Fair cake division

Selected solutions

If you have any ideas on this project, please do not hesitate to contact us by the e-mail:
Konstantin Knop kostyaknop@gmail.com, Ilya Bogdanov ilya.i.bogdanov@gmail.com

The presented solutions are arranged as follows. In section “Some sequences” we find the values of f on some sequences of pairs (m, n); notice that plenty of them also follow from more general results from the next sections. Section “Serial results” contains the solutions (or their outlines) for problems 3.4–3.8 (with problem 4.1 as a useful lemma) and bound 3.13. In section “Nonequal cakes” we extend our methods; it allows us to approach to problems 3.9 and 3.10 (it is recommended to read section on serial results before). Finally, in section “General algorithm” we describe on concrete examples the ideas of a general algorithm of solving Megaproblem (it involves some ideas from the previous two sections).

We start with the solution of problem 1.6.

1.6. a) **Answer.** $m + n - \gcd(m, n)$.

To construct an example, consider a segment of length m. Divide it by red points into m equal segments, and by blue points into n equal segments (some points will be multicolored). The segments with the red endpoints represent the cakes. We cut the cakes by all the blue endpoints. We claim that a desired division is obtained. Obviously, these pieces may be distributed among the people: it suffices to give to each person pieces between some neighboring blue points. There are $m + 1$ red points, $n + 1$ blue points, and $\gcd(m, n) + 1$ multicolored points. So the total number of points is $m + n - \gcd(m, n) + 1$ and we have the required number of pieces.

We are left to show that the number of pieces should be at least $m + n - \gcd(m, n)$. Denote $d = \gcd(m, n)$, $n = dn'$, $m = dm'$. Consider a bipartite graph with m red vertices and n blue vertices, corresponding to cakes and people. Each edge corresponds to a piece, and it connects the person getting the piece with the cake it is taken from. Consider a connected component of this graph, let it have r red vertices and b blue vertices. Then b persons eat together r cakes, which means that $b \cdot \frac{m}{n} = r$. So $\frac{r}{b} = \frac{m'}{n'}$ and hence $m' \mid r$. So, the number of connected components is at most $\frac{m}{m'} = d$. On the other hand, in each component the number of edges is at least the number of vertices decreased by 1. So the total number of edges in the graph (which is the number of pieces) is at least $m + n - d$.

b) The solution is left to the reader.

Some sequences

Here we present the solutions for some problems from section 2. Many of them follow also from more general results from section 3; nevertheless, we have put them here to show more concrete constructions.

General remark. Since the case $m \mid n$ is trivial, further we always assume that $m \nmid n$.

2.2. By 3.2b) it follows that $f(m, 2m-1) \leq \frac{m+1}{6m-3}$. Example of decomposition of cakes of weights $6m - 3$:

$$2 \times \Big(3 \cdot (2m-1)\Big) + \Big(2 \cdot (m+1) + (2m-4) + (2m-1)\Big) + 2 \times \Big(2 \cdot (m+1) + (2m-2) + (2m-3)\Big) +$$
$$\Big((m+1+i) + (m+3+i) + (2m-4-i) + (2m-3-i)\Big)_{i=1,\dots,(m-5)}$$

2.3. b) **Answer.** $\frac{4}{n}$ if $4 \mid n$; $\frac{2}{n}$ if $n = 4k+2$; $\frac{4}{n} - \frac{2}{n-1}$ if n is odd.

If $4 \mid n$ then the answer $f(4,n) = \frac{4}{n}$ is trivial. If $n = 4l+2$ is even then $f(4, 4l+2) = \frac{1}{2l+1}$ by 3.1.

Let $n = 2k+1$ is odd. By 3.3a) we may assume that every person gets exactly two pieces. So we have $4k+2$ pieces, and by the pigeonhole principle there exists a cake with not more than k pieces. So there is a piece not less than $\frac{1}{k}$, and its complement (at a person) is at most $\frac{4}{n} - \frac{1}{k} = \frac{4}{n} - \frac{2}{n-1}$; hence $f(4,n) \le \frac{4}{n} - \frac{2}{n-1}$. An example for odd n:

$$2 \times \left(\frac{n-1}{2} \cdot \frac{2}{n-1}\right) + 2 \times \left(\frac{n-1}{2} \cdot \left(\frac{4}{n} - \frac{2}{n-1}\right) + \frac{2}{n}\right) =$$
$$= (n-1) \times \left(\frac{2}{n-1} + \left(\frac{4}{n} - \frac{2}{n-1}\right)\right) + \left(2 \cdot \frac{2}{n}\right) = n \cdot \frac{4}{n}.$$

c) **Answer.** $\frac{5}{n}$ if $n \,\vdots\, 5$; $\frac{1}{\lceil 2n/5 \rceil}$ if $n = 5k+1 \ge 16$ or $n = 5k+3$; $\frac{5}{n} - \frac{1}{\lfloor 2n/5 \rfloor}$ if $n = 5k+4$ or $n = 5k+2 \ge 12$; $f(5,11) = \frac{13}{66}$. The other examples follow from the previous problems.

2.4. **Answer.** $\frac{1}{5}$ for $m \ge 6$ and $m = 2$; the other answers follow from the previous problems: $f(1,3) = \frac{1}{3}$, $f(3,7) = \frac{5}{28}$, $f(4,9) = \frac{7}{36}$, $f(5,11) = \frac{13}{66}$.

