Webs from lines and circles

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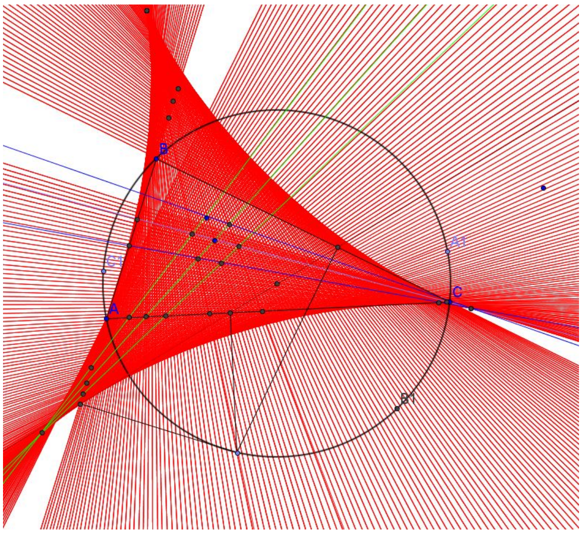

Solutions of the problems

Next lemma allows to solve problems 0.1, 0.5, 0.6.

Lemma 0. Let $A'B'C'$ be a cevian triangle of some point wrt triangle ABC (i.e the lines AA', BB', CC' concur). The line passing through an arbitrary point M_1 of side AC and parallel to $A'B'$ intersect BC in point M_2; the line padding through M_2 and parallel to $A'C'$ intersect AB in point M_3 etc. Then $M_1 = M_7$.

`Proof.`

If M_1 coincide with B_1, then M_4 and M_7 also coincide with B'. Else by Ceva theorem $\frac{\overline{AB'}}{\overline{B'C}}\frac{\overline{CA'}}{\overline{A'B}}\frac{\overline{BC'}}{\overline{C'A}} = 1$. And by Thales theorem $\frac{\overline{B'C}}{\overline{CA'}} = \frac{\overline{B'M_1}}{\overline{M_2A'}}$, $\frac{\overline{A'B}}{\overline{BC'}} = \frac{\overline{M_2A'}}{\overline{C'M_3}}$, $\frac{\overline{C'A}}{\overline{AB'}} = \frac{\overline{C'M_3}}{\overline{M_4B'}}$. Placing three last equalities into the first one we obtain that: $\frac{\overline{B'M_1}}{\overline{M_4B'}} = 1$. Thus M_1 and M_4 are symmetric wrt B'. Similarly M_4 and M_7 are symmetric wrt B'. Therefore $M_1 = M_7$.

`Solutions of problems of part 0.`

Solution of problem 0.1

First solution. Let a, b, c be the side lengths of BC, CA, and AB, respectively. Let x be the *signed* length of AA_1 (i.e., the length of AA_1 taken with positive sign, if the vectors $\overrightarrow{AA_1}$ and $\overrightarrow{AB}$ have the same orientation, and with negative sign, if they have opposite orientation). Since A_1A_2 is orthogonal to the the bissector of the angle BAC it follows that $AA_2 = AA_1 = x$ (with signs). Analogously we find consecutively $CA_3 = CA_2 = b - x$, $BA_4 = a - b + x$, $AA_5 = c - a + b - x$, $CA_6 = a - c + x$, $AA_7 = x$ (with signs). Since $AA_7 = x = AA_1$ (with signs) it follows that $A_7 = A_1$.

Second solution. Use lemma 0 to Gergonne triangle.

Solution of problem 0.2

In problems 0.2, 0.3 and 0.4 we consider the angles between the directions (vectors).

Let O be the common point of l_1, l_2, l_3. Let $(l_1, l_2) = \varphi_{1,2}$,$(l_2, l_3) = \varphi_{2,3}$, $(OA_1, l_1) = \varphi$. Since the lengths of segments OA_i are equal (the symmetry conserve the length), it is sufficiently to prove that $(OA_7, l_1) = (OA_1, l_1) = \varphi$.

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Since $(OA_2, l_1) = -(OA_1, l_1) = -\varphi$, then $(OA_2, l_2) = -\varphi + \varphi_{1,2}$. Thus $(OA_3, l_2) = \varphi - \varphi_{1,2}$. From this we obtain that $(OA_3, l_3) = \varphi - \varphi_{1,2} + \varphi_{2,3}$. Therefore $(OA_4, l_3) = -\varphi + \varphi_{1,2} - \varphi_{2,3}$. This yields that $(OA_4, l_1) = -\varphi + \varphi_{1,2} - \varphi_{2,3} - \varphi_{1,2} - \varphi_{2,3} = -\varphi - 2\varphi_{2,3}$. Similarly we obtain that $(OA_7, l_1) = -(OA_4, l_1) - 2\varphi_{2,3} = \varphi$.

