HOW DO CURVED SPHERES INTERSECT IN 3-SPACE,

OR TWO-DIMENSIONAL MEANDRA [1]

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1 Examples and main problems

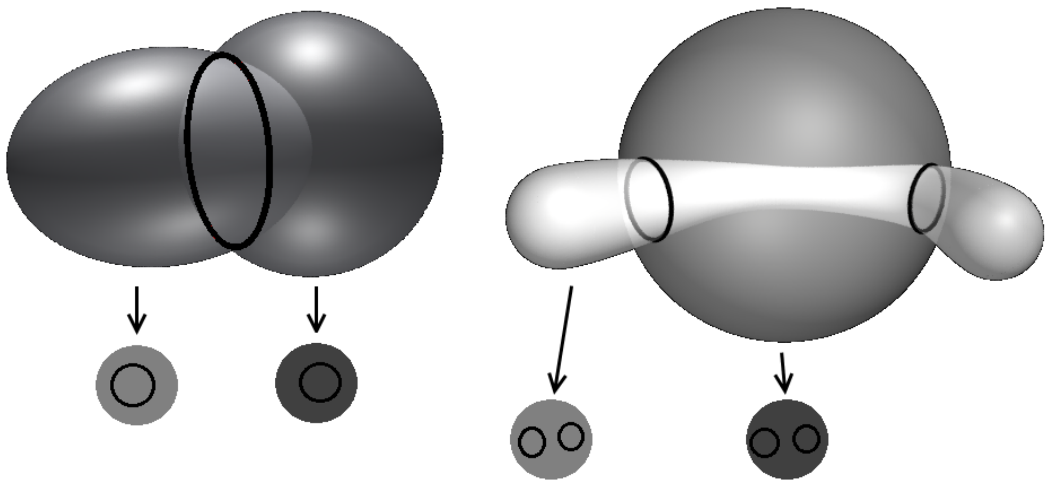

Figure 1: Curved spheres intersecting by a circle (left), by two circles (right)

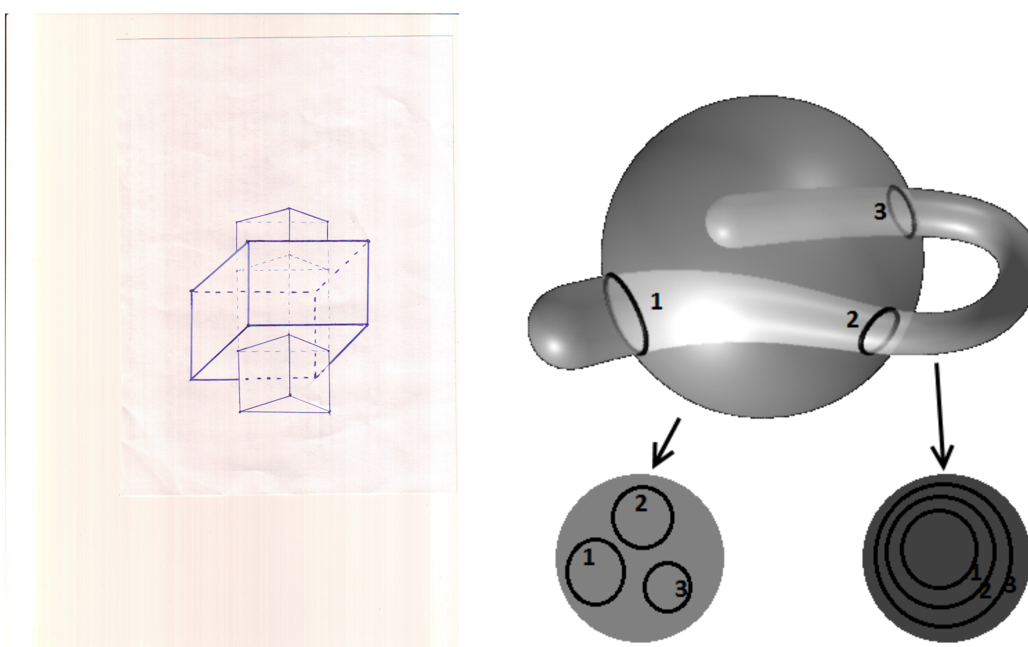

Figure 2: Curved spheres intersecting by two circles (left), by three circles (right)

How can two curved spheres intersect in 3-space? In figures 1 and 2 you see pairs of curved spheres in 3-space intersecting by a union of circles.

[1]We are grateful for useful remarks and discussions to G. Chelnokov, S. Lando and to an anonymous referee of Moscow Mathematical Conference of High-School Students.

[2]Supported by Simons-IUM Fellowship

The term *curved sphere* is possibly intuitively clear for you. If not, read rigorous definitions below. Here we only remark that in this text curved sphere *does not have self-intersections.*

You will possibly be able to solve all the problems below without really using those rigorous definitions. In your solutions you can represent curved spheres by large pictures clear to jury members, not only by formal constructions. If the formulation of a problem is a statement, it is required to prove this statement. If a problem looks like too difficult, try to solve the neighboring problems, they can contain hints. Unsolved problems are marked by stars.

1.1. Draw two curved spheres in 3-space intersecting by a disjoint union of 3 circles so that in each sphere these circles

(a) bound 3 disjoint disks (like in figure 2 left).

(b) do not bound 3 disjoint disks (like in figure 2 right).

1.2. Draw two curved spheres in 3-space intersecting by a disjoint union of 4 circles so that in each sphere these circles

(a) bound 4 disjoint disks (as in figure 3.a).

(b) are 'parallel', or 'one inside the other' (as in figure 3.b).

(c) are situated as in figure 3.c.

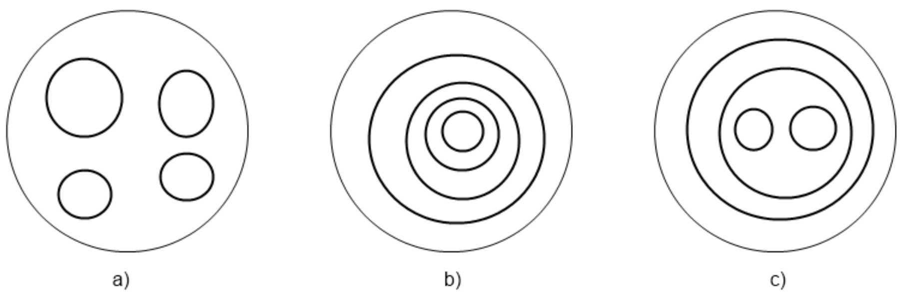

Figure 3: Four circles in a sphere

Suppose that M and N are collections of the same number of disjoint circles in curved spheres S and T. Then M *is situated in* S *as* N *in* T if there is a bijection between connected components (=connected parts) of $S - M$ and $T - N$ such that connected components of $S - M$ are neighbors if and only if the two corresponding connected components of $T - N$ are neighbors. (Or, equivalently, if pairs (S, M) and (T, N) are piecewise-linearly homeomorphic.)

1.3. (ij), $i, j \in \{a, b, c\}$. Draw two curved spheres in 3-space intersecting by a disjoint union of 4 circles so that in one sphere the circles are situated as in figure 3.i, and in the other as in figure 3.j.

In this text we study the following two problems and their generalizations. (You probably will not be able to solve the problems right away, so postpone them and try to solve other problems.)

The intersection of two curved spheres is *transversal* if near each intersection point it looks like the intersection of two planes having a common line. (See a rigorous definition below.)

1.4. (a) **The Lando Problem.** Let M and N be two unions of the same number of disjoint circles in a sphere. Do there exist two curved spheres in 3-space whose intersection is transversal and is a finite collection of disjoint circles that is situated as M in one sphere and as N in the other?

(b) Does there exist an algorithm for checking the existence of such two curved spheres? (Cf. 'relation to graphs' below.)

(c)* Does there exist such a polynomial algorithm?

In figure 1, left, two spheres intersect by a circle; each sphere is split by the circle into 2 connected components, and each connected component has one neighboring connected component (in the same sphere). In figure 1, right, (and in figure 2, left) two curved spheres intersect by 2 circles; each sphere is split by the circles into 3 connected component, of which two have one neighboring connected component and one has two neighboring connected components (in the same sphere). In figure 2, right, two curved spheres intersect by 3 circles; each sphere is split by the circles into 4 connected components, in one sphere the numbers of neighbors of connected components are $3, 1, 1, 1$, in the other sphere those numbers are $1, 2, 2, 1$.

1.5. Neighbor Sequence Problem. Given sequences $\vec{x} = (x_1, x_2, \dots, x_n)$, $\vec{y} = (y_1, y_2, \dots, y_n)$ of positive integers, does there exist two curved spheres S, T in 3-space whose intersection consists of $n-1$ circles and splits

• S into n connected components which can be numbered so that the i-th connected component has x_i neighbors in S, and

• T into n connected component which can be numbered so that the i-th connected component has y_i neighbors in T?

Some rigorous definitions.

You will probably be able to solve all the problems without really using these rigorous definitions.

We present definitions convenient in frame of this text; they could be different from common mathematical terms.

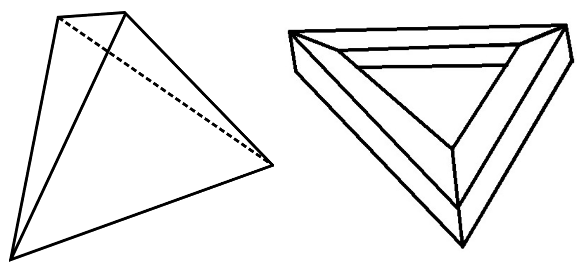

Figure 4: A curved sphere (left), not a curved sphere (right)

In this text a *curved circle* or, shortly, a *circle* is a closed broken line without self-intersections in 3-space. The definition of a polyhedron (without self-intersections, but possibly non-convex) is given in [D], see also [W]. A *curved sphere* is a polyhedron in 3-space (more precisely, 2-dimensional surface of the polyhedron), which is split into several parts by any circle lying on the polyhedron. See figure 1. (Such polyhedra are called topologically equivalent to sphere. This condition is equivalent to the condition $V - E + F = 2$.)

In order to simplify pictures, instead of a polyhedron we draw a *curved* surface 'close' to the polyhedron. For example, a curved sphere or a 'sausage' as in Figure 2. Instead of a broken line we draw a *curve* 'close' to the broken line.

A subset X of the 3-space is *connected* if each two points of X can be connected by a broken line in X. A *connected component* of a subset X of the 3-space is a maximal connected subset of X, i.e., a connected subset $Y \subset X$ such that there does not exist a connected subset $Z \subset X$ for which $Y \subset Z \subset X$ and $Y \neq Z \neq X$.

