Problems on coverings and growth functions
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Preface
This cycle is devoted to coverings of arbitrary figures by figures of a prescribed type (usually by disks or balls). The task is to estimate their number, their total area etc.

Naturally, the most difficult are the problems where the precise estimate is required. Fortunately it turns out that basic applications of these problems just don't require an exact answer, and it suffices to know the order of the respective value. The related definitions will be formulated accurately in Part B.

Part A.

A1.
It is not difficult to cover the unit square by a disk of area $\pi/2$. However, is it possible to cover the square by several disks of total area less than $\pi/2$? The disks may intersect and get out of the square.

A2.
Cover the unit square by several disks of the same radius r. Let $N(r)$ be the minimal number of disks sufficient for this.

Clearly $r \to 0$ implies $N(r) \to \infty$.

Determine the type of tending to infinity for this function. In other words: how fast does it grow?

A3.
The same task but it is required to cover the unit cube by several (possibly intersecting) balls of radius r.

Part B.

In the above problems we assumed that the notion of the "growth rate" is intuitively clear. But below we shall need the precise definition for the "growth of a function".

This definition is not quite obvious, and we will give it a bit later. Before that, having no precise definition for the "growth of a function", we will formulate certain properties of this "growth". Specifically:

In the sequel, we consider only functions $f(x)$ such that

(*) $f(x)$ is defined for all x exceeding some a (usually $a=0$, but for convenience we allow a more general case, for example, functions like $\ln(x-1)$). Moreover we assume that $f(x)$ is positive everywhere (at any rate, for $x>a$), non-strictly increases and tends to infinity.

Denote the growth of $f(x)$ by $[f]$. In the sequel, for simplicity we denote the growth of x^n by n. Thus, $n = [x^n]$.

We require that the growth of a function has the following features:

(1) If for all x exceeding some b (b not necessarily equals a) we have $f(x) > g(x)$ then $[f] \geq [g]$.

(2) Let A, B, C be arbitrary positive numbers, and $g(x) = Cf(Ax+B)$. Then $[g] = [f]$.

(3) This easily implies that a situation is possible when both relations $[f] \geq [g]$ and $[g] \geq [f]$ hold. In this case we set $[f] = [g]$ as well.

If $[f] \geq [g]$ but not $[g] \geq [f]$ then we write $[f] > [g]$.

However we still have not given a definition for growth. The correct answer is that growth is just something subject to the above conditions.

Speaking more formally, we have to consider the class of functions subject to (*) and split it into equivalence classes: $f \sim g$ iff $[f] = [g]$ (according to items (2), (3)).

Each of these classes is called the growth of all functions contained in it. Some of these classes occur to be greater than others, so that we may say that a certain function grows faster (its growth is greater) than another one.

B1.

Prove that $1 < 2$.
(A reminder: 1 and 2 are not numbers but the growth of some functions.)

For ordinary numbers there are three alternatives: either $a>b$, or $a=b$, or $a<b$. However for functions this doesn't hold. There exist "incomparable functions", that is, functions f and g such that both assertions $[f] \geq [g]$ and $[g] \geq [f]$ fail.

B2.

Find two incomparable functions.

B3.

Find a function $f(x)$ such that $1 < [f] < 3$ but f is incomparable with 2.

B4.

Find a function $f(x)$ such that for any $a>0$ we have $[f] < a$.

B5.

Find a function $f(x)$ such that $[f] = [f^2]$.

B6.

Does there exist a function $f(x)$ such that $[f] = [\ln f]$?

Part C.

This part is devoted to coverings of figures. The main task is as follows: determine the growth for the number of disks (or balls) sufficient to cover a given figure.

Definition. An epsilon-net (ε-net) is a set of points inside a given figure, such that the distance from any point of the figure to at least one of the chosen points doesn't exceed ε.
(In other words, disks or balls of radius ε cover the whole figure.)

Definition. A delta-lattice (a δ-lattice) is a set of points inside a given figure, such that the distance between any two points in the set is not less than δ. (Of course, the numbers epsilon and delta may coincide).

An ε-net is called minimal if it contains the minimal possible number of points. Denote the number of points in a minimal ε-net by $M(\varepsilon)$.

A δ-lattice is called maximal if it contains the maximal possible number of points. Denote the number of points in a maximal δ-lattice by $N(\delta)$.

Think on your own why in the first case we have to consider a minimal net, and in the second case a maximal lattice.

C1

Determine $M(\varepsilon)$ and $N(\delta)$ for a segment of length a (and arbitrary ε, δ).

C2

Give an estimate (as sharp as possible one) for $M(\varepsilon)$ and $N(\delta)$ when a net and a lattice are constructed in the unit disk, or in the unit ball. The estimate must be two-sided, that is of the form (for instance)

$2/\varepsilon < M(\varepsilon) < 5/\varepsilon$

Definition. The dimension of a figure is the growth of $M(\varepsilon)$ as of the function in $1/\varepsilon$.