2.5. **Answer.** $\frac{1}{4}$ for $k \ge 1$. The bound follows from 3.4 a). Example for cake weight $12k+8$:

$$1 \times \Big(4 \cdot (3k+2)\Big) + 2 \times \Big((4k+2) + (5k+4-2i) + (3k+2+2i)\Big)_{i=1,\dots,k} =$$
$$= k \times \Big(2 \cdot (4k+2)\Big) + 2 \times \Big((3k+2) + (5k+2)\Big) + 2 \times \Big((3k+4) + 5k\Big) + \dots + 2 \times \Big((5k+2) + (3k+2)\Big).$$

Note that the problem follows from 3.4b).

2.6. **Answer.** $\frac{2k+1}{2(4k+1)}$. The bound follows from 3.2 a). Example for cake weight $8k+2$:

$$(k+1) \times \Big(2 \cdot (4k+1)\Big) + 2 \times \Big((2k+1) + (2k+i) + (4k+1-i)\Big)_{i=1,\dots,k} =$$
$$= (2k+2) \times \Big((2k+1) + (4k+1)\Big) + 2 \times \Big((4k+1-i) + (2k+1+i)\Big)_{i=1,\dots,(k-1)} + \Big(2 \cdot (3k+1)\Big).$$

2.7. **Answer.** $\frac{1}{4}$. Bound follows from 3.4a). Example for cake with weight $32k+12$.

$$(5k+2) \times (32k+12) =$$
$$= k \times \Big(4 \cdot (8k+3)\Big) + 2 \times \Big(2 \cdot (10k+4) + (12k+4)\Big) + \Big((12k+5) + (8k+3+i) + (12k+4-i)\Big)_{i=1,\dots,4k}.$$

Note that the existence of example follows from 3.4b).

2.8. **Answer.** $\frac{6k-1}{3(9k-2)}$. The bound follows from 3.2b). Example with cake weights $27k-6$ and portions of people $15k-3$:

$$2k \times \Big(3 \cdot (9k-2)\Big) + \Big((6k-1) + (6k-1) + (6k-2+i) + (9k-2-i)\Big)_{i=1,\dots,(3k-1)} =$$
$$= 6k \times \Big((6k-1) + (9k-2)\Big) + \Big((6k-1+i) + (9k-2-i)\Big)_{i=1,\dots,3k-2}$$

2.9. **Answer.** $\frac{18k-4}{63k-15}$. The bound follows from the lemma.

Lemma. If $\frac{4}{5} < \frac{m}{n} < 1$, then $f(m,n) \le 2 \cdot \frac{m}{n} - \frac{4}{3}$.

Proof. Suppose the contrary. It is easy to check that $2 \cdot \frac{m}{n} - \frac{4}{3} \geq \frac{m}{3n}$ and $2 \cdot \frac{m}{n} - \frac{4}{3} \geq \frac{1}{4}$, so all the cakes contain two or three pieces and all people have two pieces. The number of two-piece cakes is $3m - 2n$, the number of three-piece cakes is $2n - 2m$. Two cases are possible.

1) Assume that some person gets both pieces from two-piece cakes. Then the remaining two pieces from these two cakes weigh in total $2 - \frac{m}{n}$, and one of pieces is at least $1 - \frac{m}{2n}$. Hence the completing piece is at most

$$\frac{m}{n} - \left(1 - \frac{m}{2n}\right) < 2 \cdot \frac{m}{n} - \frac{4}{3}.$$

2) Every piece of two-piece cake is completed by a piece of a three-piece cake. Let x be the minimal piece weight. Then $\frac{m}{n} - x$ is the maximal weight. Let A be some piece of a two-piece cake; then $A \geq 1 - \frac{m}{n} + x$. Hence the completing piece is $\frac{m}{n} - A \leq 2\frac{m}{n} - 1 - x$. From $\frac{m}{n} > \frac{4}{5}$ it follows that there are more two-piece cakes than three-piece ones. So there is a three-piece cake, all pieces of which are complementary to pieces of two-piece cakes. Hence all three pieces are at most $2\frac{m}{n} - 1 - x$ and

$$3\left(2 \cdot \frac{m}{n} - 1 - x\right) \leq 1 \Leftrightarrow x \leq 2 \cdot \frac{m}{n} - \frac{4}{3}.$$

□

Example. Cake weight is $63k - 15$, portion of a person is $51k - 12$, the minimal weight is $18k - 4$.

$$(17k-4) \times (63k-15) = 2k \times \Big(3 \cdot (21k-5)\Big) + 6k \times \Big((33k-8) + (30k-7)\Big) +$$
$$+\Big((33k-10-i)+(30k-7+i)\Big)_{i=1,\dots,3k-2} + 2 \times \Big((18k-3+i)+(27k-6-i)+(18k-4)\Big)_{i=1,\dots,3k-1} =$$
$$= 6k \times \Big((21k-5) + (30k-7)\Big) + 6k \times \Big((33k-8) + (18k-4)\Big) +$$
$$2 \times \Big((18k-4+i) + (33k-8-i)\Big)_{i=1,\dots,3k-2} + \Big((24k-6+i) + (27k-6-i)\Big)_{i=1,\dots,3k-1} =$$
$$= (21k-5) \times (51k-12).$$

Serial results

First, we present an estimate analogous to the Theorem on One Third.

3.13. a) If $n = 2m$, then $f(m, 2m) = \frac{1}{2}$, so we assume $\frac{m}{n} < \frac{1}{2}$. Let $8n$ be the weight of every cake, then each person receives $8m$.