Suppose that M is a collection of disjoint circles in curved sphere S. Two connected components of the complement $S - M$ are *neighbors* if their closures intersect.

Denote by $B(x, r) \subset \mathbb{R}^3$ the ball of radius r centered at x. Intersection of two curved spheres $S, T \subset \mathbb{R}^3$ is *transversal* if for each point $x \in S \cap T$ there is $r > 0$ such that both $B(x, r) - S$ and $B(x, r) \cap (T - S)$ consist of two connected components, and each connected component of $B(x, r) - S$ contains a connected component of $B(x, r) \cap (T - S)$.

You can use the following theorem and corollary without proof.

Jordan Theorem. *A curved sphere splits 3-space into exactly two connected components.*

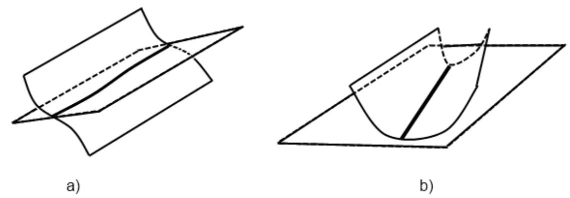

Figure 5: A transversal intersection (left), not a transversal intersection (right)

Corollary. *Suppose that S and T are curved spheres intersecting transversely by a finite union $S \cap T$ of disjoint circles. Denote by B the interior part of S (or, more precisely, the bounded part of $\mathbb{R}^3 - S$). Let Q be a connected component of $T - S$ which is situated inside S. Then Q splits B into exactly two connected components.*

Relation to graphs.

Suppose that M is a collection of disjoint circles in curved sphere S. Define ('dual to M') graph $G = G(S, M)$ as follows. The vertices are connected components of $S - M$. Two vertices are connected by an edge if the corresponding connected components are neighbors.

In figure 6 we show graphs for spheres S, T from figure 2 and collection $S \cap T$ of circles. Analogously two curved spheres intersecting by circles define a pair of graphs. Then the Lando Problem asks to describe such pairs of graphs, and Neighbor Sequence Problem asks to describe pairs of degree sequences of such pairs of graphs. [3]

Figure 6: Two graphs corresponding to Figure 2

Stars.

A team gets a star for each correct ($\geqslant +.$) written solution. *A large picture clear to jury members, or a well-structured computer program passing tests assigned by Jury, is recognized as an equivalent of a written solution.* Jury may also award stars for elegant solutions, for solutions of difficult problems and for (some) solutions written in TEX. The jury has infinite number of stars. A team may present a solution orally paying 1 star for each attempt.

We invite participants succeeding in solving these problems and working on unsolved problems to discuss their questions and ideas of solutions.

2 Neighbor Sequence Problem

A pair $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{y} = (y_1, y_2, \ldots, y_n)$ of sequences of positive integers is called *realizable* if there exist two curved spheres S, T in 3-space whose intersection consists of $n - 1$ circles and splits

• S into n connected components which can be numbered so that the i-th connected component has x_i neighbors in S, and

[3]Here is another interpretation suggested by I. N. Shnurnikov. Suppose that the unit square on the plane and (piecewise linear) function on the square are given. The function is strictly positive on the boundary of the square. The disk corresponds to the first curved sphere (with a hole), the graph of the function (above the disk) — to the second curved sphere, the zero set of the function — to the intersection of curved spheres.

• T into n connected components which can be numbered so that the i-th connected component has y_i neighbors in T.

Pair (S, T) of spheres is called a *realization* of pair $(\vec{x}, \vec{y})$.

2.1. (n), $n \in \{2, 3, 4, 5\}$. Which pairs of sequences of n positive integers are realizable?

2.2. (a) If pair $(\vec{x}, \vec{y})$ of sequences is realizable, then $x_1 + \dots + x_n = y_1 + \dots + y_n = 2n - 2$.

(b) The dual graph $G(S, M)$ to a collection M of disjoint circles in a curved sphere S is a tree.

A sequence $\vec{x} = (x_1, x_2, \dots, x_n)$ of positive integers is called *tree-like* if $x_1 + \dots + x_n = 2n - 2$.

2.3. If a sequence $\vec{x}$ is tree-like, then it has at least x_1 units.

2.4. Pair $(\vec{x}, \vec{x})$ is realizable for each tree-like $\vec{x}$.

2.5. Let $\vec{x}, \vec{y}$ be tree-like sequences in which all the units are situated at the end. If $x_1 \geq y_1$, then sequences $\vec{x}' := (x_1 - y_1 + 1, x_2, x_3, \dots, x_{n-y_1+1})$ and $\vec{y}' := (y_2, y_3, \dots, y_{n-y_1+2})$ are tree-like.

2.6. Which pairs of tree-like sequences could be obtained from pair $((1, 1), (1, 1))$ by reorderings and changes of pair ($\vec{x} = (x_1, x_2, \dots, x_n)$, $\vec{y} = (y_1, y_2, \dots, y_n)$) of vectors to pair:

(a) ($\vec{x'} = (a, x_1, x_2, \dots, x_n, 1, 1, \dots, 1)$, $\vec{y'} = (y_1 + a - 1, y_2, y_3, \dots, y_n, 1, 1, \dots, 1)$) (the number of new 1's is $a - 2$ for $\vec{x'}$ and is $a - 1$ for $\vec{y'}$; number a can be different for different changes, e.g. $((1, 1), (1, 1)) \overset{a=3}{\to} ((3, 1, 1, 1), (3, 1, 1, 1)) \overset{a=4}{\to} ((4, 3, 1, 1, 1, 1, 1), (6, 1, 1, 1, 1, 1, 1)))$.

(b) ($\vec{x'} = (x_1 + 1, x_2, x_3, \dots, x_n, 1)$, $\vec{y'} = (y_1 + 1, y_2, y_3, \dots, y_n, 1)$).

3 The Lando Problem

A pair (M, N) of two unions of the same number of disjoint circles in a sphere is *realizable* if there exist two curved spheres in 3-space intersecting transversely by a finite union of disjoint circles which union is situated as M in one sphere and as N in the other sphere.

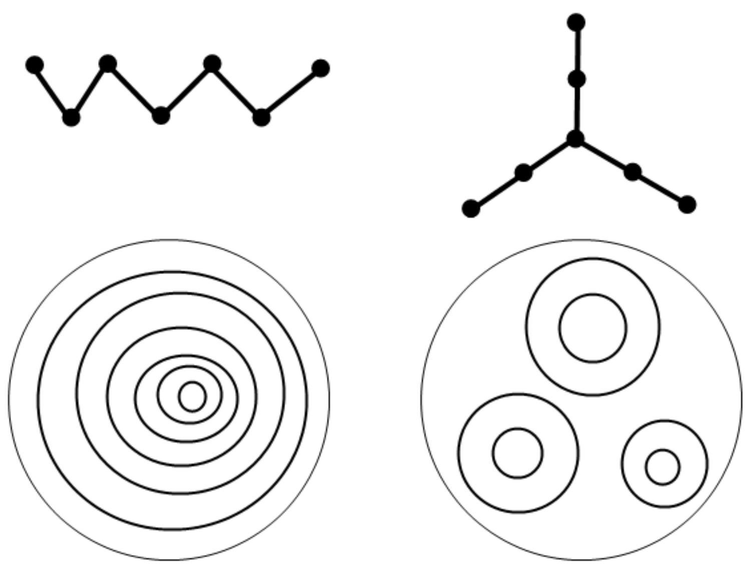

Figure 7: Is this pair realizable?

3.1. (a) Each pair of two unions of the same number $n \leq 4$ of disjoint circles is realizable.

(b) Is pair in figure 7 realizable? One graph is the path of 6 edges, the other is triod with 'rays' consisting of 2 edges.

(c) Is pair in figure 8 realizable? One graph is the star with 4 'rays', three 'rays' having two edges and one 'ray' having 1 edge. The other graph is letter 'H' with 'horizontal line' consisting of 3 edges.

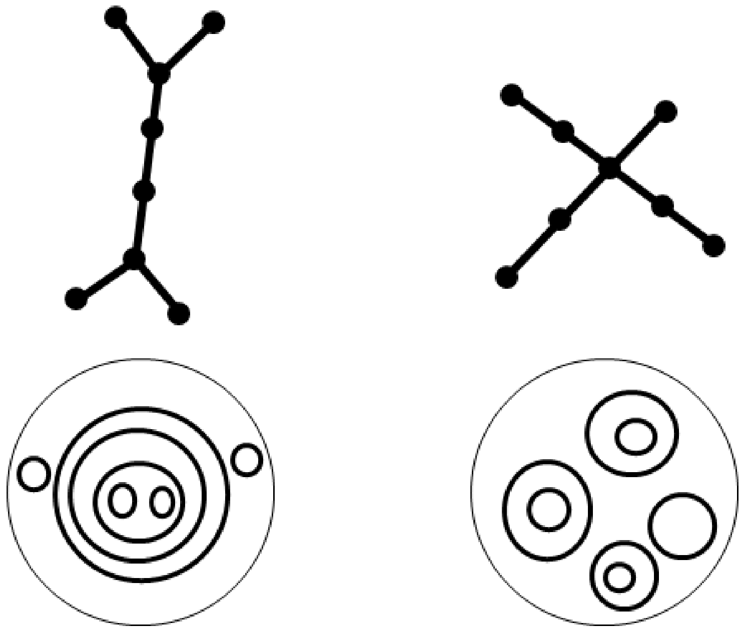

Figure 8: Is this pair realizable?

(d) There is a non-realizable pair of two unions of the same number of disjoint circles.

3.2. Suppose that n and k are given integers.

(a) Which collections of circles are realizable together with the collection of n circles bounding n disjoint disks? (Or, equivalently, which graphs are realizable together with the star of n rays?)

(b) Which graphs are realizable together with the graph that is a union, along a common edge, of the star of n rays and the star of k rays?

(c) * Which collections of circles are realizable together with the collection of n 'parallel' circles? (Or, equivalently, which graphs are realizable together with the path of length n?)

3.3. Suppose that S and T are curved spheres intersecting transversely by a finite union $S \cap T$ of disjoint circles. Then connected components of $S - T$ can be colored in black and white so that any two same colored components are not neighbors.

In the rest of this section M, N are unions of the same number of disjoint circles in curved spheres S, T. (Neither M nor N need to coincide with $S \cap T$.)

For a connected component P of $S - M$ denote by ∂P the union of boundary circles of P. Clearly, connected components P and Q of $S - M$ are neighbors if and only if $\partial P \cap \partial Q \neq \emptyset$.

