COMMENTS ON SHIRSHOV'S HEIGHT THEOREM

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In 1941 A.G. Kurosh [1] posed the problem: Is every finitely-generated algebraic associative algebra finite-dimensional? In 1964 E.S. Golod and I.R. Shafarevich [2, 3] constructed a counterexample: they presented an infinite-dimensional finitely-generated nil-algebra. This counterexample shows that in general finitely-generated algebraic associative algebras are very far from being finite-dimensional.

Every problem can be considered not only as an explicit problem but as a direction of research. In the case of Kurosh's problem such a direction can be formulated in the following way: Find the conditions which imply that a finitely generated algebra is finite-dimensional.

Before the counterexample of Golod-Shafarevich was constructed, many positive results on Kurosh's problem were obtained. In 1945 N. Jacobson [4] solved the problem of Kurosh for algebraic algebras of bounded index. In 1946 J. Levitzky [5] proved that for a finitely generated PI-algebra over a commutative ring, if each element is nilpotent then the algebra is nilpotent. Finally, in 1948 I. Kaplansky [6] solved Kurosh's problem for PI-algebras over a field. All of these results became classical and are included in textbooks on ring theory. The great role of these results in ring theory is well known. In fact, the structure theory of rings developed around the problem of A.G. Kurosh.

In 1957 A.I. Shirshov proved his famous theorem on height:

Theorem (A.I. Shirshov [7]**).** *For any finitely-generated associative PI-algebra A over a commutative ring R with 1, there exist a natural number h and elements $a_1, \dots, a_n \in A$ such that any element of A can be represented as an R-linear combination of elements of the form*

$$a_{i_1}^{\alpha_1} \cdots a_{i_k}^{\alpha_k},$$

where $k < h$.

We note that an algebra A over a commutative ring R with 1 is called a PI-algebra if A satisfies some polynomial identity $f = 0$ such that the ideal of the ring R generated by the coefficients of the highest-degree terms of the polynomial f contains 1.

The positive solution of Kurosh's problem for PI-algebras over a ring follows immediately from Shirshov's theorem. Indeed, since the elements $a_1, \dots, a_n \in A$ are algebraic (the elements $a_1, \dots, a_n$ are taken from the conclusion of the theorem on height), the degrees α_i are bounded. Hence the algebra A is a finitely-generated R-module.

Comparing the solutions of Kurosh's problem obtained by I. Kaplansky and A.I. Shirshov one notes that the solution of I. Kaplansky is based on the well-developed structure theory of rings, but makes little use of the PI-condition. In fact, the PI-condition is used in two statements: (1) The radical of a finitely-generated algebraic PI-algebra is nilpotent; (2) A matrix algebra of order n does not satisfy a polynomial identity of degree less than $2n$. These statements are quite easy from the contemporary point of view.

The solution of A.I. Shirshov does not use the structure theory at all. Moreover A.I. Shirshov also made little use of algebraicity. It follows from the above that it is sufficient to require algebraicity only for some finite set of elements. But the most important merit of the theorem on height is that it was proved for algebras over a commutative ring. Many of the results in ring theory concerning PI-algebras would not have been obtained if the theorem on height were true only for algebras over fields.

With the first results about PI-algebras it became clear that the PI-condition is a peculiar finiteness condition. In 1957 S. Amitsur [8] proved a remarkable theorem: The radical of a finitely-generated PI-algebra is a nil-ideal. This theorem once again corroborated that the PI-condition is a finiteness condition, and allowed V.N. Latyshev at that time to formulate rather boldly the problem: Is the radical of a finitely-generated PI-algebra nilpotent? (See [9]). A great contribution to the solution of this problem was made by Yu.P. Razmyslov [10] who proved that the radical of finitely-generated PI-algebra over a field is nilpotent if and only if the algebra satisfies some standard identity. To prove this statement, Yu.P. Razmyslov constructed an embedding of certain algebras into algebras which are algebraic over the center and then applied the theorem on height. Yu.P. Razmyslov was the first algebraist to apply the theorem on height very often and deeply. For algebras over a field of characteristic zero, Latyshev's problem was solved by A.R. Kemer [11] who proved that every finitely-generated PI-algebra over a field of characteristic zero satisfies a standard identity of some order. Indeed this result and the theorem of Razmyslov mentioned above imply the positive solution of Latyshev's problem in the case of characteristic zero. In 1982, A. Braun [12] solved Latyshev's problem positively for algebras over a commutative Noetherian ring. At present the theorem on the nilpotency of the radical of a finitely-generated PI-algebra is known as the theorem of Braun-Kemer-Razmyslov.