Consider the segment of length $8nm$ and divide it by red points into m equal segments (cakes). We will cut off the pieces consequently from the left end of the remaining segment. Cut off several pieces of $4m$ until the remainder will be between $6m$ and $10m$. Next, we divide the remaining part into two equal pieces of length between $3m$ and $5m$. We complete both of these two pieces to $8m$ by two pieces from the next cake. So we use at most $10m$ from the next cake, and the remainder is at least $8n - 10m \geq 6m$. Thus we may continue cutting until the last cake. Since we have combined extracted pieces to pairs of weight $8m$, and the total segment equals to $8mn$, then in the last cake will be ended by two pieces of $4m$. The distribution of pieces among people is also constructed.

Next, we present some exact values for several intervals. We start with problem 4.1, which happens to be very helpful.

4.1. a), b) **Answer.** $\frac{m}{n} \in \left[\frac{1}{k-1}, 1\right) \cup \left\{\frac{v}{(k-1)v+1}\right\}_{v=1,2,\dots}$.

Firstly, we present an example showing that the answer fits. We act as in Theorem on One Third. Consider a segment of length m. Divide it by red points into m equal segments, and by blue points into n equal segments (some points will be multicolored). The obtained segments with the red endpoints represent the cakes. Now we cut the cakes by all the blue endpoints.

We claim that a desired division is obtained. Obviously, these pieces may be distributed among the people: it suffices to give to each person pieces between some neighboring blue points. Moreover, each person gets no more than two pieces since each such segment may contain at most one red point. We are left to show that each cake is divided into at most k parts.

If $\frac{m}{n} > \frac{1}{k-1}$, then each cake contains at most $k-2$ whole portions and at most two pieces less than $\frac{m}{n}$ — thus at most k pieces at all. For $\frac{m}{n} = \frac{1}{k-1}$ the claim is obvious. Now assume that $\frac{m}{n} = \frac{v}{(k-1)v+1}$. Consider now k consecutive blue points; the distance between the first and the last of them is $\frac{(k-1)v}{(k-1)v+1}$, and their coordinates are the fractions with denominator $(k-1)v+1$. This implies that a cake can contain all these k points only if one of them is its endpoint. This means exactly that a cake is cut into no more than k pieces. The example is justified.

We are left to prove that in other cases the desired division is impossible. Let us call a person *angry* if he gets two pieces. Let us construct a graph having cakes as vertices, with each edge corresponding to an angry person and connecting two cakes he gets his pieces from. Consider any connected component of this graph; let v and e be the numbers of its vertices and edges, respectively. Then $e \geq v-1$.

These v cakes contain at most kv pieces, $2e$ of which belong to angry people. Since there are no edges from our component outside it, these pieces can be rearranged into whole portions (namely, e portions of two pieces each and, say, t of portions of one piece). Then $t \leq kv - 2e$. Next, comparing the total weight at v cakes and $e+t$ people, we get $\frac{n}{m} = \frac{e+t}{v} \leq \frac{kv-e}{v} = k - \frac{e}{v}$. If $e \geq v$, then we get $\frac{n}{m} \leq k-1$, otherwise $e = v-1$, and we have $\frac{n}{m} = \frac{v(k-1)+1}{v}$, as desired.

Important remark. Had we omitted the condition $m < n$, the answer would expand a bit. Surely it will include 1; now assume that $m > n$. In this case, all people are angry since a person's portion is greater than a cake. Now, each connected component of a graph corresponds to the division of v cakes between e people, hence $1 < \frac{m}{n} = \frac{v}{e}$ which may happen only if $e = v-1$. For such values an example can be constructed in the same way; hence the answer becomes

$$\frac{m}{n} \in \left[\frac{1}{k-1}, 1\right] \cup \left\{\frac{v}{(k-1)v+1}\right\}_{v=1,2,\ldots} \cup \left\{\frac{e+1}{e}\right\}_{e=1,2,\ldots}.$$

c) **Answer.** $\frac{m}{n} \in \left[\frac{1}{k-1}, \frac{2}{k-1}\right] \cup \left\{\frac{v}{(k-1)v+1}\right\}_{v=1,2,\ldots}$.

The solution is left to the reader.

3.4. a) As usual, we may assume that each person receives at least two pieces. Then the total number of pieces is at least $2n > 3m$. Hence some cake contains at least four pieces, one of which should not exceed $\frac{1}{4}$.

b) **Answer.** $\frac{m}{n} \in \left[\frac{5}{8}, \frac{2}{3}\right) \cup \left\{\frac{5k+2}{8k+4}\right\}_{k=1,2,\ldots}$.

Suppose that $f(m,n) = \frac{1}{4}$; then $f(m,n) > \frac{m}{3n}$, so we may assume that each person gets exactly two pieces. Next, each piece is at least $\frac{1}{4}$ and at most

$$d = \frac{m}{n} - \frac{1}{4}$$

(otherwise the other piece at the person having our one is less than $\frac{1}{4}$). Hence each cake contains at least three pieces (otherwise there exists a piece of at least $\frac{1}{2} > d$) as well as at most 4 pieces (otherwise there exists a piece not exceeding $\frac{1}{5}$). Thus, we have *fat* cakes with 4 pieces each and *usual* cakes with 3 pieces each, and the numbers of fat and usual cakes are

$$f = 2n - 3m \qquad \text{and} \qquad u = 4m - 2n$$

respectively. Since $u \geq 0$, we obtain $\frac{m}{n} > \frac{1}{2}$.

Next, each fat cake should be split into equal parts, these parts belong to $4f$ people, and the second piece at each such person weighs d. All the remaining people get both their pieces from the usual cakes; let us call these people *usual*. Then there are

$$s = n - 4f = 12m - 7n$$

usual people.