3.4. Unlinked families of circles. Suppose that S and T are curved spheres intersecting transversely by a finite union $S \cap T$ of disjoint circles. Let P and Q be two connected components of $S - T$ which are situated inside T.

(a) If Q is a curved disk (i.e., if Q has one boundary circle), then ∂P is in one component of $T - \partial Q$.

(b) If Q is a curved cylinder (i.e., if Q has two boundary circles), then ∂P is contained either in the annulus component of $T - \partial Q$ (i.e., in the component with two boundary circle), or in the union of the two disk components of $T - \partial Q$ (i.e., of the components with one boundary circle).

(c) Colour connected components of $T - \partial Q$ in black and white so that adjacent components have different colours. Then ∂P is contained in the union of same coloured components of $T - \partial Q$.

The sign $\sqcup$ means a union of disjoint sets.

3.5. Suppose that S and T are curved spheres such that $S \cap T$ is situated in S as it is shown in figure 9. Denote by A_i the 'exterior' circles, by B the 'big splitting' circle and by C the union of the

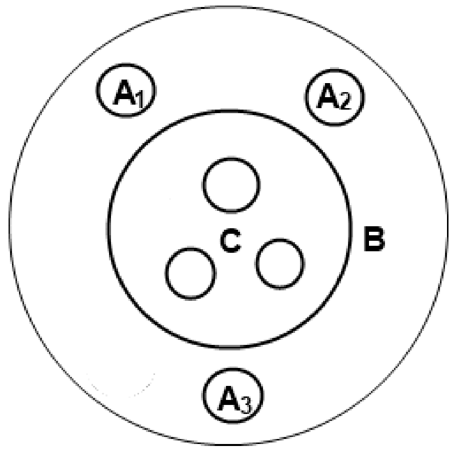

Figure 9: $S \cap T$ in S

'interior' circles, see figure 9.

(a) For each i the union $B \cup C$ is on the same side of A_i in T.

(b) The union $B \cup C$ is in the same connected component of $T - \sqcup_i A_i$.

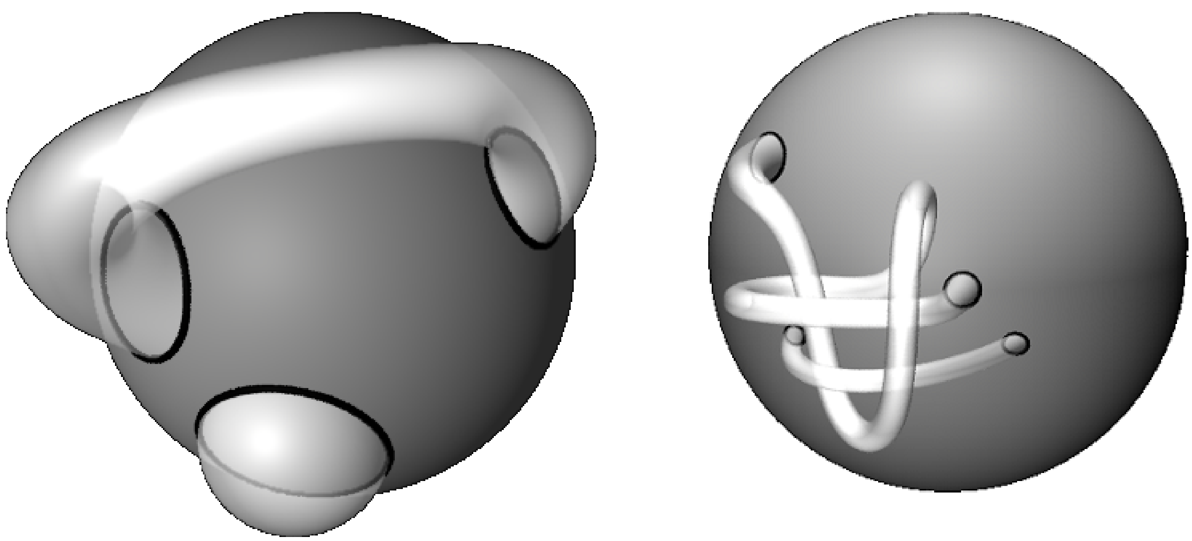

Figure 10: Disjoint curved disk and curved cylinder outside a ball (left), disjoint curved cylinders, one of them knotted, outside a ball (right)

3.6. Embedding Extension Problem. (a) Each two disjoint circles in the unit sphere bound disjoint disks inside this sphere.

(b) For which three disjoint circles p, q_1, q_2 in the unit sphere there exist disjoint *curved disks* P and *curved cylinder* Q inside this sphere such that $\partial P = p$ and $\partial Q = q_1 \sqcup q_2$? (Figure 10 left.)

(c) For which four disjoint circles p_1, p_2, q_1, q_2 in the unit sphere there exist disjoint curved cylinders P and Q inside this sphere such that $\partial P = p_1 \sqcup p_2$ and $\partial Q = q_1 \sqcup q_2$? (Figure 10 right.)

(d) For which two disjoint families p, q of disjoint circles in the unit sphere there exist disjoint curved *spheres with holes* P and Q inside this sphere such that $\partial P = p$ and $\partial Q = q$?

(e) Does there exist three disjoint families p, q, r of disjoint circles in the unit sphere such that

• each of the three pairs (p, q), (q, r) and (p, r) is extendable (to disjoint curved spheres with holes) in the sense of (d);

• there are no disjoint curved spheres with holes P, Q and R inside this sphere such that $\partial P = p$, $\partial Q = q$ and $\partial R = r$? [4]

(f) For which m disjoint families $p_1, \dots, p_m$ of disjoint circles in the unit sphere there exist disjoint curved spheres with holes $P_1, \dots, P_m$ inside this sphere such that $\partial P_i = p_i$ for each $i = 1, \dots, m$?

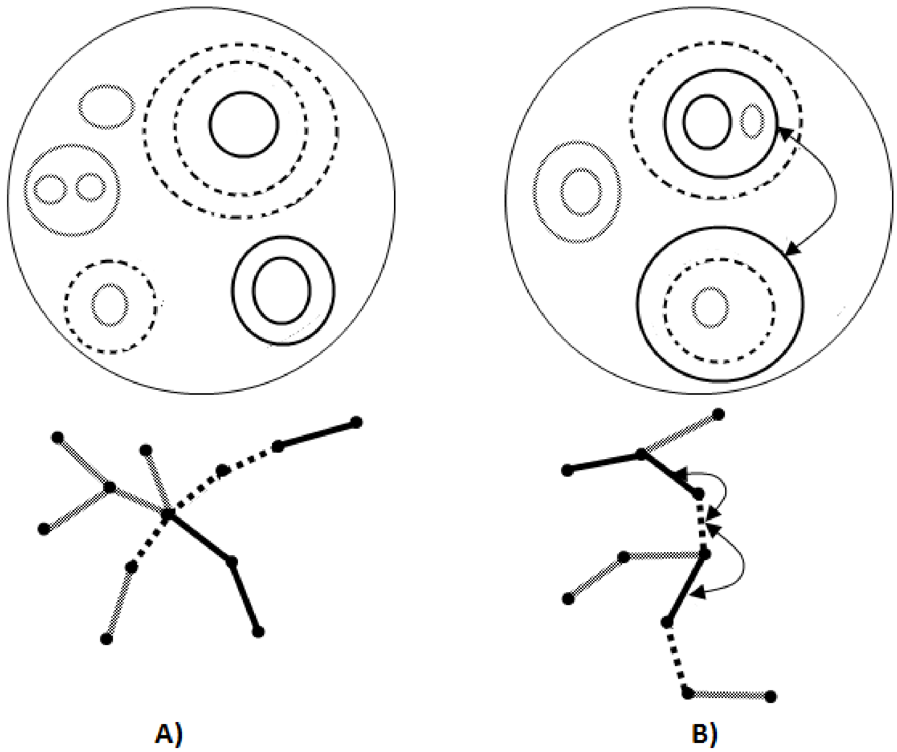

Figure 11: (A): the dotted and the bold unions of circles are unlinked.
(B): the dotted and the bold unions of circles are not unlinked because the arrowed path between two bold circles intersects the dotted circles in an odd number (one) of points.

Suppose that S and T are curved spheres such that each component of $S - T$ except one have one neighbor. (The 'exceptional' component may have one or more neighbors.) This 'exceptional' component is called *a curved sphere with holes*. A *curved disk* is a curved sphere with 1 hole (=with 1 neighbor). A *curved cylinder* is a curved sphere with 2 holes (=with 2 neighbors).

Let M and N be two unions of disjoint circles in the unit sphere S. Colour connected components of $S - q$ in black and white so that adjacent components have different colours. Union M is *on the same side the same side* (in this sphere) of N if M is contained in the union of same coloured components of $S - N$. Unions M and N are *unlinked* (in this sphere) if M is on the same side of N and N is on the same side of M. See figure 11.

3.7. (a) There are two unions M and N of disjoint circles in a sphere such that M is on the same side of N but N is not on the same side of M.

(b) Is unlinkedness transitive? That is, if M and N, N and P are unlinked, are then M and P necessarily unlinked?

(c) For a union M of disjoint circles in the unit sphere S denote by $\mathring{M}$ the unions of black connected components of $S - M$. (There are two choices of $\mathring{M}$ for given M; one of them is the complement to the other.) Two unions M and N are unlinked if and only if for each black and white colourings for M and for N such that $\mathring{M} \cup \mathring{N} \neq S$ we have either $\mathring{M} \subset \mathring{N}$ or $\mathring{N} \subset \mathring{M}$ or $\mathring{M} \cap \mathring{N} = \emptyset$.

[4] This should be compared with the well-known Borromean rings example.

4 Intermediate finish. Some solutions and new problems

1.1 and 1.2. Analogously to solution of Problem 2.4.

