac{k}{3k+1}$. Again, supposing that $s > t$, we may assume that each man has two pieces, there are exactly two cakes split into three parts, and each other cake consists of two parts. Next, constructing a graph as above, we get that it contains a path of length at least $k+1$. Again, acting as above we find that $p_k - p_0 = \frac{k}{3k+1}$, so $p_k \geq \frac{2k}{3k+1}$, and $q_k = \frac{3k}{3k+1} - p_k \leq t$. A contradiction.

To construct an example, let us multiply all the weights by $3k+1$. We have

$$\begin{aligned}
3k{\times}(3k+1) &= 2 \times \big(2 \cdot k + (k+1)\big) + \\
&+ 2 \times \big(2k + (k+1)\big) + 2 \times \big((2k-1) + (k+2)\big) + \cdots + 2 \times \big((k+1) + 2k\big) + \\
&+ \big((2k-1) + (k+2)\big) + \big((2k-2) + (k+3)\big) + \cdots + \big((k+2) + (2k-1)\big) = \\
= 2{\times}\big(k + 2k\big) &+ 2 \times \big((k+1) + (2k-1)\big) + \cdots + 2 \times \big(2k + k\big) + \\
&+ \big((k+1) + (2k-1)\big) + \big((k+2) + (2k-2)\big) + \cdots + \big((2k-1) + (k+1)\big) = (3k+1) \times 3k.
\end{aligned}$$

2.3. a) There are three cases, depending on the residue of n modulo 3.

1) $n = 3k$. Obviously, $f(3, 3k) = \frac{1}{k}$.

2) $n = 3k+1$. If $n = 1$, then $f(3, 1) = 1$. For $n = 3k+1 \geq 4$ we have $f(3, 3k+1) = \frac{3k-1}{2k(3k+1)}$. The bound follows from 3.2c) for $k' = 2k$. Example:

$$\begin{aligned}
3 \times 1 = \left(2k \cdot \frac{1}{2k}\right) + 2 \times \left(k \cdot \frac{3k-1}{2k(3k+1)} + (k+1) \cdot \frac{3}{6k+2}\right) &= \\
= 2k \times \left(\frac{3k-1}{2k(3k+1)} + \frac{1}{2k}\right) &+ (k+1) \times \left(2 \cdot \frac{3}{6k+2}\right).
\end{aligned}$$

3) $n = 3k+2$. Here we have $f(3, 3k+2) = \frac{1}{2k+2}$.

We may assume that each person has got two pieces and that the total number of pieces is $6k+4$. Then there is a cake with at least $2k+2$ pieces, one of them is not more than $\frac{1}{2k+2}$. Example:

$$\begin{aligned}
3 \times 1 = \left((2k+2) \cdot \frac{1}{2k+2}\right) + 2 \times \left((k+1) \cdot \frac{3k+4}{(3k+2)(2k+2)} + k \cdot \frac{3}{6k+4}\right) &= \\
= (2k+2) \times \left(\frac{1}{2k+2} + \frac{3k+4}{(3k+2)(2k+2)}\right) &+ k \times \left(2 \cdot \frac{3}{6k+4}\right).
\end{aligned}$$