Now we will consider some auxiliary decomposition of **negative** "cakes"; it corresponds to the division of the usual cakes and people. Let us subtract $\frac{1}{4}$ from each piece of a fat cake, and d from each piece of a usual cake. Let us forget for a while about zero pieces. Then in a new decomposition all non-usual people and all fat cakes vanish (and we forget about them too), each usual "cake" now contains not more than three **negative** "pieces" of the same total weight, while each usual person gets at most two **nonpositive** "pieces" of the same total weight. So, taking the opposites of all the obtained weights, we arrive to the situation of 4.1a) (without a condition $m < n$).

Conversely, from the division of these *deficiencies* it is easy to pass to the distribution of the original cakes. Let us cut the real cakes into three pieces with the corresponding deficiencies (if a new "cake" is divided into one or two parts, then the remaining pieces should have zero deficiencies, i.e. they should be equal to d). For the people, if someone had got two deficiencies, then we give him the two corresponding pieces; otherwise he gets one corresponding piece together with the piece of zero deficiency. Finally, all the remaining pieces with zero deficiency are paired with the pieces from the fat cakes.

Thus, we obtain that the desired decomposition exists if and only if $\frac{u}{s} = \frac{4m-2n}{12m-7n} \in \left[\frac{1}{2}, 1\right) \cup \left\{\frac{k}{2k+1}\right\} \cup \left\{\frac{k+1}{k}\right\} \cup \{\infty\}$ (see the Remark after 4.1b); we have also added the degenerate case $s = 0$). With the use of $\frac{m}{n} < \frac{2}{3}$, this leads to the condition $\frac{m}{n} \in \left[\frac{5}{8}, \frac{2}{3}\right) \cup \left\{\frac{5k+2}{8k+4}\right\}_{k=1,2,\dots}$.

Remark. In the further solutions following the same scheme we will omit repeating details, leaving them to the reader.

3.5. **Answer.** $\frac{m}{n} \in \left[\frac{2k-1}{k^2-1}, \frac{2}{k}\right) \cup \left\{\frac{d(2k-1)+2}{d(k^2-1)+k+1}\right\}_{d=1,2,\dots}$.

The solution is analogous to 3.4b).

3.6. a) By 3.2b), $f(m,n) \le \frac{m}{n} - \frac{1}{3}$. The converse inequality is proved in part b).

b) **Answer.** $\frac{m}{n} \in \left(\frac{1}{2}, \frac{5}{9}\right] \cup \left\{\frac{5k+2}{9k+3}\right\}_{k=0,1,2,\dots}$.

First, let us note that $f(m,n) \ge \frac{m}{3n} > \frac{m}{n} - \frac{1}{3}$ for $\frac{m}{n} < \frac{1}{2}$. On the other hand, $f(m,n) \le \frac{m}{2n} < \frac{m}{n} - \frac{1}{3}$ for $\frac{m}{n} > \frac{2}{3}$. So we are left to investigate the interval $\left(\frac{1}{2}, \frac{2}{3}\right)$ (the left end of the interval does not satisfy the condition, while the right end does).

We present an outline of the further solution which is similar to 3.4b). We obtain that each person gets two pieces, the sizes of pieces belong to a segment $\left[\frac{1}{3}, d\right]$ where $d = \frac{m}{n} - \frac{1}{3}$. Next, we have $f = 2n - 3m$ *fat* cakes with 4 pieces each and $u = 4m - 2n$ *usual* cakes with 3 pieces each. Each usual cake should be split into equal parts, these parts belong to $3u$ people, the second piece at each such person weighs d, and all the other $s = n - 3u = 7n - 12m$ *usual* people get both pieces from fat cakes. Notice that $\frac{m}{n} > \frac{1}{2}$ implies $\frac{f}{s} \ge \frac{1}{2}$.

Now, subtracting $\frac{1}{3}$ from each piece of a usual cake and d from each other piece we obtain the distribution of the remaining f **nonnegative** "cakes" over s usual people. The only condition remained is that each person should get not more than two pieces, while each "cake" should contain at most four pieces; this condition being satisfied, we can recover the division of the usual cakes. Hence by 4.1b) (together with the remark after it), for $\frac{f}{s} > \frac{1}{2}$ the desired division exists if and only if $\frac{f}{s} \in \left[\frac{1}{2}, 1\right] \cup \left\{\frac{v+1}{v}\right\} \cup \{\infty\}$. The answer follows.

3.7. **Answer.** $\frac{m}{n} \in \left(\frac{2}{k+1}, \frac{2k-1}{k^2}\right] \cup \left\{\frac{d(2k-1)+2}{dk^2+k}\right\}_{d=1,2,\dots}$.

The solution is analogous to 3.6b).

3.8. A particular case of 3.5.

The ideas of solution of problem 3.9 is presented in the next section.

Different cakes

Recall that $\lfloor x\rfloor$ and $\lceil x\rceil$ are the largest integer not exceeding x and the smallest integer not less than x, respectively.

4.2. a) **Answer.** $\frac{3}{N}$, if $3 \mid N$; $\max\left\{\frac{3}{N}-\frac{1}{\lfloor 2N/3\rfloor}, \frac{1}{\lceil 2N/3\rceil}\right\}$ otherwise.

If $3 \mid N$, then obviously the optimal way is to cut the cakes into the pieces of $\frac{3}{N}$ each.

Now assume that $3 \nmid N$. Then some person should get at least two pieces, hence the answer does not exceed $\frac{3}{2N}$, and we may assume that each person gets at least two pieces.

