Fair cake division

After semifinal

Additional problems to the previous sections

Just in case, we remind the problem added on the first presentation.

1.6. a) We need to divide m equal cakes and to distribute them among n people so that each gets the same total weight of pieces. Find the minimal number of pieces in such division.

b) For such a division with a minimal number of pieces, find all possible weights of the minimal piece.

Next problems form the addition to the third section. Namely, these are some bounds analogous to the theorem on one third, but more exact ones.

3.12. Prove that $f(m,n) \geq \frac{2}{5} \cdot \frac{m}{n}$, if $\frac{5}{12} \leq \frac{m}{n} \leq \frac{1}{2}$.

3.13. a) Prove that $f(m,n) \geq \frac{3}{8} \cdot \frac{m}{n}$ if $\frac{m}{n} \leq \frac{1}{2}$.

b) Find more intervals on which this inequality is true.

3.14. a) Prove that $f(m,n) \geq \frac{2}{5} \cdot \frac{m}{n}$, if $\frac{3}{5} \leq \frac{m}{n} \leq \frac{8}{13}$.

b) Try to prove this inequality for another interval, adjacent to the $\left(\frac{3}{5}, \frac{8}{13}\right)$. For example, is it true for $\frac{m}{n} \in \left(\frac{10}{17}, \frac{3}{5}\right)$? Or for $\frac{m}{n} \in \left(\frac{8}{13}, \frac{5}{8}\right)$?

c) Find more intervals (in other places of the segment $[0,1]$) where this bound is true.

3.15. For which intervals inside $\left(\frac{1}{2}, \frac{5}{8}\right)$ you can prove the bound $f(m,n) \geq \frac{2}{3} \cdot \frac{m}{n} - \frac{1}{6}$?

Testing area.

This section is useful for everyone who wants to have some nontrivial pairs (m,n), here are some of these pairs. **Attention!** We can check answers and examples for these pairs, but we will not check a proof, unless it includes some general ideas; so this pairs are not formed as a problem.

Here are these pairs (This list can be replenished):

$$(17,29); \quad (31,70); \quad (17,47); \quad (117,133); \quad (27,61); \quad (566,643); \quad (3130,6813).$$

Good luck!

4 Variations of the problems setting

In this section we generalize original setting in different ways. Solutions of these problems can be very useful in solving Megaproblem.

4.1. a) We have m cakes with weight 1 and $n > m$ people. We have to cut cakes and give it to the people so that each gets the same total weight of pieces. And herewith each gets at most two pieces and each cake is cut into at most three parts. Find all such pairs (m,n).

b) The same problem, but each cake is cut into at most k parts.

c) The same problem, but each cake is cut into either $k-1$ or k parts.

Next problems deal with the situation when cakes can be different.

4.2. a) Two cakes with weights 1 kg and 2 kg are divided between N people so that each gets the same total weight of pieces. What is the maximal weight of the minimal piece?

b) The same question for two cakes with weights 2 kg and 5 kg.

4.3. a) Let $k > 1$. There are $3k$ cakes (each of weight 3), $k-1$ cakes (each of weight 4), and $3k-1$ cakes (each of weight 7). We have to cut each cake of the weight 3 into two pieces, and the each of the others — into three pieces so that the pieces may be distributed among several people and each of them will have two pieces with the same total weight. What is the maximal possible size of the minimal piece?

b) The same question for $3k$ cakes of the weight 3, $k+2$ cakes of the weight 4, and $3k+2$ cakes of the weight 7.

c) The same question for $3k$ cakes of the weight 3, $2k-1$ cakes of the weight 4, and $4k-1$ cakes of the weight 7, where $k \geq 10$. What can you say for other values of k (for example, $k = 7$)?

4.4. a) We have a cake with weight 59, a cake with weight 89 and two cakes of weight 41. We have to cut the first cake into 4 pieces, the second cake into 6 pieces, and the each of the last into 5 pieces so that they may be distributed among 10 persons so that all persons will have the same number of pieces and the same total weight of their pieces. What is the maximal value of the minimal piece?

b) There are two cakes with weight 41, three cakes with weight 35, and 11 cakes with weight 29. Each cake of the first group should be divided into 5 pieces, each cake of the second group — into 4 pieces, and the each cake of the third group — into 2 pieces. The pieces should be distributed among 22 persons so that all persons have the same number of pieces and the same total weight of their pieces. What is the maximal possible value of the minimal piece?

c) Find $f(23, 29)$.