C3

Prove that the dimension equals the growth of $N(\varepsilon)$ as of the function in $1/\varepsilon$.

Thus the dimension can be defined both in terms of minimal nets and of maximal lattices.

C4

Give an example of a figure of dimension 3/2 .

C5

What are other possible values for the dimension of various figures? (It suffices to provide several examples.)

C6

A *helix* is a curve which is located inside some disk and tends to its center; in polar coordinated, it can be defined by the equation $r = f(\varphi)$ where f is defined for $\varphi \geq 0$ and decreases tending to 0 at infinity.

Determine the dimension of a helix. Does this dimension depend on the specific function f, or it is always the same?

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Part D.

Refinement of Problem A1.

Now, we are interested in a sharp estimate in Problem A1. In other words: we look for a number α such that (1) the unit square cannot be covered by disks of total area less than α, and (2) for any $\beta > \alpha$, this square can be covered by disks of total area β. (Try also to determine whether the square can be covered by disks of total area equal to α.)

We split this problem into three different ones. To formulate the first problem, we recall one of the solutions for problem A1: tile the square by the regular hexagons, and cover each hexagon by a disk. The ratio of areas of a hexagon and a disk is $\gamma = 2\pi / 3\sqrt{3}$. Since the total area of hexagons is larger than the area of the square, the total area of disks β will be greater than γ, but the difference $\beta - \gamma$ can be made arbitrarily small by taking sufficiently small hexagons.

D1.

Suppose that all the radii of the disks should be equal. Prove that the above construction is optimal, that is, $\alpha = \gamma$.

D2.

Find α if it is allowed to use the disks of two distinct radii.

D3.

Find α if it is allowed to use the disks of arbitrary radii without any restrictions.

(**Warning.** The problems are NOT necessarily ordered by the ascending difficulty!)

D4 – D6.

The same problem but the task is to cover the unit cube by balls which of course may intersect. (Here it suffices to give some good estimates; the precise answer is not known.)

D7.

We need to cover the square with some disks of equal radii so that each point is covered by at least N disks. Prove that there exists some N such that it is possible to make this with total area of the disks smaller than $N\gamma$.

D8.

Does the statement from D7 hold for $N = 2$?

Part E.

E1.

A checked sheet of paper lies on a table. Another checked sheet of paper is put on it; the squares on both sheets have the same size but the lines on the second list aren't parallel to those on the first sheet. The upper sheet is transparent, so that we see how its lines dissect some fixed square of the lower sheet.

Determine the maximum and the minimal possible number of parts in a dissection of a lower square.

E2.

A checked sheet of paper of size 10000*10000 lies on a table. Above it, another checked sheet of paper of size 1000*2000 is put; the squares on both sheets have the same size. The upper sheet is transparent, so that we see how the lines of the lower sheet dissect the squares of the upper one. So, the number of parts of the upper list occurs to be greater than two millions.

Prove that this number is less than ten millions.

Part F.

Now we return to the problems of the same type as in cycle B. However, we consider a somewhat different construction. We assume the number ε to be fixed (for instance $\varepsilon=1$) but consider an unbounded figure Φ, for instance, the whole plane, a half-plane etc.

Consider an arbitrary point O of the given figure, and construct the figure $\Phi(R)$ consisting of all points of the given figure whose distance from O does not exceed R (in other words, $\Phi(R)$ is the intersection of the given figure with a disk or a ball with center O and radius R).

Denote by $N(R)$ the number of points in a minimal ε-net in the figure $\Phi(R)$. Let χ be the growth rate of the function $N(R)$; if this function does not tend to infinity as $R \to \infty$, we conventionally say that $\chi = 0$.

The number χ is called *the volume characteristics* of the given figure.

F1.

Which figures have a zero volume characteristics?

F2.

Determine the volume characteristic of the following figures: (a) the plane, (b) a half-plane, (c) a strip bounded by parallel lines.

F3.

Prove that the volume characteristic of a figure does not depend on the choice of point O.

F4.

Given two positive numbers ε and δ. Is it true, that the volume characteristics of a figure Φ calculated using the disks of radius ε is the same as that calculated using the disks of radius ε?

F5.

Determine the volume characteristic of the interior and of the exterior of a parabola.

F6.

Determine the volume characteristic for each of three parts of the plane obtained by its dissection by the hyperbola $xy=1$.

F7.

What are possible numerical values of the volume characteristic of a figure (in the plane or in the space)?

F8.

Can the volume characteristic of a figure be not a number (in other words, can $N(R)$ be a function incomparable with some numbers)? For instance, can it be greater than 1, less than 2 and incomparable with 3/2, like the function from Problem B4?