Next, the smaller cake is divided either into $\leq \lfloor 2N/3\rfloor$ parts, or into $\geq \lceil 2N/3\rceil$ parts (for $N=2$, the second case necessarily holds). In the first case, one of these pieces should be $\geq \frac{1}{\lfloor 2N/3\rfloor}$, so its complement (at a person) is $\leq \frac{3}{N}-\frac{1}{\lfloor 2N/3\rfloor}$. In the second case, the smaller cake contains a piece which is $\leq \frac{1}{\lceil 2N/3\rceil}$. So, in any case the minimal weight does not exceed one of the numbers $\frac{3}{N}-\frac{1}{\lfloor 2N/3\rfloor}$ and $\frac{1}{\lceil 2N/3\rceil}$, i.e. it does not exceed $D=\max\left\{\frac{3}{N}-\frac{1}{\lfloor 2N/3\rfloor}, \frac{1}{\lceil 2N/3\rceil}\right\}$.

We are left to present an example with D as the minimal piece weight. Assume that $D = \frac{3}{N}-\frac{1}{\lfloor 2N/3\rfloor}$. Let us divide the smaller cake into the pieces of $\frac{1}{\lfloor 2N/3\rfloor}\geq D$, cut away the same number of pieces of D each from the larger cake, and divide the rest into the whole portions. Obviously, this division fits. In the second case, the example is constructed analogously.

Remark. One may check that $D=\frac{3}{N}-\frac{1}{\lfloor 2N/3\rfloor}$ if $N=3k+2$, and $D=\frac{1}{\lceil 2N/3\rceil}$ otherwise.

b) **Answer.** $\frac{7}{N}$, if $7 \mid N$; $\max\left\{\frac{7}{N}-\frac{2}{\lfloor 4N/7\rfloor}, \frac{2}{\lceil 4N/7\rceil}\right\}$ otherwise.

The solution is completely analogous and is left to the reader.

The next problem gives a hint of how does the general algorithm work. We need to introduce some

Definitions and notation. Recall that a *hypergraph* is a pair (V,E) where V is the set of *vertices*, and E is the set of *(hyper)edges* which are some nonempty subsets of V. A hypergraph is *homogeneous* if all its edges have the same cardinality. For any hypergraph $G=(V,E)$ we can construct its *underlying graph* $G'=(V,E')$ with the same set of vertices, connecting every two vertices belonging to one hyperedge of G. A hypergraph is *connected* if its underlying graph is connected.

Further we will denote by $[b:c]$ the following situation: we have a cake of weight b which should be divided into c parts. So, the notation $2\times[4:3]+3\times[7:4]$ will denote the collection of two cakes of weight 4 which should be divided into three parts each together with three cakes of weight 7 which should be divided into four parts each.

4.4. a) **Answer.** $\frac{49}{6}$.

In our notation, we have $[59:4]+[89:6]+2\times[41:5]$. One of the pieces in $[89:6]$ is at least $\frac{89}{6}$, and its complement is $\leq \frac{49}{6}$, as desired. It remains to provide an example:

$$\left(4\cdot\frac{59}{4}\right)+\left(6\cdot\frac{89}{6}\right)+2\times\left(3\cdot\frac{49}{6}+2\cdot\frac{33}{4}\right)=4\times\left(\frac{59}{4}+\frac{33}{4}\right)+6\times\left(\frac{89}{6}+\frac{49}{6}\right).$$

b) **Answer.** $\frac{49}{6}$.

In our notation, we have $2\times[41:5]+3\times[35:4]+11\times[29:2]$. We say that the cakes of weight 29 are *small*, and the others are *large*. Notice that each person should get two pieces of total weight 23. Assume that each piece weighs at least than $\frac{49}{6}$; then each piece should also be at most than $23-\frac{49}{6}=\frac{89}{6}$.

Assume that a person gets two pieces from small cakes (surely these two cakes are distinct), then the average weight of the remaining two pieces in these cakes is $\frac{2\cdot 29-23}{2}>\frac{89}{6}$ which is impossible.

Hence all 22 pieces of small cakes come to different people, and therefore all the pieces from the large cakes also come to different people.

Now let us call the cakes of weight 41 *fat*, and the cakes of weight 35 *usual*. Construct a hypergraph with small cakes as vertices; each edge will correspond to a usual cake and consist of all the small cakes containing the people's complements of the pieces of this usual cake. This hypergraph contains at least two connected components.

Now let us remove all the pieces of the usual cakes, as well as their complements in small cakes. Next, we glue the remaining pieces of each connected component into one new "cake". Let us calculate a number of pieces and a weight of this "cake".

Assume that a component contains v vertices and e edges. Due to each edge, we have removed 4 pieces of total weight $4 \cdot 23 - 35 = 57$; hence the number of the pieces removed from our component is $4e$, while their total weight is $57e$. Thus the average weight of the remaining pieces is $\frac{29v-57e}{2v-4e}$ which should be $\leq \frac{89}{6}$, which rewrites as $2v \geq 7e$. On the other hand, since the component is connected, we have $v \leq 3e+1$. The two obtained inequalities hold only if the pair (v,e) is either $(4,1)$ or $(7,2)$. Hence our hypergraph should contain one component of type $(4,1)$ and one of type $(7,2)$. In the latter component, one of the pieces will be at least $\frac{7\cdot 29-2\cdot 57}{14-8} = \frac{89}{6}$ which provides the desired estimate.