Solution of problem 0.3

Let $(l_1, l_2) = \varphi_{1,2}, (l_2, l_3) = \varphi_{2,3}$ and $(l_{(1,2)}, l_1) = \varphi$, where $l_{(i,i+1)}$ ix the vector from O to the projection of O to the line A_iA_{i+1}. Since the distances from O to all lines A_iA_{i+1} are equal (the symmetry conserve the length), it is sufficiently to prove that $(l_{(7,8)}, l_1) = (l_{(1,2)}, l_1) = \varphi$.

Since $(l_{(2,3)}, l_1) = -\varphi$ then $(l_{(2,3)}, l_2) = (l_{(2,3)}, l_1) + (l_1, l_2) = -\varphi + \varphi_{1,2}$. Similarly $(l_{(3,4)}, l_2) = \varphi - \varphi_{1,2}$. Thus $(l_{(3,4)}, l_3) = (l_{(3,4)}, l_2) + (l_2, l_3) = \varphi - \varphi_{1,2} + \varphi_{2,3}$. Also $(l_{(4,5)}, l_3) = -\varphi + \varphi_{1,2} - \varphi_{2,3}$. Therefore $(l_{(4,5)}, l_1) = (l_{(4,5)}, l_3) + (l_1, l_3) = -\varphi + \varphi_{1,2} - \varphi_{2,3} + (-\varphi_{1,2} - \varphi_{2,3}) = -\varphi - 2\varphi_{2,3}$. Similarly we obtain that $(l_{(7,8)}, l_1) = -(l_{(4,5)}, l_1) - 2\varphi_{2,3} = \varphi$.

Solution of problem 0.4.

Let $(l_1, l_2) = \varphi_{1,2}, (l_2, l_3) = \varphi_{2,3}$, $(A_1A_2, l_1) = \varphi$. Since the circumradii of triangles OA_iA_{i+1} are equal (by the sinus theorem), it is sufficiently to prove that $(A_7A_6, l_3) = -(A_1A_2, l_2) = -\varphi - \varphi_{1,2}$ (by the inverse sinus theorem).

Since $(A_3A_2, l_3) = -(A_1A_2, l_1) = -\varphi$ then $(A_3A_2, l_2) = -\varphi - \varphi_{2,3}$. Since $(A_3A_4, l_1) = -(A_3A_2, l_2) = \varphi + \varphi_{2,3}$ then $(A_3A_4, l_3) = \varphi + \varphi_{1,2} + \varphi_{2,3} + \varphi_{2,3}$. Since $(A_5A_4, l_2) = -(A_3A_4, l_3) = -\varphi - \varphi_{1,2} - 2\varphi_{2,3}$ then $(A_5A_4, l_1) = -\varphi - \varphi_{1,2} - 2\varphi_{2,3} - \varphi_{1,2}$. Since $(A_5A_6, l_3) = -(A_5A_4, l_1) = \varphi + 2\varphi_{1,2} + 2\varphi_{2,3}$ then $(A_5A_6, l_2) = \varphi + 2\varphi_{1,2} + 2\varphi_{2,3} - \varphi_{2,3}$. Since $(A_7A_6, l_1) = -(A_5A_6, l_2) = -\varphi - 2\varphi_{1,2} - \varphi_{2,3}$ then $(A_7A_6, l_3) = -\varphi - 2\varphi_{1,2} - \varphi_{2,3} + \varphi_{1,2} + \varphi_{2,3} = -\varphi - \varphi_{1,2}$.

Solution of problem 0.5.

Use lemma 0 to the orthotriangle.

NOTE. The problem can be reformulated in the next way.

Let point A_1 lies on AB. The circumcircle of triangle A_1AC secondary meets BC in point A_2. The circumcircle of triangle A_2BA secondary meets CA in point A_3 etc. Prove that $A_1 = A_7$.

Solution of problem 0.6

Use lemma 0 to the medial triangle.

Solution of problem 0.7

a) The Pappus theorem in an equivalent statement is proved in the book [1, Chapter 1].

b) Consider the hexagon $A_1A_2A_3A_6A_5A_4$: the lines A_1A_2, A_3A_6, A_5A_4 concur in point R, and the lines A_4A_1, A_2A_3, A_6A_5 concur in point G. Therefore "the diagonals" A_2A_5, A_3A_4 (these two lines pass through B) and A_6A_1 concur (in point B). Thus $A_7 = A_1$.

