Fair cake division

I.I. Bogdanov, G.R. Chelnokov, K.A. Knop, I.N. Shnurnikov

1 General setting

Firstly, we present several model problems of this project. In every problem, a first question is easy; but then the difficulty grows fast. For instance, problem 1.3e) is very hard!

Warning! If you are stuck with some question for a long time, we advice to switch to another problems of the project. Perhaps you will find there some clues or hints. For instance, you may find that problem 3.3b) is very useful.

1.1. a) Three small cakes, each of weight 200 g, are divided into some pieces. It happens that one can distribute all the pieces to 4 children so that each gets the pieces of the same total weight. Prove that the minimal piece weight is not more than 50 g. Is it possible to replace the number 50 by a smaller one?

b) Four small cakes, each of weight 210 g, are divided into some pieces. It happens that one can distribute all the pieces to 7 children so that each gets the pieces of the same total weight. Prove again that the minimal piece weight is not more than 50 g. Is it possible to replace the number 50 by a smaller one?

c) Now we have four large cakes, each of weight 3 kg, and we divide them into pieces so that it is possible to distribute all the pieces to 25 children so that each gets the same total weight. Prove that the minimal piece weighs not more than 230 g. Is it possible to replace the number 230 by a smaller one?

1.2. a) We have five cakes of 1 kg each, and we need to cut them and to distribute the pieces to seven people. We need the minimal piece weight to be the largest possible. Find this largest possible minimal weight.

b) The same problem for 7 cakes and 9 people.

1.3. a) We need to cut eight cakes of 1 kg each and to distribute the pieces to 9 people. Find the largest possible weight of the smallest piece in this division.

b) The same problem for 11 cakes and 14 people.

c) The same problem for 14 cakes and 17 people.

d) The same problem for 13 cakes and 16 people.

e) The same problem for 31 cakes and 52 people.

Surely, all these problems are the particular cases of the following general setting. (We always assume that the variables denote some positive integers.)

Megaproblem. There are m cakes (of weight 1 kg each) and n people. We need to divide the cakes into pieces and to distribute them all to the people so that each gets the same total weight of pieces. We aim to maximize the minimal piece weight. So we need to find this largest possible weight of the minimal piece.

Definition. Denote by $f(m,n)$ the answer to the Megaproblem.

Although the problem seems to be quite innocent, it is very hard to solve it in this general setting. It happens that in several iterations one can find the answers for "most" values of m and n, but at every step there still remain pairs (m,n) for which the answer is unknown.

It seems that there is no answer in a closed form. Thus our main purpose is to construct an algorithm of solving Megaproblem for every particular case. We do not formulate the search of this algorithm as a separate problem, but please keep in mind that this is the guiding star of this project.

Attention! If you think that you have invented the general algorithm (or even some well-formulated conjectures of how should it look like) — we are always open to discuss that! Moreover, it also concerns to the algorithm which works for some (large enough) interval of values of the ratio m/n.

We finish this introduction with formulating three general problems. The first one is quite easy; surprisingly, the answer for the second one is yet unknown. Formally speaking, the third one is not connected with the project, but its solution may be helpful.

1.4. Given the value of $f(m,n)$, determine $f(n,m)$.

Remark. In view of this problem, it is enough to investigate only the case $m < n$. Hence, further we deal with this case only.

1.5*. Determine whether the relation $f(tm,tn) = f(m,n)$ always holds.

Remark. The authors are almost sure that the answer is affirmative; this is confirmed in all known particular cases. So, in many further problems we deal with the ratio m/n instead of the pair (m,n).

1.6. a) We need to divide m equal cakes and to distribute them among n people so that each gets the same total weight of pieces. Find the minimal number of pieces in such division.

b) For such a division with a minimal number of pieces, find all possible weights of the minimal piece.

In the next two sections we collect the questions mainly concerning the particular cases of Megaproblem: in Section 2 — for some sequences of values of m/n, and in Section 3 — (mostly) for some intervals of values. In each Section, the problems are arranged (more or less) by the difficulty in an increasing order. We recommend to try to solve the problems from all Sections simultaneously.