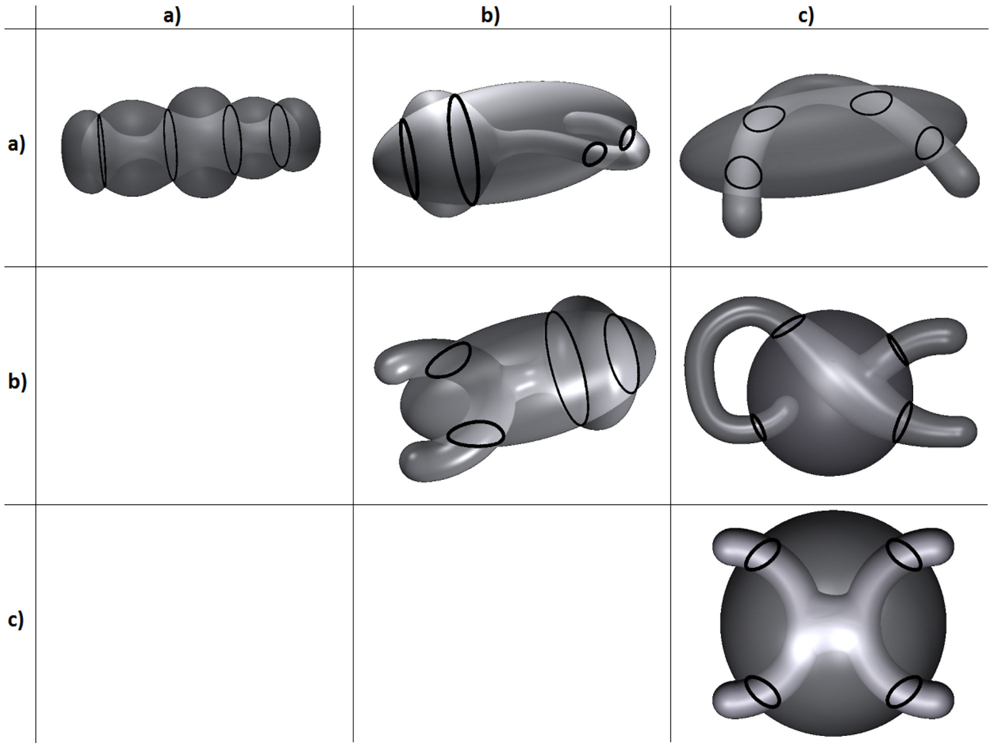

Figure 12: To the solution of Problem 1.3.

1.3. The case $i = j$ follows analogously to Problem 2.4. The cases ab, ac and bc are shown in figure 12.

1.4. (a) The answer is given by the answer to Problem 4.5.

(b) Such an algorithm is given by the answer to Problem 4.5. (Clearly, it is not polynomial.)

1.5. Theorem 1. *Let n be a positive integer and $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{y} = (y_1, y_2, \ldots, y_n)$ be sequences of positive integers. There exist curved spheres S, T in 3-space whose intersection consists of $n-1$ circles and splits*

• *S into n connected components which can be numbered so that the i-th connected component has x_i neighbors in S, and*

• *T into n connected components which can be numbered so that the i-th connected component has y_i neighbors in T*

if and only if $x_1 + \cdots + x_n = y_1 + \cdots + y_n = 2n - 2$.

This follows from Theorem 1' (Problem 4.3) below.

2.1. *Answer.* Pairs $(\vec{x}, \vec{x})$ are realizable for $\vec{x}$ up to reordering equal to

$$(1,1), (2,1,1), (3,1,1,1), (2,2,1,1), (4,1,1,1,1), (3,2,1,1,1), (2,2,2,1,1).$$

Other realizable pairs are up to reordering pairs of two sequences of the same number of elements from this list.

2.2. (a) Suppose that curved spheres S,T realize pair $(\vec{x},\vec{y})$. Recall definition of a graph $G = G(S, S\cap T)$ from §1. The number of the vertices is n. Out of k-th vertex there issues x_k edges. Hence the number of the edges is $(x_1+\dots+x_n)/2$. It is obvious that G is connected. By the Jordan Curve Theorem [5] G is split by any edge. So G is a tree. Hence the number of edges is $n-1 = (x_1+\dots+x_n)/2$. Analogously $n-1 = (y_1+\dots+y_n)/2$. QED

Sketch of an alternative solution of (a) by T. Nowik. By induction on the number of circles. The statement is true for one circle (there are only 2 disks on each sphere hence $n = 2$). Each additional circle splits one connected component into two, and adds two boundary circles.

(b) Clealy G is connected. By the Jordan Curve Theorem G is split by any edge. So G is a tree.

2.3. If the number of units is s, then $2n-2 = x_1+\dots+x_n \ge x_1+2(n-1-s)+s = 2n-2+x_1-s$. So $s \ge x_1$.

2.4. Let S be the unite cube. Take a family M of circles on S 'realizing' $\vec{x}$. (The existence of such a family is proved by induction; the inductive step is proved using deletion of a hanging vertex.) Color the complements in S to these circles into black and white so that neighboring components have different colors. Take a sphere T close to S and such that $S\cap T = M$, each black component of T is inside S, and each white component of T is outside S. Then S,T realize $(\vec{x},\vec{x})$.

2.5. By Problem 2.3 $x_1 \le s$. Then

$$x_{n-y_1+1} = x_{n-y_1+2} = \dots = x_{n-y_1+1} = \dots = x_n = y_{n-y_1+1} = y_{n-y_1+2} = \dots = y_n = 1.$$

$$\text{Hence}\quad \Big(\sum_{i=1}^{n-y_1+1} x_i\Big) - y_1 + 1 = \Big(\sum_{i=1}^{n} x_i\Big) - y_1 + 1 - (y_1-1) = 2(n-y_1+1) - 2$$

$$\text{and}\quad \Big(\sum_{i=2}^{n-y_1+2} y_i\Big) = \Big(\sum_{i=1}^{n} y_i\Big) - y_1 - (y_1-2) = 2(n-y_1+1) - 2.$$

So the new sequences are tree-like.

2.6. *Answer:* each pair.

(a) Induction on n. The inductive base $n = 2$ is clear. Suppose that the statement is true for each $n < k$, let us prove it for $n = k$.

If sequences $\vec{u} = (u_1,\dots,u_k)$ and $\vec{v} = (v_1,\dots,v_k)$ are the same up to reordering, then pair $(\vec{u},\vec{v})$ is realizable by Problem 2.4. If sequences $\vec{u}$ and $\vec{v}$ are not the same up to reordering, then reorder the sequences so that $u_1 \ne v_1$ and the units are at the end of the sequences. Without loss of generality $u_1 < v_1$. Denote

$$a := u_1,\quad n := k-u_1+1,\quad x_i = u_{i+1} \text{ for } i = 1,\dots,n,\quad y_1 := v_1-u_1+1,\quad y_i = v_i \text{ for } i = 2,\dots,n.$$

By Problem 2.3 $u_i = 1$ for each $i \ge n+2$ and $v_i = 1$ for each $i \ge n+1$. Thus pair $(\vec{u},\vec{v})$ is obtained from pair $(\vec{x},\vec{y})$ by given transformation. We have

$$x_1+\dots+x_n = u_2+\dots+u_{n+1} = 2k-2-u_1-(u_1-2) = 2n-2 \quad\text{and}$$

$$y_1+\dots+y_n = v_1+\dots+v_n-(u_1-1) = 2k-2-(u_1-1)-(u_1-1) = 2n-2.$$

So the sequences $\vec{x}$ and $\vec{y}$ are tree-like. Since $u_1 > 1$, we have $n < k$. Inductive step is proved.

(b) By induction on n.

[5] **Jordan Curve Theorem.** *A circle on a sphere splits the sphere into exactly two parts. Two points of the sphere not lying on the circle both lie in the same part if and only if they can be connected them by (spherical) broken line not intersecting the circle.*

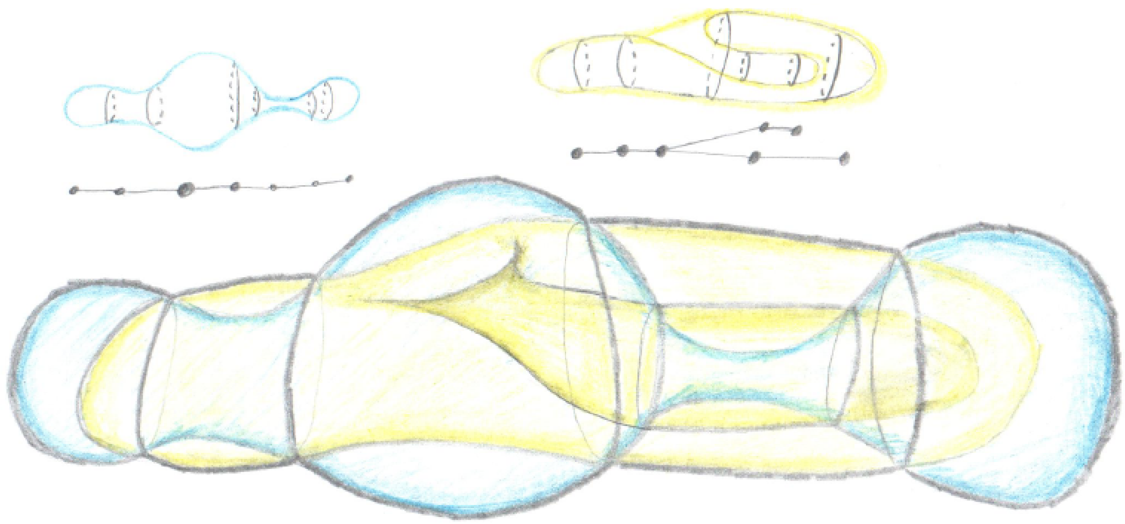

Figure 13: Two spheres realizing the pair in figure 13.

3.1. (a) Follows from Problems 1.1 and 1.3.

(b) Yes. See figure 13. An alternative construction is as follows. Let S and T' be the curved spheres from figure 2. Let T'' be a sphere inside T' 'close and parallel' to T'. Take the connected component X of $\mathbb{R}^3 - S - T'$ lying inside T' whose boundary contains a disk connected component of $T' - S$. Let T be a curved sphere obtained by joining T' and T'' by a tube in X. Then S and T are as required.

(c) No. This fact is obtained using a computer program based on solution of Problem 4.5.

(d) An example is shown in figure 14. Hint: use Problem 3.5 (or Problem 3.4 in the form 4.7 below).

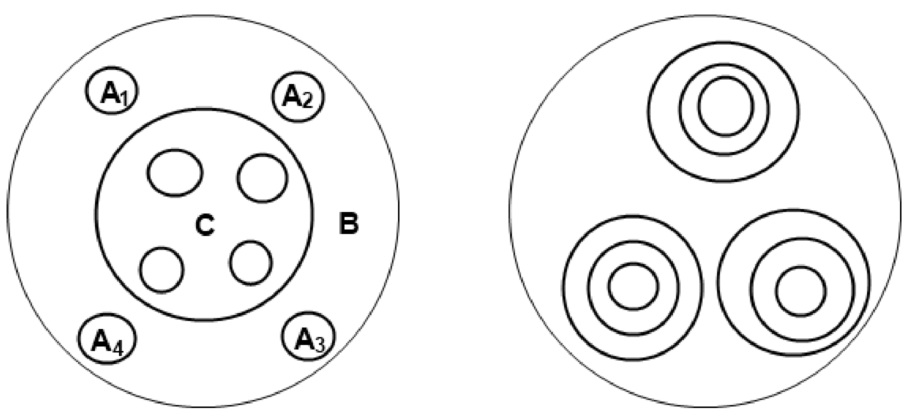

Figure 14: Nine circles (thick) situated in a sphere (thin) in two different ways (left, right).

