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NOTE

On Tilings by Ribbon Tetrominoes[1]

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1. INTRODUCTION

A *ribbon polyomino* is a polyomino which has at most one square (i, j) in every diagonal $i-j=c$. A tetromino is a polyomino with four squares. Up to translations there are exactly 8 different ribbon tetrominoes, which we denote $\tau_1, ..., \tau_8$ as in Fig. 1. Let $\mathbf{T}=\{\tau_1, ..., \tau_8\}$.

Now let Γ be a simply connected region (a finite connected set of squares), and let v be a *tiling* of Γ by ribbon tetrominoes. This means that Γ is covered without intersection by parallel translations of ribbon tetrominoes. Denote by $a_i(v)$ the number of times tetromino τ_i occurs in the tiling v. While numbers a_i may be different for different tilings, this is no longer true for certain linear combinations of them.

THEOREM 1.1. *For every simply connected region Γ and a tiling v of Γ we have*

$$a_2(v)+a_3(v)-a_6(v)-a_7(v)=C_1(\Gamma)$$

and

$$a_1(v)+a_2(v)+a_7(v)+a_8(v)=C_2(\Gamma) \qquad (\text{mod } 2),$$

where $C_{1,2}(\Gamma)$ are functions of Γ and do not depend on v.

The theorem was conjectured by the second author in [P], where it was proved for all row (column) convex regions. A more general version of the conjecture for all ribbon polyominoes remains open (see [P] for details).

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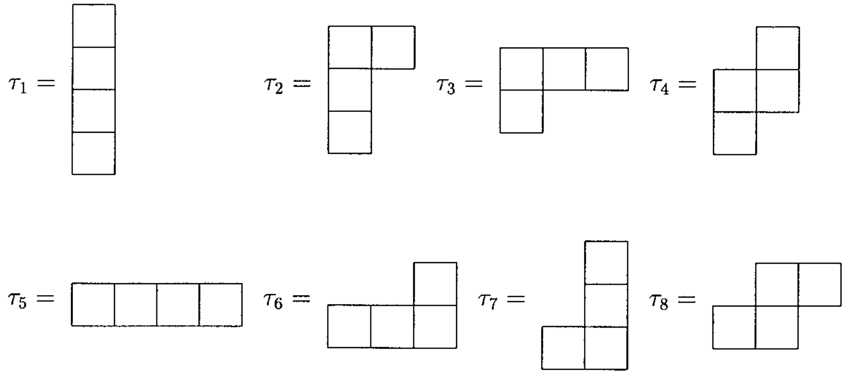

FIGURE 1

While the result for row (column) convex regions was obtained by use of the Young tableau technique, here we rely on the technique developed by Conway and Lagarias in [CL].

It remains open whether there exists a finite set of "moves" such that by using these moves one can start with any tiling and get to any other tiling of a given simply connected region. Such a set of moves was proposed in [P] where this property was shown for Ferrer's shapes. In case of domino tilings and lozenges the result is known for all simply connected regions (see [ST, T]).

It is important to note that as shown in [P] the theorem cannot be obtained by use of the coloring arguments (see [G, CL]). Thus our result lays in line with other "hard" results for trominoes (see [CL]), **T**-tetrominoes (see [W]), skew and square tetrominoes (see [Pr]), and rectangles (see [K]).

2. PROOF OF THE THEOREM

Observe that all tiles $\tau \in \mathbf{T}$ are simply connected. This fact is crucial in the induction we present below. Our proof relies on the following lemma.

LEMMA 2.1. *Let Γ be a compact simply connected region. Assume that v is a tiling of Γ by tiles $\tau_i \in \mathbf{T}$. Then there exists a tile τ in the tiling v such that $(\Gamma - \tau)$ is simply connected.*

Versions of the lemma have appeared previously in [CL, Pr]. We give here a new rigorous proof of the claim.

Proof. Denote by $|v|$ the number of tiles in a tiling v. The result is trivial for $|v| = 1, 2$. Now suppose $|v| > 2$. We say that two regions are *attached* if the intersection of their boundaries contains an interval. Note that two regions can be attached from either inside or outside.