But we can also get the example from this construction! Namely, from the component of type $(4,1)$ we have obtained a "cake" of 4 pieces and total weight $4\cdot 29-57 = 59$, while from the remaining component we get a "cake" of 6 pieces and total weight $7 \cdot 29 - 2 \cdot 57 = 89$. Also we have $2 \times [41:5]$ remained. Thus we come to the situation of 4.4a), so we may take the division from that example and then find the weights of the removed pieces. The resulting example is

$$11 \times 29 + 2 \times 41 + 3 \times 35 = 4 \times \left(\frac{59}{4} + \frac{57}{4}\right) + 6 \times \left(\frac{89}{6} + \frac{85}{6}\right) + \left(2 \cdot \frac{29}{2}\right) +$$
$$+ 2 \times \left(3 \cdot \frac{49}{6} + 2 \cdot \frac{33}{4}\right) + \left(4 \cdot \frac{35}{4}\right) + 2 \times \left(3 \cdot \frac{53}{6} + \frac{17}{2}\right) =$$
$$= 4 \times \left(\frac{59}{4} + \frac{33}{4}\right) + 6 \times \left(\frac{89}{6} + \frac{49}{6}\right) + 4 \times \left(\frac{57}{4} + \frac{35}{4}\right) + 6 \times \left(\frac{85}{6} + \frac{53}{6}\right) + 2 \times \left(\frac{29}{2} + \frac{17}{2}\right).$$

c) **Answer.** $\frac{49}{138} = \frac{1}{23} \cdot \frac{49}{6}$ (could you guess it?).

Let us multiply all the weights by 29. As usual, we may assume that each person gets exactly two pieces, and each cake is divided into either two or three parts. Then the numbers of cakes of both types can be found, and we arrive to the situation $12 \times [29:3] + 11 \times [29:2]$. We say that the cakes with three pieces are *fat*, and the others are *usual*. Assume that each piece weighs at least than $\frac{49}{6}$; then each piece should also be at most than $23 - \frac{49}{6} = \frac{89}{6}$.

By the same reasons as above, none of the people gets two pieces from a usual cake. Hence all 22 pieces of usual cakes come to different people, and their complements belong to fat cakes. The remaining 14 pieces of fat cakes come to 7 remaining people; let us call these people *fat*. Now construct a graph with fat cakes as vertices; each edge will correspond to a fat person and connect two fat cakes containing the cakes containing the pieces of this person. This graph contains at least five connected components.

Now let us remove all the pieces of fat people. Next, we glue the remaining pieces of each connected component into one new "cake". Let us calculate a number of pieces and a weight of this "cake".

Assume that a component contains v vertices and e edges (then $v \leq e+1$). Due to each edge, we have removed 2 pieces of total weight 23; hence the number of the pieces removed from our component is $2e$, while their total weight is $23e$. Thus the average weight of the remaining pieces is $\frac{29v-23e}{3v-2e}$ which should be $\geq \frac{49}{6}$, which rewrites as $27v \geq 40e$. This is impossible if $v \leq e$, so we get $v = e+1$ and hence $27 \leq 13e$, or $e \leq 2$. Thus, each component is a tree (so there are exactly five of them) and has at most two edges.

The most "regular" case is when there are two components with two edges and three components with one edge; so the obtained new "cakes" will look as $2 \times [41:5] + 3 \times [35:4]$. So by 4.4b) the answer will be at most $\frac{49}{6}$.

In any other case, an isolated vertex appears; this means that all three pieces of this cake are paired up (in portions) with the pieces from usual cakes. Consider these three complements, and take three usual cakes containing them. The average of three remaining pieces of these cakes is $\frac{4\cdot 29-3\cdot 23}{3} > \frac{89}{6}$, which is impossible. Hence the estimate is established.

The example again can be obtained from the example for 4.4b) by filling up the removed pieces:

$$\begin{aligned}11\times 29+12\times 29 &= 4\times\left(\frac{59}{4}+\frac{57}{4}\right)+6\times\left(\frac{89}{6}+\frac{85}{6}\right)+\left(2\cdot\frac{29}{2}\right)+\\ &\quad +4\times\left(\frac{49}{6}+\frac{33}{4}+\frac{151}{12}\right)+2\times\left(\frac{49}{6}+2\cdot\frac{125}{12}\right)+\\ &\quad +2\times\left(2\cdot\frac{35}{4}+\frac{23}{2}\right)+2\times\left(2\cdot\frac{53}{6}+\frac{34}{3}\right)+2\times\left(\frac{53}{6}+\frac{17}{2}+\frac{35}{3}\right)=\\ &= 4\times\left(\frac{59}{4}+\frac{33}{4}\right)+6\times\left(\frac{89}{6}+\frac{49}{6}\right)+4\times\left(\frac{57}{4}+\frac{35}{4}\right)+6\times\left(\frac{85}{6}+\frac{53}{6}\right)+\\ &\quad +2\times\left(\frac{29}{2}+\frac{17}{2}\right)+4\times\left(\frac{151}{12}+\frac{125}{12}\right)+\left(2\cdot\frac{23}{2}\right)+2\times\left(\frac{34}{3}+\frac{35}{3}\right).\end{aligned}$$

3.9. a) **Answer.** $\frac{5}{4}\cdot\frac{m}{n}-\frac{1}{2}$.

First, let us prove the upper bound. As usual, we may assume that each person gets two pieces, all the cakes are divided into 3 or 4 parts, hence there are $u = 4m-2n$ *usual* cakes of three parts and $f = 2n-3m$ *fat* cakes of four parts. Since $4f < n$ (provided by $\frac{m}{n} > \frac{7}{12}$), some person should get both pieces from the usual cakes. Consider two cakes containing these pieces; the average weight of the rest four pieces in them will be $t = \frac{1}{4}\left(2-\frac{m}{n}\right)$, so one of these pieces weighs at least t. So its complement weighs at most $d = \frac{m}{n} - t = \frac{5}{4}\cdot\frac{m}{n}-\frac{1}{2}$, as desired.