3.2. (a) *Answer:* each collection is. *Hint.* Take n disjoint spheres intersecting given sphere S at the n circles of given collection M. Connect them by $n-1$ disjoint tubes ('along a tree') inside S to obtain sphere T. Check that T is as required.

(b) *Conjecture.* The pair of such a graph and a tree is realizable if and only if this tree is the union of two trees with n and k edges intersecting by exactly one edge.

(c) *Conjecture.* Any collection of n circles is realizable together with the collection of n 'parallel' circles.

3.3. Colour in black and white the connected components inside and outside T, respectively.

3.4. (a) and (b) are intuitively obvious and follow by (c).

(c) We may assume that T is a round sphere and that circles of ∂Q are round circles, none of them being an equator. For each circle of ∂Q take the round sphere passing through this circle and the center of T. The union of Q and the parts of such spheres lying outside T is a curved sphere, say, Q'. Sphere Q' splits $\mathbb{R}^3$ into two connected components. Since Q is connected, intersection of both connected components with the interior of T is connected. These connected components intersect T by black and white parts, respectively. Since P lies in one of the components, ∂P either lies in black part of $T - \partial Q$ or lies in white part of $T - \partial Q$.

3.5. (a) is analogous to (b).

(b) Consider disks $\overline{A_1}, \overline{A_2}, \overline{A_3} \subset S$ bounded by $A_1, A_2, A_3 \subset S$ and not containing other circles. Without loss of generality we may assume that the interiors of these disks lie inside T. Then the interior of the component of $S - M$ bounded by $B \cup C$ lies inside T as well (because the intersection $S \cap T$ is transversal). This interior lies in one of the connected components of $\mathbb{R}^3 - (T \cup \overline{A_1} \cup \overline{A_2} \cup \overline{A_3})$. So all the 4 circles of $B \cup C$ lie in the same connected component of $T - (A_1 \cup A_2 \cup A_3)$.

Note that this is a particular case of Problem 3.4.c (cf. Problem 4.6.a).

3.7. (a) See definitions of A_0, A_+ and A_- in figure 15 and the Example below in §4. Set $\{A_0\}$ is on the same side of set $\{A_+, A_-\}$ but not vice versa.

(b) No.

Some new problems on the Neighbor Sequence Problem.

A pair $\vec{x} = (x_1, x_2, \dots, x_n)$, $\vec{y} = (y_1, y_2, \dots, y_n)$ of sequences of positive integers is called *strongly realizable* if there exist two curved spheres S, T

(1) whose intersection consists of $n-1$ circles and splits

• S into n connected components which can be numbered so that the i-th connected component has x_i neighbors in S, and

• T into n connected components which can be numbered so that the i-th connected component has y_i neighbors in T;

(2) there is a circle of $S \cap T$ that bounds a disk and a component with x_1 neighbors in $S - T$, as well as bounds a disk and a component with y_1 neighbors in $T - S$.

Pair (S, T) of spheres is called a *strong realization* of pair $(\vec{x}, \vec{y})$.

4.1. Let $\vec{x}, \vec{y}$ be tree-like sequences in which all the units are situated at the end. If pair of sequences $\vec{x}' := (x_1 - y_1 + 1, x_2, x_3, \dots, x_{n-y_1+1})$, $\vec{y}' := (y_2, y_3, \dots, y_{n-y_1+2})$ is strongly realizable, then pair $(\vec{x}, \vec{y})$ is strongly realizable.

4.2. If pair $(\vec{x}, \vec{y})$ is strongly realizable, then for each positive integer a pairs $(\vec{x'}, \vec{y'})$ are strongly realizable for:

(a) $\vec{x'} = (a, x_1, x_2, \dots, x_n, 1, 1, \dots, 1)$, $\quad \vec{y'} = (y_1 + a - 1, y_2, y_3, \dots, y_n, 1, 1, \dots, 1)$.

(The number of new 1's is $a-1$ for $\vec{y'}$ and is $a-2$ for $\vec{x'}$; number a can be different for different changes.)

(b) some choice of $\vec{x'} \in \{(1, x_1 + 1, x_2, x_3, \dots, x_n), (x_1 + 1, x_2, x_3, \dots, x_n, 1)\}$ and $\vec{y'} \in \{(1, y_1 + 1, y_2, y_3, \dots, y_n), (y_1 + 1, y_2, y_3, \dots, y_n, 1)\}$.

4.3. Theorem 1'. *A pair of sequences is strongly realizable if and only if both sequences are tree-like.*

Hint: Use Problems 2.5 and 4.1 (or, alternatively, Problems 2.6 and 4.2).

Some new problems on the Lando Problem.

4.4. Each pair of unions of

(5) 5; (6) 6;

disjoint circles is realizable.

The Lando problem is solved via solution of its *numbered*, or *colored*, analogue. Let us introduce definitions necessary to formulate the analogue.

Assume that M and N are two sets of disjoint circles in spheres S and T, and that in each set the circles are numbered by $1, 2, \ldots, n$. Pair (M, N) is *realizable* if there exist two curved spheres S' and T' intersecting transversely by a finite union $S' \cap T'$ of disjoint circles, and a numbering of these circles such that

- $S' \cap T'$ in S' and M in S are numbered equivalent;
- $S' \cap T'$ in T' and N in T are numbered equivalent.

(Numbered sets M in S and N in T are *numbered equivalent* if there is a 1-1 correspondence between connected components of $S - M$ and of $T - N$ such that two connected components of $S - M$ are adjacent along a circle of M if and only if the two corresponding connected components of $T - N$ are adjacent along the corresponding circle of N.)

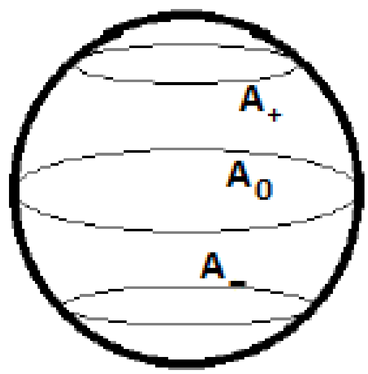

Figure 15: Numbered sets (A_-, A_0, A_+) and (A_0, A_-, A_+) are not numbered equivalent.

Example. On the unit sphere (or on the Earth sphere) let $A_0 = A_2$ be the equator, $A_+ = A_1$ the parallel of sixty degrees northern latitude, $A_- = A_3$ the parallel of sixty degrees southern latitude. See figure 15. Then

- unnumbered (or, equivalently, unordered) sets $\{A_+, A_0, A_-\}$ and $\{A_0, A_+, A_-\}$ are the same (or unnumbered equivalent).
- numbered (or, equivalently, ordered) sets (A_-, A_0, A_+) and (A_+, A_0, A_-) are numbered equivalent.
- numbered sets (A_+, A_0, A_-) and (A_0, A_+, A_-) are not numbered equivalent.
- pair $(A_+, A_0, A_-), (A_0, A_+, A_-)$ of numbered sets in the unit sphere S and in its copy T is non-realizable.

Proof of the last assertion. Suppose to the contrary that there are spheres S' and T' realizing given pair. Denote by

- B_k the copy A_k on the copy T of S;
- A'_k the circle on S' corresponding to A_k;
- B'_k the circle on T' corresponding to B_k;
- $D' \subset S'$ the disk in $S' - T'$ bounded by A'_+;
- $C' \subset S'$ the cylinder in $S' - T'$ bounded by A'_0 and A'_-.

See figure 16. Since S' and T' realize pair $(A_+, A_0, A_-), (B_0, B_+, B_-)$, we have $A'_+ = B'_0$, $A'_0 = B'_+$ and $A'_- = B'_-$.

Clearly, C' and D' lie in 3-space on the same side from sphere T'. (Cf. Problem 3.3.) We have $\partial D = A'_+ = B'_0$. The boundary $\partial C' = A'_0 \sqcup A'_- = B'_+ \sqcup B'_-$ does not lie in one component of $T' - \partial D' = T' - B'_0$. This contradicts to the assertion of Problem 3.4.a for $P = C'$ and $Q = D'$. QED

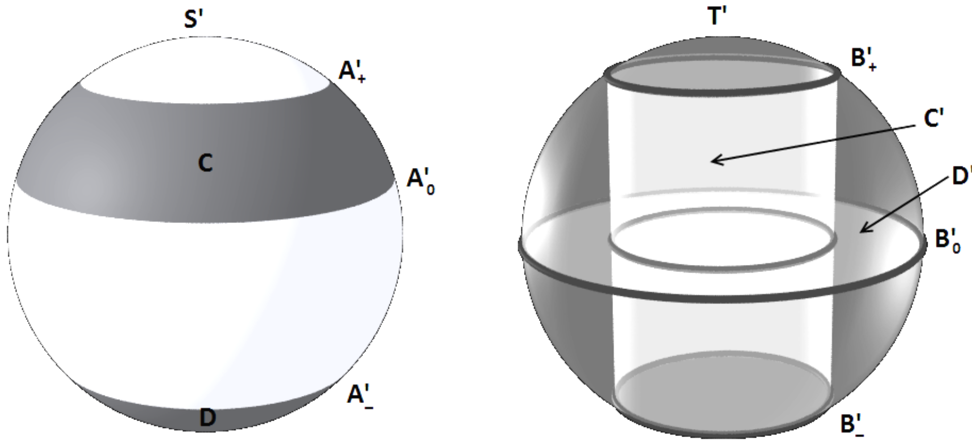

Figure 16: Curved spheres S' and T' drawn apart

4.5. The Numbered Lando Problem. Which pairs of disjoint unions of numbered circles are realizable?

4.6. (a) If pair (M, N) of disjoint unions of numbered circles in spheres S and T is realizable, then connected components of $S - M$ can be colored in black and white so that for each two same coloured components P and Q of $S - M$ unions in T corresponding to ∂P and ∂Q are unlinked in T.

(b) Does the converse to (a) hold?

Let p and q be two sets of edges of a tree G. Colour connected components of the complement in G to the interiors of edges of q. Set p is *on the same side* (in this tree G) of q if p is contained in the union of same-coloured connected components of $G - q$ (or, equivalently, if $p \cap q = \emptyset$ and for each two vertices of edges of p there is a path in the tree connecting these two points, and containing an even number of edges of q). Sets p and q are *unlinked* (in this tree) if p is on the same side of q and q is on the same side of p.

For a vertex P of a graph denote by δP the union of edges issuing out of P.