Solution of problem 0.8

a) The Brianchon theorem is proved in the book [1, Chapter 1].

b) It is clear that the lines A_4A_5, A_5A_6, A_6A_7 are the reflections of A_4A_3, A_3A_2, A_2A_1 in OI. Therefore $A_1 = A_7$.

Solution of problem 0.9

First solution (D. Yakutov). Let us compute the angle $\angle GA_4R$:

$$\begin{aligned}
\angle GA_4R &= \pi - \angle GA_4B - \angle BA_4R \\
&= \pi - \angle GOB - \angle BA_5R \\
&= \pi - \angle GOB - (\pi - \angle GA_5R - \angle GA_5B) \\
&= \angle GA_5R + \angle GA_5B - \angle GOB \\
&= \angle GOR - \angle GOB + \angle GA_6B \\
&= \angle GOR - \angle GOB + (\pi - \angle GA_6R - \angle BA_6R) \\
&= \angle GOR - \angle GOB + (\pi - \angle GA_7R - \angle BOR) \\
&= (\pi - \angle BOR - \angle GOB + \angle GOR) - \angle GA_7R.
\end{aligned}$$

Thus $\angle GA_4R + \angle GA_7R = \pi - \angle BOR - \angle GOB + \angle GOR$.

Similarly $\angle GA_1R+\angle GA_4R=\pi-\angle BOR-\angle GOB+\angle GOR$. Thus $\angle GA_1R=\angle GA_7R$. Also $\angle GA_1B=\angle GOB=\angle GA_7B$. Hence the points G,B,A_1,A_7 belong to one circle and the points G,R,A_1,A_7 also belong to one circle. But these two points have at most two common point, one of which is G. Since $A_1\neq G$ and $A_7\neq G$ it follows that $A_1=A_7$.

Second solution. Consider an arbitrary inversion with center O. It transforms the red, green and blue circles to three lines. Let the image of the red point be C, the image of the blue point be A, and the image of the green point be B. Now look after points A_i. Through point A_1 ($\in AB$) we take a green (passing through A and C) circle, which secondary meets in A_2 the blue "circle" — line BC. Through A_2 we take a red circle secondary meeting in A_3 the green "circle" — line AC. Through A_3 we take a blue circle secondary meeting in A_4 the red "circle" — line AB etc.

Thus we obtained the reformulating of problem 0.5 given in the note to this problem. Therefore $A_1=A_7$.

`Solutions of problems of part 1.`

Most of the solutions of problems from sections 1–3 are based on Problem 4.4.

Solution of problem 1.1.

Use problem 4.4.

- For each $t\in\mathbb{R}$ take a homothety $H_O^{2^t}$ with center O and coefficient 2^t. It is clear that for any point A $H_O^{2^{t+s}}(A)=H_O^{2^t}(H_O^{2^s}(A))$. The sets γ_A are a rays with origin O.
- Draw through point $A(1,1)$ the lines $y=1$ (this is γ_1) and $x=1$ (this is γ_2). Draw through an arbitrary point $B\in\gamma_1$ a ray γ_B. Paint all such rays red. Now paint green and blue respectively the lines $H_O^{2^t}(\gamma_1)$ ans $H_O^{2^t}(\gamma_2)$, i.e the lines parallel to Ox and Oy. It is evident that the rays or the lines of each color don't intersect.
- Consider a disc with radius 1 and center (1;1). It is clear that exactly one ray or line of each color passes through each point of this disc.

Therefore by problem 4.4 the constructed rays and lines form a web. It is clear that the lines given in the problem also form a web.

Solution of problem 1.2.

First solution. Follows from the assertion of problem 0.7b.

Second solution. This follows from problem 1.1 by a projective transformation taking the line through 2 points from the 3 given ones to the infinitely distant line.

Solution of problem 1.3.

Follows from the assertion of problem 0.8b).

Solution of problem 1.4.

Follows from the assertion of problem 0.8a).

Solution of problem 1.5.

These lines don't form a web.

Solution of problem 1.6.

These lines don't form a web.