The example will follow from part b).

b) We investigate only the case $\frac{m}{n}\in\left(\frac{7}{12},\frac{2}{3}\right)$ where the upper bound from part a) holds. We claim that on this interval $f(m,n) = \frac{5}{4}\cdot\frac{m}{n}-\frac{1}{2}$ if and only if $\frac{m}{n}\in\left(\frac{7}{12},\frac{22}{37}\right]\cup\left\{\frac{22d-3}{37d-2}\right\}_{d=1,2,\dots}$; here is an outline of the proof.

We multiply all the weights by $4n$; so the weight of the minimal piece should be $d = 5m-2n$, while the maximal weight should be $t = 2n-m$.

In our case, each person should get two pieces, there are $f = 2n-3m$ *fat* cakes with four pieces each and $u = 4m-2n$ *usual* cakes with three pieces each. Next, it is easy to see that no person gets two pieces from the fat cakes, so there are $4f$ persons getting a piece from the fat cake and $s = n-4f$ *usual* persons with two pieces from usual cakes.

Construct a graph G having usual cakes as vertices, with the edges formed by the two pieces of a usual person. Remove all the usual people's pieces and glue each component into a new cake. If some connected component contains more than one edge, then deleting all the usual people's pieces from this component we get some pieces of average weight $> d$ which is impossible. Hence we get s new cakes of weight $8n-4m = 4t$ consisting of four pieces and $u-2s$ old cakes of weight $4n$ consisting of three pieces. Notice that new cakes should be divided into four equal parts each.

Now we act as in 3.4b). Subtract d from each remaining piece of usual cake, and t from each piece of a fat cake. Then the new cakes vanish, all remaining usual cakes turn into positive "cakes" of three (or less) pieces, and all fat cakes turn into negative "cakes" of four (or less) pieces. Taking the absolute values of negative pieces, we come to the following situation:

We have $u-2s$ equal cakes, and we need to cut each of them into at most three parts and redistribute into f groups of equal weight having at most four pieces each.

Moreover, one can see that if the desired division of the "cakes" is possible then one may recover the division of the initial cakes. Hence it remains to determine when the new problem has a solution. This can be made in the same way as in 4.1.

General algorithm

Finally, we show how the general algorithm works on some nontrivial example — that is, we will find $f(31, 52)$.

1.3. e) Unlike in the other problems, we do not start with an answer, but we wish to see how to *find* it from the very beginning.

Part I. Firstly, we will perform some strange process which provides neither an example nor the bound. But we will get an answer; and then we will check that this answer is achievable and optimal.

During the process, we will make some assumptions on how should the optimal example look like. So, after the example is constructed we will need to check that lacking these assumptions we will obtain a worse division. We mark these assumprions by a bold font and number them consecutively.

Preliminaries. Multiply all the weights by 52. We will **assume(1)** that each person has two pieces, and since $\frac{1}{2} < \frac{m}{n} < \frac{2}{3}$ we will **assume(2)** that each cake is divided into three of four parts. Then we have

$$11 \times [52 : 4] + 20 \times [52 : 3].$$

Let us call the cakes with four pieces *fat*, and the other cakes *usual*.

Initial step. Now we have 44 pieces in fat cakes, which is smaller than the number of people. We **assume(3)** that all of them come to different persons. Hence there will be exactly 8 *usual* persons with both pieces in usual cakes. So we construct a graph on usual cakes as vertices, where each usual person induces an edge. This graph contains 20 vertices and 8 edges, so it has at least 12 connected components.

In such a situation we **assume(4)** that (i) all components are trees (so there are exactly 12 of them), and (ii) the edges are distributed between the components almost uniformly (that is, the numbers of edges in any two components differ by at most 1). In our case, this means that there are 8 components with one edge and 4 isolated vertices. Now, removing the pieces belonging to usual people and gluing the pieces of one component, we come to a situation

$$11 \times [52 : 4] + 8 \times [73 : 4] + 4 \times [52 : 3].$$

Regular step 1. Now we have 44 *small* pieces in 11 fat cakes, and 44 *large* pieces in remaining cakes; each person should get one piece of each type. Notice that the average weight of a piece in [52 : 3] is smaller than that in [73 : 4]. Informally speaking, this means that we need to cut the latter cakes as uniformly as we can. So we postpone them and deal with the remaining ones.

Consider a hypergraph on the fat cakes as vertices, with each [52 : 3] cake inducing an edge (this edge consists of the fat cakes containing the complements of the pieces of our [52 : 3] cake). Thus we have a hypergraph on 11 vertices with 4 edges of cardinality ≤ 3. Such hypergraph has at least $11 - 4 \cdot (3 - 1) = 3$ connected components.

As before, we **assume(5)** that (i) each component has the maximal possible number of vertices for its number of edges (so there are exactly three of them), and (ii) the edges are distributed between the components almost uniformly (that is, the numbers of edges in any two components differ by at most 1). In our case, this means that there are two components with three vertices and one edge, as well as one component with two edges and five vertices. Now, removing all the pieces of [52 : 3] cakes together with their complements, and gluing the pieces of one component, we come to a situation

$$2 \times [105 : 9] + [178 : 14] + 8 \times [73 : 4].$$

Regular step 2. Now we have 32 *large* pieces in [73 : 4] cakes, and 32 *small* pieces in remaining cakes; each person should get one piece of each type. Notice that the average weight of a piece in [105 : 9] is larger than that in [178 : 14]. Again, this means that we need to cut the latter cakes as uniformly as we can, so we deal with [105 : 9] cakes.