In 1974, Yu. P. Razmyslov introduced a new concept of trace identity, and proved that each trace identity of the matrix algebra of order n over a field of characteristic 0 follows from the Cayley-Hamilton trace identity of degree n and the identity $\mathrm{Tr}(1) = n$ [13]. Little later C. Procesi [14] proved actually the same result in the terms of invariants.

The Cayley-Hamilton identity of degree n has the form

$$X_n(x) = x^n + b_1(x)x^{n-1} + \cdots + b_n(x) = 0,$$

where the coefficient $b_m(x)$ is a form of degree m. In the case of characteristic zero the coefficients $b_m(x)$ can be represented as linear combinations of trace monomials of the form

$$\mathrm{Tr}(x^{i_1})^{\alpha_1}\mathrm{Tr}(x^{i_2})^{\alpha_2}\cdots\mathrm{Tr}(x^{i_k})^{\alpha_k}.$$

Of course this theorem of Yu.P. Razmyslov does not concern the theorem on height directly, but the idea of trace identities gives a way of embedding (if possible) a finitely-generated PI-algebra over a field into a finite-dimensional algebra (a matrix algebra) over a larger field (such algebras are called representable). Indeed, let a finitely-generated algebra A over a field F be embeddable into the matrix algebra $M_n(K)$, $F \subseteq K$. Consider the F-subalgebra $C = SA$, where S is the F-subalgebra (with unity) of the field K generated by all the elements $b_m(a)$ ($a \in A$) where the elements $b_m(a)$ are the coefficients of the Cayley-Hamilton identity of degree n. It follows from this that in the case of characteristic zero the algebra A is embeddable into the algebra

$$D = A \otimes T\langle A\rangle/J,$$

where $T\langle A\rangle$ is the commutative algebra generated by the symbols $\mathrm{Tr}(a), a \in A$, the trace on the algebra $A \otimes T\langle A\rangle$ is defined by the formula

$$\mathrm{Tr}\Big(\sum a_k \otimes t_k\Big) = \sum \mathrm{Tr}(a_k) t_k,$$

and the ideal J is generated by the elements $X_n(d)$ $(d \in A \otimes T\langle A\rangle)$. In the case of characteristic p the algebra $A \otimes T\langle A\rangle$ is generated by the symbols $b_m(a)$ $(a \in A)$. The forms $b_m(x)$ are defined in the same manner but with more complicated formulas.

Assume that the algebra A is embeddable into the algebra D. Then the algebra A is embeddable into the algebra

$$D' = A \otimes T'\langle A\rangle / J \cap A \otimes T'\langle A\rangle,$$

where $T'\langle A\rangle$ is the subalgebra of $A \otimes T\langle A\rangle$ generated by the elements $b_m(a_i)$ (the elements a_i are taken from the conclusion of the theorem on height). The algebra D' is finitely-generated and algebraic over the commutative algebra $T'\langle A\rangle$ because it satisfies the Cayley-Hamilton identity. By the theorem on height the algebra D' is a finitely-generated $T'\langle A\rangle$-module. Since the algebra $T'\langle A\rangle$ is noetherian, by a theorem of K. Beidar [15] the algebras D' and A are representable. In 1995 the theorem of Razmyslov in the case of characteristic p was proved by A.R. Kemer at the multilinear level [16] and little later A.N. Zubkov proved this theorem at the homogeneous level [17].

A very important problem in the theory of PI-algebras was posed by W. Specht [18] in 1950: Does every associative algebra over a field of characteristic zero have a finite basis of identities? The finite basis problem makes sense for algebras over any field, and even for rings, groups and arbitrary general algebraic systems. A positive solution of the finite basis problem for a given class of algebraic systems is a sort of classification of these algebraic systems in the language of identities.