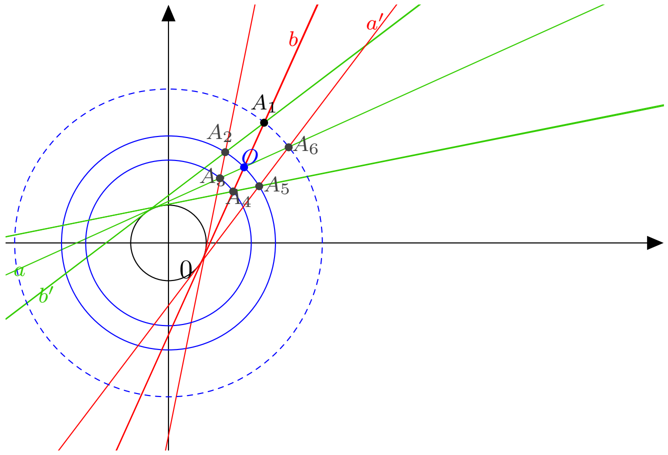
`Solutions of problems of part 2.`

Solution of problem 2.1.

Apply for example an arbitrary inversion to the web formed by the lines parallel to the sidelines of some triangle.

Solution of problem 2.2.

First solution (E. Streltsova). Let us prove that these lines and circles form a wweb; see figure below. Take a disc in the first coordinate quarter above the line $y=1$ so that it has no common points with the unit disk. Let the radius of the disk be <1. Through each point of the disk there is exactly one red and one green line because there is exactly one tangent line of each color from each point. Through each point T there is exactly one line with the center at the origin (Z) because the radius (ZT) and the center (Z) uniquely determine a circle. The circles with the center Z cannot coincide with the tangents to the unit circle. And the green and the red line cannot coincide because the disk is above the line $y = 1$. Concentric circles cannot touch each other. And the green and the red lines cannot touch the circles with the center Z because the latter have radius > 1 and our disk have no common points with the unit disk. The tangents intersect the blue circles because they contain points inside the circles. Thus the foliation condition holds.



The green (a) and the red (b) lines through the point O are symmetric with respect to the line ZO. Thus $A_3 = S_{ZO}(A_4)$. Thus the red line through A_3 is symmetric to the green line through A_4 with respect to the line ZO. Hence $A_2 = S_{ZO}(A_5)$. Hence the green line through A_2 (b') is symmetric to the red line through A_5 (a').

Further, $a = S_{ZO}(a'), b = S_{ZO}(b')$. Thus $A_6 = a \cap a' = S_{ZO}(b \cap b') = S_{ZO}(A_1)$. Thus $A_6 = S_{ZO}(A_1)$. Hence $ZA_6 = S_{ZO}(ZA_1)$. Therefore $ZA_6 = ZA_1$, hence the blue circle through the point A_6 passes through A_1. The closure condition has been checked.

Second solution. Use problem 4.4.

- For each $t \in \mathbb{R}$ take a rotation $R_O^{\pi t}$ around the origin O to the angle πt. It is clear that for any point A $R_O^{\pi(t+s)}(A) = R_O^{\pi t}(R_O^{\pi s}(A))$, if $t, s \in \mathbb{R}$. The sets γ_A are an arcs of the circles with center O.
- Draw through the point $A(0, 2)$ the rays $y = \sqrt{3}x+2, x > -\sqrt{3}/2$ (this is γ_1) and $y = -\sqrt{3}x+2, x < \sqrt{3}/2$ (this is γ_2). These rays touche the unit semicircle. Draw through each point $B \in \gamma_1$ an arc γ_B of the circle with the center in the origin. Paint all such arcs the red. Now paint the rays $R_O^{\pi t}(\gamma_1)$ and $R_O^{\pi t}(\gamma_2)$ green and blue respectively, these rays touche the unit semicercles.
- Consider a disc with radius $1/2$ and center $(0; 2)$. It is clear that exactly one arc or ray of each color passes through each point of this disc.

Therefore by problem 4.4 the constructed arcs and rays form a web. It is clear that the lines and the circles given in the problem also form a web.

Solution of problem 2.3.
Use problem 4.4.

- For each $t \in \mathbb{R}$ take a rotation $R_O^{\pi t}$ around the origin O to angle πt. It is clear that for any point A $R_O^{\pi(t+s)}(A) = R_O^{\pi t}(R_O^{\pi s}(A))$, if $t, s \in \mathbb{R}$. The sets γ_A are an arcs of the circles with center O.
- Draw through the point $A(0, 2)$ the rays $y = \sqrt{3}x + 2, x > -\sqrt{3}/2$ (this is γ_1) and $x = 0, y > 0$ (this is γ_2). One of these rays touches the unit semicircle and the other passes through the origin. Draw through each point $B \in \gamma_1$ an arc γ_B of the circle with center O. Paint all such arcs the red. Now paint the rays $R_O^{\pi t}(\gamma_1)$ and $R_O^{\pi t}(\gamma_2)$ green and blue respectively.
- Consider a disc with radius $1/2$ and center $A(0; 2)$. It is clear that exactly one line of each color passes through each point of this disc.