Consider a hypergraph on the [73 : 4] cakes as vertices, with each [105 : 9] cake inducing an edge. Unlike the previous cases, this hypergraph may be connected; so we **assume(6)** that it is connected, and we are ready to finish. We make this graph connected, hence by the standard removing we arrive to the situation

$$[178 : 14] + [256 : 14].$$

But this situation is trivial, and the maximal possible smallest piece is $\frac{178}{14}$: it is enough to divide each cake into equal parts and give to each person one piece from each cake. Notice that our last aim (to cut $[178:14]$ with the maximal uniformity) is completely reached.

Thus, under all our assumptions we obtain that the minimal piece is at most $d = \frac{89}{7}$.

Part II. Now we are to construct an example, moving backwards in our process. Recall that in our example we have

$$[178:14] + [256:14] = \left(14 \cdot \frac{89}{7}\right) + \left(14 \cdot \frac{128}{7}\right).$$

Regular step 2. The cake $[256:14]$ is obtained from $8 \times [73:4]$ by removing the pieces complementary to the ones from $2 \times [105:9]$. Now we reconstruct the division of these cakes — for instance, with the help of intervals; now it is easy to apply:

$$8 \times [73:4] + 2 \times [105:9] = 8 \times \left(3 \cdot \frac{128}{7} + \frac{127}{7}\right) + 2 \cdot \left(5 \cdot \frac{89}{7} + 4 \cdot \frac{90}{7}\right).$$

Regular step 1. The cakes $2 \times [105:9]$ were obtained from $3 \times [52:4] + 3 \times [52:4]$ by removing the complements of the pieces from $[52:3] + [52:3]$; analogously, the cake $[178:14]$ was obtained from another $4 \times [52:3]$ by removing the complements of the pieces from $2 \times [52:3]$. Now we will reconstruct them; it is immediate after we split the pieces in $[105:9]$ and $[178:14]$ into the $[52:4]$ cakes they were taken from. Making this arbitrarily we get

$$\begin{aligned} 11 \times [52:4] + 4 \times [52:3] = 4 \times \left(2 \cdot \frac{89}{7} + \frac{90}{7} + \frac{96}{7}\right) + 2 \times \left(\frac{89}{7} + 2 \cdot \frac{90}{7} + \frac{95}{7}\right) + \\ + 4 \times \left(3 \cdot \frac{89}{7} + \frac{97}{7}\right) + 2 \times \left(2 \cdot \frac{89}{7} + 2 \cdot \frac{93}{7}\right) + \\ + 2 \times \left(2 \cdot \frac{121}{7} + \frac{122}{7}\right) + 2 \times \left(2 \cdot \frac{120}{7} + \frac{124}{7}\right). \end{aligned}$$

Initial step. We are left to reconstruct the last cakes $16 \times [52:3]$ from $8 \times [73:4]$ by adding the usual people; it also goes automatically:

$$16 \times [52:3] = 8 \times \left(2 \cdot \frac{128}{7} + \frac{108}{7}\right) + 8 \times \left(\frac{128}{7} + \frac{127}{7} + \frac{109}{7}\right).$$

So the whole example is reconstructed:

$$\begin{aligned} 11 \times [52:4] + 20 \times [52:3] = 4 \times \left(2 \cdot \frac{89}{7} + \frac{90}{7} + \frac{96}{7}\right) + 2 \times \left(\frac{89}{7} + 2 \cdot \frac{90}{7} + \frac{95}{7}\right) + \\ + 4 \times \left(3 \cdot \frac{89}{7} + \frac{97}{7}\right) + 2 \times \left(2 \cdot \frac{89}{7} + 2 \cdot \frac{93}{7}\right) + \\ + 2 \times \left(2 \cdot \frac{121}{7} + \frac{122}{7}\right) + 2 \times \left(2 \cdot \frac{120}{7} + \frac{124}{7}\right) + \\ + 8 \times \left(2 \cdot \frac{128}{7} + \frac{108}{7}\right) + 8 \times \left(\frac{128}{7} + \frac{127}{7} + \frac{109}{7}\right). \end{aligned}$$

Part III. We are left to check that all our assumptions were necessary. It can be done easily within the methods involved in the previous sections. Denote $d = \frac{89}{6}$, $t = 31 - d = \frac{128}{7}$. If $f(31, 52) > d$ then all the piece weights should belong to the interval (d, t).

Assumption (1) should hold since otherwise the least piece is at most $\frac{31}{3} < d$.

Assumption (2) is necessary since otherwise we get a piece of $\leq \frac{52}{5} < d$ or $\geq 31 - \frac{52}{2} > t$.

Assumption (3) should hold, otherwise let us consider a person with two pieces in two fat cakes. Then the average weight of the remaining six pieces in these cakes is $\frac{52 \cdot 2 - 31}{6} < d$.

Assumption (4): suppose it fails; then two edges have a common vertex. Consider three cakes participating in these edges; the average of 5 their pieces not corresponding to our edges is $\frac{3\cdot 52-2\cdot 31}{5} > t$.

Assumption (5) may be checked in the same way as in 4.4b).

Finally, we do not need to check the last **Assumption (6)** at all: to this step, we already have a $[178 : 14]$ cake, so the minimal piece cannot exceed $\frac{178}{14} = d$.

We are done!

We suggest you to apply this algorithm to the pairs from the Testing area to see how it works!