On the other hand, if you will solve one of the Sections (almost) up to the end, we are always ready to add some more difficult problems.

2 Some special sequences of values

2.1. a) Determine $f(3k-1,3k)$.
b) Determine $f(3k+1,3k+2)$.
c) Determine $f(3k,3k+1)$.

2.2. Prove that $f(m,2m-1) = \frac{m+1}{6m-3}$ for all $m \geq 4$.

2.3. a) Determine $f(3,n)$.
b) Determine $f(4,n)$.
c) And also $f(5,n)$.

2.4. Determine $f(m,2m+1)$.

2.5. Determine $f(2k+1,3k+2)$.

2.6. Determine $f(3k+1,4k+1)$.

2.7. Determine $f(5k+2,8k+3)$.

2.8. Determine $f(5k-1,9k-2)$.

2.9. Determine $f(17k-4,21k-5)$.

3 Serial results

We consider the problems of this Section as some steps towards the general algorithm (but surely not all of them!). So, if you get some different **serial** results — do not hesitate to submit them!

Recall that we always assume $m < n$.

3.1. Given that m does not divide n, prove that $f(m,n) \le \frac{m}{2n}$. Determine also all the pairs (m,n) for which the equality is achieved.

3.2. a) Assume that $\frac{3}{4} < \frac{m}{n} < 1$. Prove that $f(m,n) \le \frac{m}{n} - \frac{1}{2}$.

b) Assume that $\frac{1}{2} < \frac{m}{n} < 1$. Prove that $f(m,n) \le \frac{m}{n} - \frac{1}{3}$.

c) Assume that $\frac{2}{k+1} < \frac{m}{n} < 1$ with $k \ge 4$. Prove that $f(m,n) \le \frac{m}{n} - \frac{1}{k}$.

Attention! Part b) of the next problem is very important!

3.3. a) Suppose that $f(m,n) > \frac{m}{3n}$. Prove that in every optimal distribution for the pair (m,n) each man gets not more than 2 pieces.

b) (Theorem on One Third) Prove that $f(m,n) \ge \frac{m}{3n}$.

c) Given that $\frac{2}{3} < \frac{m}{n} \le \frac{3}{4}$, prove that $f(m,n) = \frac{m}{3n}$. (See also problem 3.11.)

3.4. a) Given that $\frac{m}{n} < \frac{2}{3}$, prove that $f(m,n) \le \frac{1}{4}$.

b) Determine all pairs (m,n) (with $\frac{m}{n} < \frac{2}{3}$) such that $f(m,n) = \frac{1}{4}$.

3.5. Determine all pairs (m,n) such that $\frac{2}{k+1} < \frac{m}{n} < \frac{2}{k}$ and $f(m,n) = \frac{1}{k+1}$.

3.6. a) Prove that $f(m,n) = \frac{m}{n} - \frac{1}{3}$ if $\frac{1}{2} < \frac{m}{n} \le \frac{5}{9}$.

b) Determine all pairs (m,n) such that $f(m,n) = \frac{m}{n} - \frac{1}{3}$.

3.7. Determine all pairs (m,n) such that $\frac{2}{k+1} < \frac{m}{n} < \frac{2}{k}$ and $f(m,n) = \frac{m}{n} - \frac{1}{k}$ (for $k \ge 4$).

3.8. a) Given that $\frac{7}{15} < \frac{m}{n} < \frac{1}{2}$, find $f(m,n)$.

b) Find all values of m/n for which $f(m,n)$ has the same form.

3.9. a) Given that $\frac{7}{12} < \frac{m}{n} < \frac{22}{37}$, find $f(m,n)$.

b) Find all values of m/n for which $f(m,n)$ has the same form.

3.10. a) Given that $\frac{14}{17} < \frac{m}{n} < \frac{5}{6}$, find $f(m,n)$.

b) Find all values of m/n for which $f(m,n)$ has the same form.

3.11*. Determine all pairs (m,n) such that $f(m,n) = \frac{m}{3n}$.