Therefore by problem 4.4 the constructed rays and arcs form a web. It is clear that the lines and the circles given in the problem also form a web.

Solution of problem 2.4.

Use problem 4.4.

- For each $t \in \mathbb{R}$ take a translation $T_{(0,t)}$ to the vector $(0,t)$. It is clear that for any point A $T_{(0,t+s)}(A) = T_{(0,t)}(T_{(0,s)}(A))$, if $t, s \in \mathbb{R}$. The sets γ_A are lines parallel to Oy.
- Draw through the point $A(1/2 + 1/\sqrt{8}, 1/2 + 1/\sqrt{8})$ the arc $(x - 1/2)^2 + (y - 1/2)^2 = 1/4, x > 1/2, y > 1/2$ (this is γ_1) and the line $y = 1/2 + 1/\sqrt{2}$ (this is γ_2). Through each point $B \in \gamma_1$ draw a line γ_B parallel to Oy. Paint all such lines red. Now paint the arcs $T_{(0,t)}(\gamma_1)$ and the lines $T_{(0,t)}(\gamma_2)$ green and blue respectively.
- Consider a disc with radius $1/2 - 1/\sqrt{8}$ and center $A(1/2 + 1/\sqrt{8}; 1/2 + 1/\sqrt{8})$. It is clear that exactly one arc of each color passes through each point of this disc.

Therefore by problem 4.4 the constructed arcs form a web. Then the generalized circles considered in the problem also form a web.

It is clear that the lines and the circles given in the problem also form a web.

Solution of problem 2.5.

Use problem 4.4. Consider as the maps the homotheties with center O. Then the red lines pass through the origin. The green circles touch both segments of the firs pair. The blue circles touch both segment of the second pair. By Problem 4.4 it follows that certain arcs of these circles form a web. Then the generalized circles considered in the problem also form a web.

By problem 4.4 these lines and circles form a web.

Solution of problem 2.6.

Use problem 4.4. Consider as the maps the homotheties with center O. Then the red lines pass through the origin. The green circles are the circles with center O. The blue circles touche two given segments.

By problem 4.4 these lines and circles form a web.

`Solutions of problems of part 3.`

The assertions the analogous of the problem 4.4 are true for the torus and the hyperboloid of revolution.

Solution of problem 3.1.

Use problem 4.4. Consider as the maps the rotations around the axis of the torus. Then the parallels are the red circles. The Villarceau circles are the green and blue circles.

By problem 4.4 these circles form a web.

Solution of problem 3.2. Take an arbitrary point O of the torus. Draw through it the meridean γ_1 and the Villarceau circles γ_2, γ_3. Paint the meridians red, paint the Villarceau circles obtained from the circle γ_2 by rotation green, paint the Villarceau circles obtained by rotation the circles γ_2 blue. Take a sphere with center O and radius $R = \frac{r}{100}$ (r is the distance between γ_1 and the axis of the torus). It is clear that any two the Villarceu circles have at most one common point inside the sphere. By Ω denote the intersection of the sphere and the torus.

Consider an arbitrary point O' inside $\mathbf{\Omega}$. Draw through it the red w_1, green w_2, and blue w_3 circles. Let all constructed points A_i lie inside Ω. Let $A_1 \in w_1$. Draw through A_1 the green circle w_2'. Let A_2 be the common point of w_2' and w_3. Draw through A_2 the red circle w_1'. Let A_3 be the common point of w_1' and w_2. Draw through A_3 the blue circle w_3'. Let A_4 be the common point of w_3' and w_1. Draw through A_4 the green circle w_2''. Let A_5 be the common point of w_2'' and w_3. Draw through A_5 the red circle w_1'. Let A_6 be the common point of w_1'' and w_2. Draw through A_6 the blue circle w_3'. Let A_7 be the common point of w_3'' and w_1. Let α be a plane such that w_1 lie in. It is clear that the circles w_3',w_1', w_2' are the reflections of w_2'', w_1'', w_3'' in α. Therefore $A_1 = A_7$.

Solution of problem 3.3.