Graphs $G(S, M)$ and $G(T, N)$ are defined in §1. Numberings of circles in M and of circles in N give numberings of edges in $G(S, M)$ and of edges in $G(T, N)$.

4.7. (a) If pair (M, N) of disjoint unions of numbered circles in spheres S and T is realizable, then the vertices of $G(S, M)$ can be colored in black and white so that for each two same coloured vertices P, Q of $G(S, M)$ the unions in $G(T, N)$ corresponding to δP and δQ are unlinked in $G(T, N)$.

(b)* Given two trees G and G' having the same number of edges, is there a polynomial algorithm for checking the existence of numberings of their edges such that the vertices of G can be colored in black and white so that for each two same coloured vertices P, Q of G the unions in G' corresponding to δP and δQ are unlinked in G'?

5 More spheres and spheres with handles

Let n_1, n_2, n_3 be positive integers. A triple

$$\vec{x_1} = (x_{11}, x_{12}, \ldots, x_{1n_1}), \quad \vec{x_2} = (x_{21}, x_{22}, \ldots, x_{2n_2}), \quad \vec{x_3} = (x_{31}, x_{32}, \ldots, x_{3n_3})$$

of sequences of positive integers is called *realizable* if there exist three curved spheres S_1, S_2, S_3 in 3-space pairwise intersecting by circles and such that $S_1 \cap S_2 \cap S_3 = \varnothing$ and for each $k = 1, 2, 3$ the complement $S_k - S_{k+1} - S_{k+2}$ has n_k connected components which can be numbered so that the i-th connected component has x_{ki} neighbors in S_k for each $i = 1, \ldots, n_k$.

In this section (and in corresponding solutions) subscripts $k, k+1, k+2$ are considered mod 3. Triple (S_1, S_2, S_3) of spheres is called a *realization* of triple $(\vec{x_1}, \vec{x_2}, \vec{x_3})$.

5.1. Triple Neighbor Sequence Problem. Which triples of tree-like sequences are realizable?

5.2. Which triples of tree-like sequences, each having at most 4 numbers, are realizable?

5.3. If a triple of sequences of lengths n_1, n_2, n_3 is realizable, then
(a) $n_1 + n_2 + n_3$ is odd;
(b) $n_k < n_{k+1} + n_{k+2}$ for each $k = 1, 2, 3$.

5.4. Let $x_1 \geq x_2 \geq \dots \geq x_n$ be a tree-like sequence. Let p, q be positive integers such that $p \geq q > 1$ and $p + q = n + 1$. Then there exist two tree-like sequences $a_1, a_2, \dots, a_p$ and $b_1, b_2, \dots, b_q$ such that $a_1 + b_1 = x_1$ and ordered sets $(a_2, a_3, \dots, a_p, b_2, b_3, \dots, b_q)$ and $(x_2, x_3, \dots, x_n)$ are the same up to reordering.

What are analogs of characterizations of neighbor sequences (of Theorems 1 and 2) for intersections of *more than three* curved spheres?

5.5. * **Conjecture.** Let $n_1, n_2, n_3, \dots, n_s$ be positive integers and

$$x_{11}, x_{12}, \dots, x_{1n_1}, \quad x_{21}, x_{22}, \dots, x_{2n_2}, \quad \dots, \quad x_{s1}, x_{s2}, \dots, x_{sn_s}$$

sequences of positive integers. There exist s curved spheres $S_1, S_2, \dots, S_s$ pairwise intersecting by circles and such that

• no three of them intersect;

• for each $k = 1, \dots, s$ and $j = 1, \dots, n_k$ the complement $S_k - S_{k+1} - S_{k+2} - \dots - S_{k+s-1}$ has n_k connected components, of which the j-th has x_{kj} neighbors in S_k;

if and only if each of s sequences is tree-like, and $n_1 + n_2 + \dots + n_s - s$ is an even number greater or equal to $2n_k$ for each $k = 1, \dots, s$.

For $s < 4$ this conjecture is proved (see Theorems 1 and 2), the first unknown case is $s = 4$.

What can be neighbor sequences if there can be 'triple points', i.e. intersection points of of three spheres?

5.6. * **Conjecture.** *Let n_1, n_2, n_3 be positive integers and*

$$x_{11}, x_{12}, \dots, x_{1n_1}, \quad x_{21}, x_{22}, \dots, x_{2n_2}, \quad x_{31}, x_{32}, \dots, x_{3n_3}$$

be sequences of positive integers. Then there exist curved spheres S_1, S_2, S_3 in 3-space

• *pairwise intersecting by circles,*

• *having $2T$ triple intersection points and*

• *such that for each $k = 1, 2, 3$ the complement $S_k - S_{k+1} - S_{k+2}$ has n_k connected components and the i-th connected component has x_{ki} neighbors in S_k for each $i = 1, \dots, k$*

if and only if $n_1 + n_2 + n_3 + T$ is odd, $x_{k1} + x_{k2} + \dots + x_{kn_k} = 2n_k - 2 + 2T$ and $n_k + T < n_{k+1} + n_{k+2}$ for each k.

What are analogs of Theorems 1 and 2 for intersections of *curved spheres with handles*?

5.7. * **Conjecture.** *Let g_1, g_2, n be positive integers and*

$$x_{11}, x_{12}, \dots, x_{1n}, \quad x_{21}, x_{22}, \dots, x_{2n}$$

two sequences of positive integers. There exist curved sphere with g_1 handles S_1 and curved sphere with g_2 handles S_2 such that they intersect by circles splitting S_k into n connected components, of which the j-th has x_{kj} neighbors in S_k for each $k = 1, 2$ and $j = 1, \dots n$

if and only if $s := x_{11} + x_{12} + \dots + x_{1n} = x_{21} + x_{22} + \dots + x_{2n}$ is even and $2n - 2 \leq s \leq 2n - 2 + 2g_k$ for each $k = 1, 2$.

It would be interesting to solve analogous problems in case when self-intersections are allowed. Both cases are interesting — either with triple self-intersection points or without them.

HOW DO CURVED SPHERES INTERSECT IN 3-SPACE,

OR TWO-DIMENSIONAL MEANDRA

S. Avvakumov, A. Berdnikov, A. Rukhovich and A. Skopenkov

6 Solutions after finish

In this section curved spheres are shortly called *spheres*.

Neighbor sequence problem

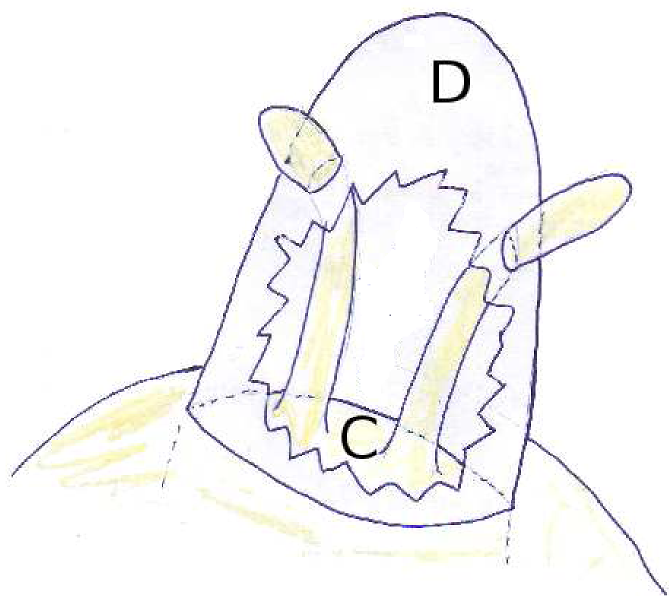

Figure 17: Inductive construction

4.1. We may assume that $x_1 \geq y_1$. Take spheres S', T' realizing pair $(\vec{x}', \vec{y}')$ of sequences. Take a circle of $S' \cap T'$ from condition (2). This circle that bounds

• in $S' - T'$ a connected component, say C, that has $x_1 - y_1 + 1$ neighbors,

• in $T' - S'$ a disk, say D.

We modify spheres S', T' by joining C and D by $y_1 - 1$ fingers, see Figure 17. Denote the new spheres by S and T. Let us prove that they realize pair $(\vec{x}, \vec{y})$ of sequences.

Condition (1) is satisfied for S, T because

• each component of $S' - T'$ except C is also a component of $S - T$,

• C is separated by $y_1 - 1$ circles of $(S \cap T) - (S' \cap T')$ into $y_1 - 1$ disks and a component with $(x_1 - y_1 + 1) + (y_1 - 1) = x_1$ neighbors.

and

• each component of $T' - S'$ except D is also a component of $T - S$,

• D is separated by $y_1 - 1$ circles into $y_1 - 1$ disks and a component with y_1 neighbors.

Any circle of $(S \cap T) - (S' \cap T')$ satisfies condition (2).

4.3. *Proof of Theorem 1'.* Proof by induction on the length n of the sequences. For each tree-like sequence of n numbers we have $n \geq 2$. The induction base is $n = 2$ and is clear.

Let us prove the induction step. Suppose Theorem 1' is proved for $2, 3, \ldots, n-1 \geq 2$. Let us prove it for n.

Reorder our sequences so that the 1's will be at the end. By Problems 2.5 and 4.1 and by the induction hypothesis the new sequences are realizable. Take spheres S, T realizing the new sequences. So S, T satisfy condition (1) from the definition of the strong realizability for the old sequences. Also,

• if $x_1 \neq 1$, then conditions (2) for the new and for the old sequences $\vec{x}$ are equivalent;

• if $x_1 = 1$, then the circle from condition (2) for the new sequence $\vec{x}$ bounds a disk, so it bounds a component with $x_1 = 1$ neighbor.

Same holds for $\vec{x}$ replaced by $\vec{y}$. So condition (2) is also satisfied for the old sequences. Thus S, T strongly realize the old sequences.

Lando problem

3.1. (d) *A direct solution.* Assume to the contrary that there exist two spheres S' and T' realizing pair (M, N) from figure 14. Denote the connected components of $S' - T'$ as shown in fig. 14 left.

Without loss of generality we may assume that the interiors of disks $A_1, \dots, A_4 \subset S'$ lie inside T'. Then the interior of component $C \subset S'$ lies inside T' as well (because the intersection of S' and T' is transversal). Since $C, A_1, \dots, A_4$ are disjoint, C lies in one of the connected components of $\mathbb{R}^3 - T' \cup \bigsqcup A_i$. So all the 5 circles of ∂C lie in the same connected component of $T' - \bigsqcup \partial A_i$. (Here we use a trivial particular case of the Embedding Extension Theorem.)