Observe that if we remove any tile $\tau \in \nu$ which is attached to Γ, then we obtain a region which is a union of simply connected regions. Indeed, this follows from $\Gamma^c + \tau$ being connected since Γ is simply connected, and τ is attached to Γ^c. (Γ^c is a complement of Γ.)

Denote by $l(\tau)$ the number of tetrominoes in the smallest connected component in $\Gamma - \tau$, and by $n(\tau)$ the number of connected components of $\Gamma - \tau$. We will show that there exists a tile $\tau \in \nu$ such that *either* $n(\tau) = 1$ or $l(\tau) = 1$. This implies the lemma. Indeed, in the first case tile τ is the desired tile while in the second case we can simply remove a unique tile τ' in either of the smallest connected components and obtain the desired simply connected region $\Gamma - \tau'$.

Now, let τ be a tile attached to Γ. Let Γ_1 be any smallest connected component obtained after removing τ. Observe that the boundary of Γ_1 is made up of pieces of the boundary of Γ and τ. As τ is simply connected, Γ_1 has a common boundary with Γ, otherwise the boundary of Γ_1 lies inside the boundary of τ. Consider any tile τ' in Γ_1 which is attached to Γ. Consider removing tile τ' instead of τ. In this case, the component of $\Gamma - \tau'$ which contains τ also contains all components of $\Gamma - \tau$ other than Γ_1 simply because they are attached to τ. We call it a *big* component of $\Gamma - \tau'$. Observe that besides the big component all the other components must be of size smaller than $l(\tau)$. If there are no components other than the big component, then $n(\tau') = 1$ and tile τ' is the one desired in the lemma. If there exists such a component, we have $l(\tau') < l(\tau)$. Now proceed by induction until either $n(\tau) = 1$ or $l(\tau) = 1$.

This finishes proof of the lemma.

Let $F_2 = \langle A, B \rangle$ be a free group generated by A, B. A represents the direction from left to right and B represents the up direction.

For any region Γ and a point x on the boundary $\partial\Gamma$ define a word $\omega(\Gamma)$ obtained by reading $\partial\Gamma$ counterclockwise starting from x. For example for τ_2 starting at the lowest left corner $\omega(\tau_2) = AB^2ABA^{-2}B^{-3}$. Any region has more than one representation depending on the starting point. However, it is easy to see that all these presentations are conjugates of each other.

Consider a subgroup $G = \langle A^4, B^4, (AB)^2 \rangle$ of F_2, generated by the elements as shown, and let $H = N(G)$ be the smallest normal subgroup of F_2 which contains H. Finally, consider a quotient F_2/H and its Cayley graph representation given in Fig 2. Here we have an edge correspond to a generator A or B if it belongs to the corresponding square.

LEMMA 2.2. *If Γ is tileable by tiles* **T** *then $\omega(\Gamma)$ is in H.*

Proof. By Lemma 2.1, it is sufficient to check that for every tile $\tau \in \mathbf{T}$ we

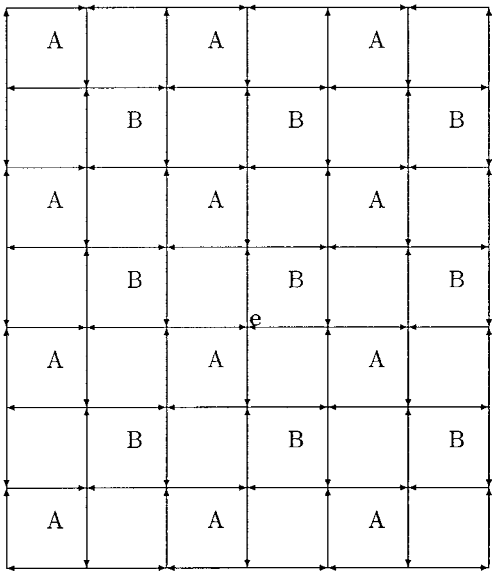

FIGURE 2

have $\omega(\tau) \in H$. Indeed, if this is true, we can use induction to show that $\omega(\Gamma) \in H$.