Use problem 4.4. Consider as the maps the rotations around the axis of the torus. Then the parallels are the red circles. The line lying on the hyperboloid are the green and blue lines. By problem 4.4 these circles form a web.

`Hints and solutions of problems of part 4.`

Hint to problem 4.1.

Let us list a few possible examples of the sets of blue general circles:

- (B) an arbitrary pencil of lines (Problems 1.1 and 1.2);
- (B) circles with a common center;
- (B) circular arcs obtained from a given one by parallel translations along either the Ox or the Oy axis (Problem 4.3);

By the Graf–Sauer theorem (Problem 4.12) there are no other examples of sets of blue lines. By the Shelekhov classification of all webs from pencils of general circles [3, Theorem 0.1] it follows that there are no other examples in which the set of blue circles is a pencil. Description of all possible examples, not necessarily pencils, is an open problem.

Hint to problem 4.2.

Let us list a few possible examples of the sets of blue general circles:

- (B) an arbitrary pencil of lines (Problems 1.1 and 1.2);
- (B) pencil of circles with a limit point at the origin O and the common radical axis parallel to the Ox axis;
- (B) circular arcs obtained from a given one by homotheties with center at the origin (Problem 4.3).

By the Graf–Sauer theorem (Problem 4.12) there are no other examples of sets of blue lines. By the Shelekhov classification of all webs from pencils of general circles [3, Theorem 0.1] it follows that there are no other examples in which the set of blue circles is a pencil. Description of all possible examples, not necessarily pencils, is an open problem.

Hint to problem 4.3.

Perform an inversion with the center at one of the limit points. The obtained pencils of general circles form a web by Problem 4.4.

Solution of problem 4.4.

The foliation condition is true by the third condition of the problem. Let us show that the closure condition also is true.

Take an arbitrary point O inside the disc. Draw trough it the red (w_1), green (w_2) and blue (w_3) arcs of general circles. Let all constructed points A_i lie inside $\mathbf{\Omega}$. Let $A_1 \in w_1$ and $t \in \mathbb{R}$ is such that $R_t(O) = A_1$ (such t exists by the first condition of the lemma: if $w_1 = \gamma_X$, where $X \in \gamma_1$, then there exist such $y, z \in \mathbb{R}$, that $R_y(X) = O$ and $R_z(X) = A_1$, thus $R_{z-y}(O) = R_{z-y}(R_y(X)) = R_z(X) = A_1$. Therefore $t = z - y$.). Draw through A_1 the green arc w'_2. Let A_2 be the common point of w'_2 and w_3. Draw through A_2 the red arc w'_1. Let A_3 be the common point of w'_1 and w_2. Draw through A_3 the blue arc w'_3. Let A_4 be the common point of w'_3 and w_1 etc.

Now let us show that $R_t(O) = A_7$, this yields that $A_1 = A_7$.

We know that $R_t(O) = A_1$, thus $R_t(w_2) = w'_2$ (it is true because $R_t(w_2)$ is the red arc passing through A_1, and the unique such arc is w'_2), therefore $R_t(A_3) \in w'_2 \cap w'_1 = A_2$. Since $R_t(A_3) = A_2$, then $R_t(A_4) = O$ (similarly). Since $R_t(A_4) = O$, then $R_t(A_5) = A_6$. Since $R_t(A_5) = A_6$, then $R_t(O) = A_7$. The assertion is proved.

Hints to problems 4.5–4.6.

These problems are discussed in [4].

Hints to problem 4.7.

The solution of this problem is given in the book by Prasolov and Solov'ev [2].

Hint. Let the equations of red lines be $a_1x + b_1y - 1 = 0$, $a_2x + b_2y - 1 = 0$, $a_3x + b_3y - 1 = 0$, and let the equation of blue ones be $c_1x + d_1y - 1 = 0$, $c_2x + d_2y - 1 = 0$, $c_3x + d_3y - 1 = 0$. Prove that the equation of the curve has the form

$$p(a_1x + b_1y - 1)(a_2x + b_2y - 1)(a_3x + b_3y - 1) + q(c_1x + d_1y - 1)(c_2x + d_2y - 1)(c_3x + d_3y - 1) = 0$$

for some real numbers p and q.

Hints to problem 4.8.

This problem is obtained from the previous one by the projective duality.

Hints to problem 4.9.

Use Problem 4.8.

Hints to problem 4.10–4.11.

Use Problem 4.9. A figure to Problem 4.10 by A. Ghaneiyan Sebdani and E. Ashourioun is shown at the first page of this document.

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