Let us restate the previous statement in terms of graph $G := G(T', N)$ (fig. 18). Denote by $G(C)$ the union of 5 edges of G corresponding to the circles of ∂C. Then $G(C)$ lies completely in one of the connected components of the compliment of G to the 4 edges corresponding to the circles of $\bigsqcup \partial A_i$. Since G has only 9 edges, this means that $G(C)$ is a subtree of G. Denote by $G(B)$ the union of 5 edges of G corresponding to the circles of ∂B. Likewise, $G(B)$ is a subtree of G.

Since $G(B) \cup G(C) = G$, at least two of the three edges a, b, c of G (fig. 18) belong to one of subtrees $G(B)$ or $G(C)$. Without loss of generality we may assume that $a, b \in G(B)$. But any subtree of G containing both a and b has at least 6 edges while $G(B)$ has only 5 edges. Contradiction.

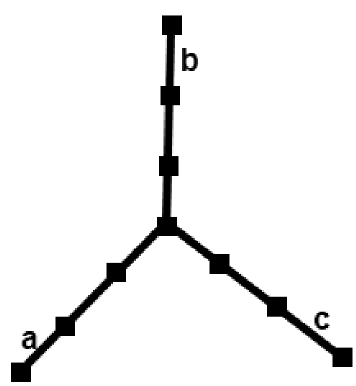

Figure 18: Graph $G := G(T', N)$.

3.6. (a) Clear.

(b) ... if and only if circles q_1 and q_2 are on the same side of p.

(c) ... if and only if $p_1 \sqcup p_2$ and $q_1 \sqcup q_2$ are unlinked.

(d) ... if and only if p and q are unlinked. *Hint:* generalize solution of Problem 3.6.d 3.2.a. *Formal solution* is obtain by taking $m = 2$ in the solution of (f).

(e) No, by the answer to (f).

(f) ... if and only if p_i and p_j are unlinked for each $i \neq j$.

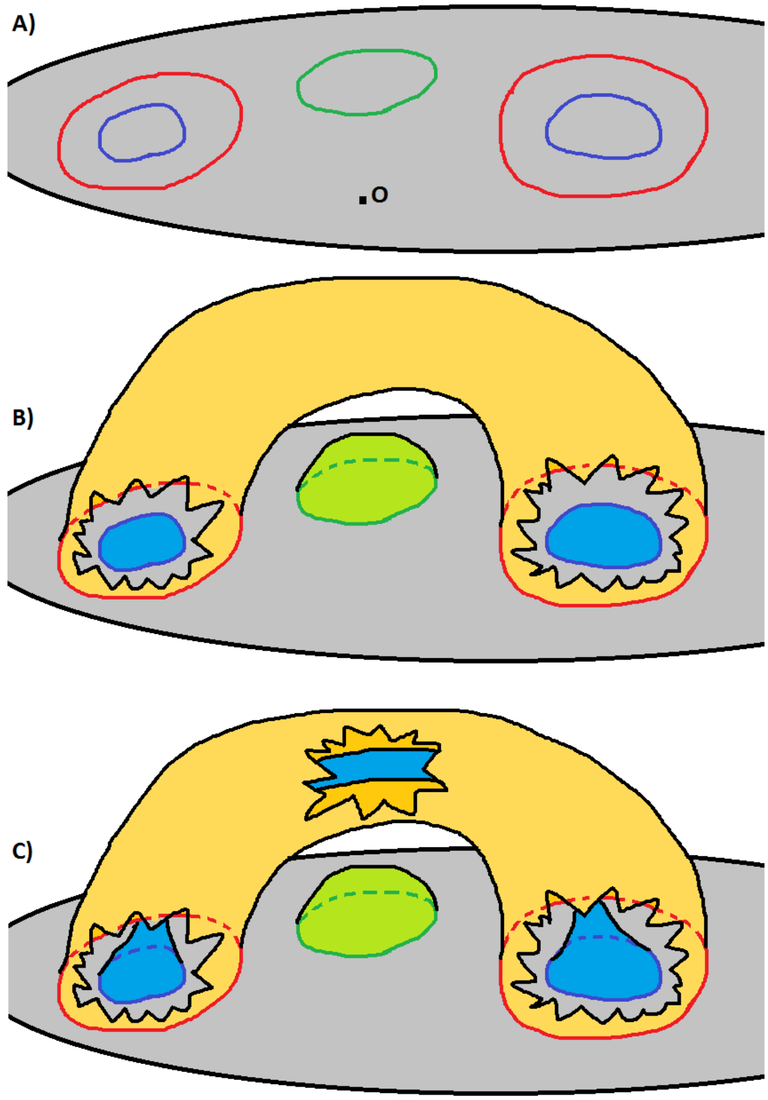

Figure 19: To the solution of Problem 3.6.f. (A) We have S (gray), p_1 (red), p_2 (green), p_3 (blue). (B) We have that $\mathring{p}_3$ (blue) is the 'smallest'. We construct P_1 (yellow) and P_2 (green) by induction. (C) Connected components of $\mathring{p}_3$ (blue) can be connected by a path disjoint with $P_1 \cup P_2$. So we connect them by a tube and obtain P_3 (blue).

Embedding Extension Theorem. *Unions $p_1, \dots, p_m$ of disjoint circles in the unit sphere S are pairwise unlinked if and only if there exist disjoint curved spheres with holes $P_1, \dots, P_m$ whose interiors are inside S and such that $\partial P_i = p_i$ for each $i = 1, \dots, m$.*

Proof. The necessity is essentially proved in Problem 3.4.c. The sufficiency is proved by induction on m. Base $m = 1$ is essentially proved in the solution of Problem 3.2.a. Let us prove the inductive step. Take a point $O \in S - \bigsqcup\limits_{i=1,\dots,m} p_i$. For each i take a black and white colouring of $S - p_i$ such that O is white. Recall that $\mathring{p}_i$ is the union of black components of $S - p_i$. Since O is white and p_i and p_j are unlinked, by Problem 3.7.b for each $i \neq j$ either $\mathring{p}_i \subset \mathring{p}_j$ or $\mathring{p}_j \subset \mathring{p}_i$ or $\mathring{p}_i \cap \mathring{p}_j = \emptyset$. So there is a 'smallest' $\mathring{p}_i$, i.e. $\mathring{p}_i$ such that $\mathring{p}_j \not\subset \mathring{p}_i$ for each $j \neq i$. We may assume that $i = m$. Then $\mathring{p}_m \cap \bigsqcup\limits_{i=1}^{m-1} p_i = \emptyset$, $\mathring{p}_m$ is a collection of curved spheres with holes, $\partial\mathring{p}_m = p_m$ and $\mathring{p}_m \subset S$. Denote by Δ the closed 3-ball bounded by S (i.e., 'the interior part' of S). By the inductive hypothesis there are disjoint spheres with holes $P_1, \dots, P_{m-1} \subset \Delta$ such that $\partial P_i = p_i$ for each $i = 1, \dots, m-1$.

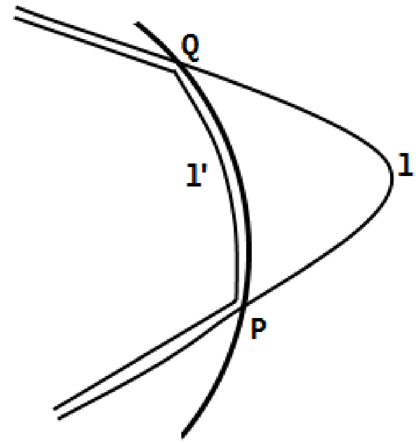

Figure 20: Proof of Claim

Claim. *Union p_m lies in one connected component of $\Delta - (P_1 \sqcup \dots \sqcup P_{m-1})$.* [6]

Proof of Claim. Take any two points $A, B \in p_m$. Denote by l a path inside S connecting A and B such that $\bar{l} := \#(l \cap \bigsqcup_{i=1}^{m-1} P_i)$ is minimal (minimal by l, objects $A, B, p_m, S, P_1, \dots, P_{m-1}$ are fixed). Assume to the contrary that l is not as required, i.e., $\bar{l} > 0$. Since p_m is on the same side of ∂P_i, points A and B are in the same connected component of $\Delta - P_i$, so $\#(l \cap P_i)$ is even for each i. (If $m = 2$, we may even obtain that $\#(l \cap P_1) = 0$ and stop here.) Then $\#(l \cap P_i) \geq 2$ for some i. Denote by Q and R two consecutive points of $l \cap P_i$. Denote by Q' the point of l slightly before Q and by R' the point of l slightly after R. Since P_i is connected, Q and R can be connected by a path in P_i. So Q' and R' can be connected by a path l' very close to P_i but not intersecting P_i. Path l' does not intersect any of $P_1, \dots, P_{m-1}$ because it is very close to P_i and $P_1, \dots, P_{m-1}$ are pairwise disjoint. Substitute the part of l between Q' and R' by l'. Denote the obtained path by l''. Then $\overline{l''} = \bar{l} - 2$. This contradicts to the minimality of $\bar{l}$. Thus l is as required. QED

Completion of the proof of Embedding Extension Theorem. Let $\mathring{p}'_m$ be a disjoint union of curved spheres with holes obtained from $\mathring{p}_m$ by a slight deformation so that the interior of $\mathring{p}'_m$ is inside the interior of Δ and $\partial \mathring{p}'_m = \partial \mathring{p}_m = p_m$. By Claim each two points of $\mathring{p}'_m$ can be connected by a path inside S disjoint with $P_1, \dots, P_{m-1}$. So we can connect all the connected components of $\mathring{p}'_m$ by tubes inside S disjoint with $P_1, \dots, P_{m-1}$. The number of the tubes is one less than the number of the connected components of $\mathring{p}'_m$, so that there are no 'cycles of tubes'. Then we obtain a sphere with holes. Denote it by P_m. We have $\partial P_m = p_m$, $P_m \subset \Delta$ and P_m is disjoint with $P_1, \dots, P_{m-1}$. The inductive step is proved. QED

4.4. This fact is obtained using a computer program based on Theorem 3 below.

4.5. The answer is given by Problem 4.6 and is as follows.

Theorem 3. *Pair (M, N) of disjoint unions of numbered circles in spheres S and T is realizable if and only if connected components of $S - M$ can be colored in black and white so that for each two same coloured components P and Q of $S - M$ unions in N corresponding to ∂P and ∂Q are unlinked in T.*

4.6. (a) This is a restatement of Problem 3.4.

[6]This assertion for $m = 2$ is essentially the definition of the comparability (or, rather, of 'p_2 is on the same side of ∂P_1'). This case $m \geq 3$ is interesting because in general the union of two subsets could split the ambient set even if each subset alone does not split the ambient set.

(b) Yes. The idea is to prove and use the answer to Embedding Extension Problem 3.6.e.

Let T' be the unit cube. Numberings give a 1–1 correspondence h between circles of M and circles of N.