On the other hand, for the Cayley graph above it is easy to check that every tile in **T** is mapped into a closed path on the graph. This proves the lemma.

Now, to each simply connected region Γ which is tileable by tiles **T** we can assign a closed path $\omega(\Gamma)$ on the Cayley graph of F_2/H, although this path is not uniquely defined. By assigning weights to each cell in Fig. 2 and counting the winding numbers of the path of $\partial\Gamma$ with respect to these weights we will show that the identities in the theorem hold. (cf. [CL]).

LEMMA 2.3. *Assign values* 0 *to each cell that correspond to* A^4 *and* B^4 *and values* 1 *to each cell of* $ABAB$. *Then*

$$a_2(v) + a_3(v) - a_6(v) - a_7(v)$$

is equal to $-\frac{1}{2}$ *times the winding number of* $ABAB$ *cells.*

Proof. Use induction on the number of tiles covering Γ. For $n = 1$, check that the paths associated to tiles τ_2 and τ_3 enclose two cells $ABAB$

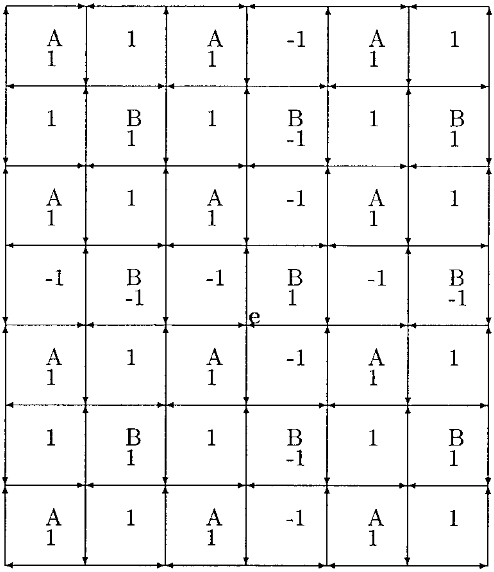

FIGURE 3

going clockwise. Similarly, paths for tiles τ_6 and τ_7 enclose two cells $ABAB$ going counterclockwise. Paths for tiles τ_1 and τ_5 enclose no $ABAB$ cells. Finally, paths for tiles τ_4 and τ_8 enclose 2 $ABAB$ cells, one in the clockwise direction and one the in counterclockwise direction. Thus for $n=1$ the statement is true.

Assume the statement is true for $n=k$. Let Γ be covered by $n=k+1$ tiles. By Lemma 2.1, there exists a tile τ such that $\Gamma-\tau$ is a simply connected region. Call it Γ_1. Then by a suitable conjugation $\omega(\Gamma)=\omega(\Gamma_1)\circ\omega(\tau)$ (here $\circ$ is a group operation in F_2). Now use the additivity property of winding numbers and the induction assumption for the region Γ_1. This proves the lemma.

Note that the first part of the Theorem 1.1 follows immediately from Lemma 2.3. Similarly, the second part is implied by the following result.

LEMMA 2.4. *Assign the values to each cell as shown on the Fig.* 3. *Namely, assign* -1 *to squares* (i, j) *with exactly one coordinate divisible by* 4. *Assign* 1 *to the remaining squares. Then*

$$a_1(v)+a_2(v)+a_7(v)+a_8(v) \qquad (\text{mod } 2)$$

is equal to $\frac{1}{2}$ *times the winding number of the region* Γ.

Proof. The proof is similar to the proof of Lemma 2.3. It is easy to check that the winding numbers are: $2 \pmod 4$ for τ_1, $2 \pmod 4$ for τ_2, $0 \pmod 4$ for τ_3, $0 \pmod 4$ for τ_4, $0 \pmod 4$ for τ_5, $0 \pmod 4$ for τ_6, $2 \pmod 4$ for τ_7, $2 \pmod 4$ for τ_8. The rest of the proof goes along the lines of the proof of Lemma 2.3. We omit the details.

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