Strange propeties from strange condition.
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General statement of problem.

Given a triangle ABC. Let A_0, B_0 and C_0 be the middle points of its sides, A_1, B_1, C_1 be the touching points of incircle with sides BC, AC and AB respectively. Denote intersection point of lines AA_1, BB_1, CC_1 by G (we will call it Gergonne point). Let I be the incenter of triangle ABC, O is circumcenter of $\triangle ABC$, H is common point of triangle's altitudes, F is Feuerbach point of $\triangle ABC$.

We impose only one condition on the triangle: points A, I, G and C are concyclic.

Quantifier $\forall$ in brackets before the statement of a problem means that following fact is true for any triangle (not only for triangle with our condition). Note that absence of this sign does not mean that this condition is needed.

Part A.

1. ($\forall$)Prove that angle $\angle AIC = 90° + \frac{\angle B}{2}$.

2. ($\forall$)Prove that $\angle AIC + \angle A_1B_1C_1 = 180°$.

3. ($\forall$)Prove that line passing through A_1 parallel to bissector of angle $\angle C$ intersects line through C_1 parallel to bissector of angle $\angle A$ in point which is opposite to point B_1 in incircle.

4. Prove that if points A, G, I and C are concyclic then forth vertex F' of parallelogram A_1GC_1F' lies on incircle.

5. ($\forall$)Prove that G is a Lemoine point of triangle $A_1B_1C_1$. Lemoine point is a point which is isogonal conjugative to center of gravity of triangle.

6. Prove that in our case center of gravity of triangle $A_1B_1C_1$ lies on circumcircle of triangle A_1IC_1.

7. Construct a triangle with our condition and given angle $\angle B$. For instance by vertex B and incircle.

8. Prove that angle $\angle B$ is less than $60°$.

Part B.

Let intersection point of AA_1 with incircle which is different from A_1 be Q, and the same for CC_1 is P. Let K be moving point on incircle. Denote intersection point of KP with BC by P', and KQ with AB by Q'.

This part is not necessary for following parts of project.

1. Prove that line passing through P parallel to BC and line passing through Q parallel to AB intersect in a point on incircle. Denote this point by T.

2. Find locus of midpoint of segments $P'Q'$.

3. Let A_1C_1 intersect AC in point R. We draw lines parallel to AB and BC through point R and intersect them with BC and AB in points P_1 and Q_1 respectively. Prove that lines PP_1 and QQ_1 have a common point on incircle.

4. ($\forall$)Prove that points P, Q and R are collinear.

5. Let P_2 be intersection point of B_1C_0 and BC and Q_2 be intersection point of B_1A_0 and AB. Prove that lines PP_2 and QQ_2 intersect on incircle.

6. Prove that line which passing through midpoint of segment AC_1 and Q and line which passing through midpoint of segment CA_1 and P intersect on incircle. From this time point K is this intersection point. It start be fixed. Points P' and Q' are fixed also and equals P' and Q' for this position of K.

7. Prove that points T, K and B lie on one line.

8. Prove that lines BK and BF' are isogonal conjugative in angle $\angle B$.

9. Prove that points K, G and F' are collinear.

10. Let E be midpoint of the segment A_1C_1. Prove that E is incenter of triangle $P'Q'B$.

Later we will not be needed in points P, Q, P', Q' and R (may be sometimes we will use last one). We recommend you draw your picture again without these points.

Part C.

Three lines passing through Feuerbach point.

Still this part is very important for the project but very difficult we allow using in solutions previous facts without proof. Also it will be allowed using in future solutions items from this part without proof.

In this part there are fact which are true for all nonisosceles triangles so we omit sign $\forall$ before items.

1. Let points A', B', C' be the intersection points of the respective sides of triangles $A_0B_0C_0$ and $A_1B_1C_1$. Prove that the vertices of triangle ABC lie on the sides of triangle $A'B'C'$

2. Prove that triangle $A'B'C'$ is autopolar with respect to (later wrt) the incircle. I. e. each of its sides is the polar line of the opposite vertex.

3. Prove that lines AA', BB', CC' are parallel.

4. Prove that line IG passes through the common points of the corresponding sides of triangles $A_1B_1C_1$ and $A'B'C'$.

5. Let M be a centroid of triangle ABC. Prove that line IM passes through the intersection points of the respective sides of triangles $A_0B_0C_0$ and $A'B'C'$.

6. Prove that lines AA', BB', CC' are parallel to a line passing through the intersection points of the respective sides of triangles $A_1B_1C_1$ and ABC. I. e. they are parallel to a polar of G in incircle.

7. Let C_A be an intersection point of CI and C_1A_1 and C_B be an intersection point of CI and C_1B_1. Points A_B, A_C, B_A, B_C are defined by the same way. Prove that points A, C_1, C_A, I are concyclic.

8. Prove that C_A lie on B_0C_0.

9. Prove that circumcenter of triangle $C_AC_BC_1$ is point C_0.

10. Prove that points C_A, C_B, A_B, A_C lie on one circle which is orthogonal to the incircle (two circles are orthogonal if their tangents in common point are orthogonal). Denote this circle by ω_B. Three analogous circles denote by ω_A and ω_C respectively.

11. Let circles ω and γ be orthogonal and let P and Q be their centers respectively. Prove that the polar of P wrt γ, the polar of Q wrt ω and the radical axis of γ and ω coincide.

12. Given inscribed quadrilateral $ABCD$. Let N be a common point of AB and CD, M be a common point of BC and AD, and P be a common point of AC and BD. Prove that NM is the polar line of P wrt circumcircle of $ABCD$.

13. Denote the center of ω_B by O_B. Points O_A and O_C are defined similarly. Prove that O_B lie on $A'C'$.

14. Prove that the midpoints of the sides of triangle ABC lie on the sides of triangle $O_AO_BO_C$.

15. Prove that triangle $A_0B_0C_0$ is orthotriangle of triangle $O_AO_BO_C$. That is points A_0, B_0, C_0 are the feet of the altitudes of triangle $O_AO_BO_C$.

16. Let M_B be a midpoint of side O_AO_C. Points M_A and M_C are defined by the same way. Prove that triangles $A_1B_1C_1$ and $M_AM_BM_C$ are homothetic.

17. Prove that points B', M_B and B_1 are collinear.

18. Prove that lines A_1A', B_1B', C_1C' pass through the Feuerbach point of triangle ABC.

19. Prove the Feuerbach theorem. I.e. the incircle and the Euler's circle of a nonisosceles triangle touche.

20. Prove that the tangent to the incircle through F passes through the common points of the corresponding sides of triangles ABC and $A'B'C'$.

21. Prove that points B_C, C_B, A_1, A_0 are concyclic.

22. Prove that F lies on the circle from previous item.

23. Invent another proof of the fact that line A_1A' passes through F.