Denote by $A_1, \dots, A_m$ the white connected components of $S - M$. By the assumption $h(\partial A_1), \dots, h(\partial A_m)$ are pairwise unlinked in T'. By the answer to Embedding Extension Problem 3.6.e there exist disjoint curved spheres with holes $A'_1, \dots, A'_m$ whose interiors are *inside* T' and such that $\partial A'_i = h(\partial A_i)$ for each $i = 1, \dots, m$.

Denote by $B_1, \dots, B_n$ the black connected components of $S - M$. Analogously there exist disjoint curved spheres with holes $B'_1, \dots, B'_n$ whose interiors are *outside* T' and such that $\partial B'_i = h(\partial B_i)$ for each $i = 1, \dots, n$.

Let $S' := (A'_1 \cup \dots \cup A'_m) \cup (B'_1 \cup \dots \cup B'_n)$. By construction S' does not have self-intersections. We have that A'_i has the same number of holes as A_i, and B'_i has the same number of holes as B_i. Since $S = (A_1 \cup \dots \cup A_m) \cup (B_1 \cup \dots \cup B_n)$ is a curved sphere, S' is a curved sphere. (A rigorous proof is obtained using Euler characteristic.) Clearly, S' and T' realize given pair M, N.

4.7. (a) This is a restatement of Problems 3.4 and 4.6.

More spheres and spheres with handles

5.1. Theorem 2. *Let n_1, n_2, n_3 be positive integers and*

$$x_{11}, x_{12}, \dots, x_{1n_1}, \quad x_{21}, x_{22}, \dots, x_{2n_2}, \quad x_{31}, x_{32}, \dots, x_{3n_3}$$

be sequences of positive integers. There exist curved spheres S_1, S_2, S_3 in 3-space pairwise intersecting by circles and such that

- *$S_1 \cap S_2 \cap S_3 = \varnothing$;*
- *$S_k - S_{k+1} - S_{k+2}$ has n_k connected components, which can be numbered so that the i-th component has x_{ki} neighbors in S_k, for each $k = 1, 2, 3$*

if and only if the sequences are tree-like, $n_1 + n_2 + n_3$ is odd and $n_k < n_{k+1} + n_{k+2}$ for each $k = 1, 2, 3$.

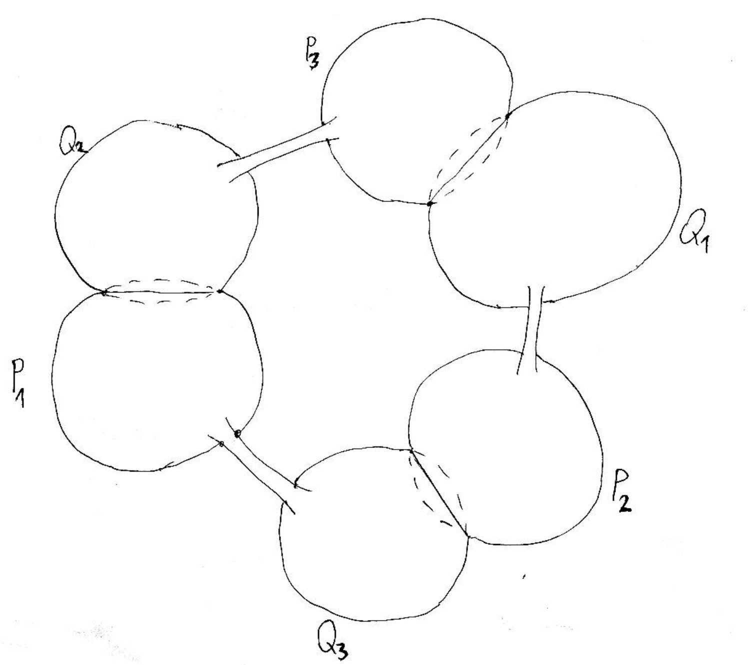

Figure 21: Construction of three spheres

Proof. The 'only if' part follows by Problems 2.2 and 5.3. Let us prove the 'if' part. Let

$$m_1 := (n_2 + n_3 - n_1 + 1)/2, \quad m_2 := (n_1 + n_3 - n_2 + 1)/2, \quad m_3 := (n_1 + n_2 - n_3 + 1)/2.$$

So

$$m_1 + m_2 = n_3 + 1, \quad m_1 + m_3 = n_2 + 1, \quad m_2 + m_3 = n_1 + 1.$$

By Problem 5.4 there exist sequences

$$p_{11}, p_{12}, \dots, p_{1m_3}, \quad p_{21}, p_{22}, \dots, p_{2m_1}, \quad p_{31}, p_{32}, \dots, p_{3m_2},$$

$$q_{11}, q_{12}, \dots, q_{1m_2}, \quad q_{21}, q_{23}, \dots, q_{2m_3}, \quad q_{31}, q_{32}, \dots, q_{3m_1},$$

such that $p_{k-1,1} + q_{k+1,1} = x_{k1}$ and ordered sets

$$(p_{k-1,2}, p_{k-1,3}, \dots, p_{k-1,m_{k+1}}, q_{k+1,2}, q_{k+1,3}, \dots, q_{k+1,m_{k-1}}) \quad \text{and} \quad (x_{k2}, x_{k3}, \dots, x_{kn_k})$$

are the same up to reordering. By Theorem 1' there exist spheres

$$Q_1, P_1, Q_2, P_2, Q_3, P_3 \subset \mathbb{R}^3 \quad \text{such that} \quad Q_k \cap Q_{k+1} = \varnothing, \quad Q_k \cap P_l = \varnothing \quad \text{if} \quad l \neq k-1 \quad \text{and}$$

- $Q_k - P_{k-1}$ is the disjoint union of m_{k+1} connected components, i-th one has q_{ki} neighbors
- $P_{k-1} - Q_k$ is the disjoint union of m_{k+1} connected components, i-th one has $p_{k-1,i}$ neighbors
- the boundary of some connected component of $\mathbb{R}^3 - P_{k-1} - Q_k$ contains a component $\widetilde{q}_k$ with q_{k1} neighbors on Q_k and a component $\widetilde{p}_{k-1}$ with $p_{k-1,1}$ neighbors on P_{k-1}.

For $k = 1, 2, 3$ let S_k be the connected sum of spheres Q_{k+1} and P_{k-1} along a small tube joining the two components $\widetilde{q}_{k+1}$ and $\widetilde{p}_{k-1}$ from the third condition, see Figure 21. This can be done without intersections of the three tubes.

Then $S_k - S_{k+1} - S_{k+2}$ is as required for each $k = 1, 2, 3$. QED.

5.2. *Answer:* these triples are

$$\{(2,1,1),(2,1,1),(2,1,1)\}, \quad \{(3,1,1,1),(3,1,1,1),(2,1,1)\}, \quad \{(3,1,1,1),(2,2,1,1),(2,1,1)\},$$

$$\{(3,1,1,1),(2,1,1),(1,1)\}, \quad \{(2,2,1,1),(2,2,1,1),(2,1,1)\},$$

$$\{(2,2,1,1),(2,1,1),(1,1)\}, \quad \{(2,1,1),(1,1),(1,1)\}.$$

Proof. There exist only 4 tree-like sequences of length at most 4. They are

$$(1,1),\ (2,1,1),\ (3,1,1,1),\ (2,2,1,1).$$

According to Problem 5.3 the number of odd length sequences in a realizable triple is odd. So in each realizable triple of sequences of length at most 4, except triple $\{(2,1,1),(2,1,1),(2,1,1)\}$, there are one sequence $(2,1,1)$ and two sequences of even length. According to the answer to Problem 5.1 all these 7 triples are realizable.

5.3. Let m_3, m_2, m_1 be the numbers of the circles in $f_1 \cap f_2$, $f_1 \cap f_3$ and $f_2 \cap f_3$. Then $n_1 = m_3 + m_2 + 1$, $n_2 = m_3 + m_1 + 1$, $n_3 = m_2 + m_1 + 1$.

So $n_1 + n_2 + n_3 = 2(m_3 + m_2 + m_1) + 3$ is odd.

Since $2m_k + 1 > 0$ we have $n_k < n_{k+1} + n_{k+2}$ for each $k = 1, 2, 3$.

5.4. Let $r = r(\vec{x})$ be the number of those x_i's that are greater than 1. Let $z_s = x_2 + x_3 + \dots + x_s$. For each $s \leq r$ let

$$a_1 = p - (z_s - s + 3) + 1, \quad a_i = x_i \quad \text{for} \quad 2 \leq i \leq s \quad \text{and} \quad a_i = 1 \quad \text{for} \quad s+1 \leq i \leq p,$$

$$b_1 = x_1 - a_1, \quad b_i = x_{i+s-1} \quad \text{for} \quad 2 \le i \le r-s+1, \quad b_i = 1 \quad \text{for} \quad r-s+2 \le i \le q = n+1-p.$$

Since $s \le r$, the sequence $b_1, b_2, \ldots, b_q$ is well-defined. For each i we have that a_i and b_i depend on s.

We have

$$a_1 + a_2 + \cdots + a_p = p - (z_s - s + 3) + 1 + z_s + p - s = 2p - 2,$$

i.e. the sequence $a_1, a_2, \ldots, a_p$ is tree-like. Also

$$b_1 + b_2 + \cdots + b_q = z_n - a_1 - a_2 - \cdots - a_p = 2n - 2 - 2p + 2 = 2q - 2,$$

i.e. the sequence $b_1, b_2, \ldots, b_q$ is tree-like.

It remains to prove that there exists $s \le r$ such that $1 \le a_1 \le x_1 - 1$. For each $i < r$ we have $x_1 \ge x_i$, so

$$z_i - i + x_1 + 1 \ge (z_{i+1} - (i+1) + 3) - 1.$$

In other words,

$$\begin{gathered} 2 = z_1 - 1 + 3, \\ z_1 - 1 + x_1 + 1 \ge (z_2 - 2 + 3) - 1, \\ z_2 - 2 + x_1 + 1 \ge (z_3 - 3 + 3) - 1, \\ \ldots, \\ z_{r-1} - (r-1) + x_1 + 1 \ge (z_r - r + 3) - 1, \\ z_r - r + x_1 + 1 = n - 1. \end{gathered}$$

Here the last equality is not analogous to the previous equalities but follows because sequence $x_1, x_2, \ldots, x_n$ is tree-like and $1 = x_{r+1} = \cdots = x_n$. Since $2 \le p \le n-1$, there exists $s \le r$ such that

$$z_s - s + 3 \le p \le z_s - s + x_1 + 1 \quad \Leftrightarrow \quad 1 \le a_1 \le x_1 - 1. \quad QED$$

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