A rather large number of papers have been devoted to Specht's problem for associative algebras over a field of characteristic zero. We note the most important results. In 1977 V.N. Latyshev [19] proved that any associative algebra over a field of characteristic zero satisfying a polynomial identity of the form

$$[x_1, \ldots, x_n] \cdots [y_1, \ldots, y_n] = 0,$$

has a finite basis of identities. This result was also obtained independently by G. Genov [20] and A. Popov [21].

In 1982 A.R. Kemer reduced the Specht problem to the finite basis problem for graded identities of finitely-generated associative PI-superalgebras [22] and in 1986 he solved the Specht problem positively [23]. The first proof of the theorem on the finite basis of identities was rather complicated. A little later in 1987 A.R. Kemer [24] proved that relatively free finitely-generated associative PI-superalgebras over a field of characteristic zero are representable. This theorem implies the theorem on the finite basis, and explains the reason why the Specht problem has a positive solution. This reason is that finite-generated PI-algebras over a field of characteristic zero cannot be distinguished in the language of identities from finite-dimensional algebras. More precisely, for every finitely-generated PI-algebra A there exists a finitely-dimensional algebra C such that the ideals of identities of these algebras are equal. In 1988 A.R. Kemer proved the same result for algebras over an infinite field of characteristic p [25].

The main idea of the proof of this theorem is to approach step-by-step the given T-ideal Γ by the ideals of identities of finite-dimensional algebras. At the first step there is constructed a finite-dimensional algebra C_0 such that

$$T[C_0] \subseteq \Gamma.$$

The existence of this algebra follows from the theorem on nilpotency of Braun-Kemer-Razmyslov and the theorem of J. Lewin [33]. The most difficult part of the proof is the following statement: If $T[C] \subseteq \Gamma$, $T[C] \neq \Gamma$ (C is finite-dimensional) then there exists a finite-dimensional algebra C' such that

$$T[C] \subseteq T[C'] \subseteq \Gamma, \qquad T[C] \neq T[C'].$$

The proof of this statement uses identities with forms and the standard application of the theorem on height which was described above.

Examples of infinitely-based algebras in the case of characteristic p were constructed in 1999 by V.V. Schigolev [26] and A.Ya. Belov [27].

In 1998 A.Ya. Belov [28] announced a positive solution of the local finite basis problem for algebras over a commutative noetherian ring, and announced a result about the representability of the relatively free algebra over a commutative noetherian ring in some weak sense: The relatively free finitely-generated PI-algebra A over a commutative noetherian ring R is embeddable into some algebra A' over a commutative noetherian ring R' such that A' is a finitely-generated R'-module ($R \subseteq R'$). In other words the algebra A is embeddable into the algebra of endomorphisms of some finitely generated R'-module.

Regarding the methods of A.Ya. Belov we should note that most of the ideas of A.Ya. Belov are combinatorial, and come from the theorem on height and other results of A.I. Shirshov. A.Ya. Belov developed the combinatorial ideas of A.I. Shirshov which made it possible to consider more complicated combinatorial situations than in the theorem on height. In this sense one can call A.Ya. Belov a successor of A.I. Shirshov.

Another nice idea is applying Zariski closure. This idea was new for PI-theory. The algebras of endomorphisms of finitely generated modules over a ring have a more complicated structure than finite-dimensional algebras, but applying Zariski closure A.Ya. Belov proved that a finitely-generated PI-algebra A over a commutative noetherian ring R has the same identities as some algebra C over a commutative noetherian ring R', $R \subseteq R'$, satisfying the property that the radical of the algebra C splits off and is nilpotent, i.e., $C = P + \operatorname{Rad} C$, where the subalgebra C is semisimple. Applying Zariski closure A.Ya. Belov also obtained a lot of information about the semiprime part P. We note that the main results of A.Ya. Belov are not yet published.

We also mention the results devoted to the estimation of height in the theorem of A.I. Shirshov. The height $h(A)$ of an algebra A depends on the number of generators s and the minimal degree of identities $m = \deg(A)$. The estimate for the height which follows from the proof of the theorem on height is not satisfactory. In 1982 A.G. Kolotov [29] obtained the estimate

$$h(a) \leq s^{s^m}.$$

In [30] E.I. Zelmanov raised a question about the exponential estimate of the height. The positive answer was obtained by A. Ya. Belov in 1988 [31, 32].

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