Products of consecutive Integers

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Problems

We shall consider the following problem:

Could a product of some consecutive integers be a power of an integer?

In other words we ask whether the equation

$$x(x+1)(x+2)\dots(x+k-1) = y^m \tag{1}$$

has a solution in positive integers (for each $k \geqslant 2$ and $m \geqslant 2$).

A solution of the problem for any partial case will be accepted as a progression. We recommend to consider the following cases first.

1 Some partial cases

1.1. $k = 2$, $m = 2$.
1.2. $k = 2$, m is arbitrary.
1.3. $k = 3$, $m = 2$.
1.4. $k = 3$, m is arbitrary.
1.5. $k = 4$, $m = 2$.
1.6. $k = m$.
1.7. $k = 8$, $m = 4$.
1.8. $k = 8$, $m = 2$.
1.9. $k = 4$, m is arbitrary.
1.10. $k = 5$, $m = 2$.

2 Variations of the question

2.1. Prove that for $m = 2$ and even k the equation does not have infinitely many solutions (x, y).
2.2. We take 5 consecutive integers, choose 4 of them and multiply. Is it possible the result to be an exact square?
2.3. Prove that the equation $x(x+d)(x+2d) = y^2$ has infinitely many solutions (x, y, d) in nonnegative integers.
2.4. Prove that for every $k \neq 2, 4$ a polynom of a form $x(x+1)(x+2)\dots(x+k-1)+c$, where c is a rational number, is not a square of a polynom.

3 Convenient numbers

We call a number k *convenient* if among each k consecutive positive integers there is at least one which is relatively prime to the others.

We shall refer to our main equation (1) by the notation (k, m). For example, in the very first problem we spoke about the equation (2, 2).

3.1. Prove that the equation (k, m) could not have infinitely many solutions for convinient k and $m > k$.

3.2. Prove that for each convinient k there is a number $m_0(k)$, such that for $m > m_0(k)$ the equation (k, m) has no solutions.

3.3. Prove that the equation (k, m) has no solutions for convinient k and $m \geqslant 2k$.

3.4. Prove that the equation (k, m) has no solutions for convinient k and $m \geqslant k + 2\log_2 k$.

3.5. Prove that the equation $(5, 7)$ has no solutions.

3.6. Prove that all positive integers less than or equal to 16 are convenient.

3.7. Prove that 17 is not convenient.

3.8. Prove that all positive integers greater than 17 are not convenient.

4 Common properties of solutions

You may apply the following two theorems in your solutions.

Tchebyshev theorem (Bertrand postulate). There are at least two primes between integers n and $2n$ if $n > 5$.

Sylvester theorem. There is a prime $p > k$ that divides $(n + 1)(n + 2) \ldots (n + k)$ if $n > k$.

Let us write the factors from left hand side of the equation (k, m) in form

$$x + i = a_i z_i^m, \qquad 0 \leqslant i \leqslant k - 1\,,$$

where integers a_i are free of m-th powers, i.e. the power of each prime factor of a_i is less than m.

4.1. Prove that $x > k$ for any solution of the equation (k, m).

4.2. Prove that $x > k^m$ for any solution of the equation (k, m).

4.3. Prove that all prime factors of integers a_i are less than k.

4.4. Solve the equation (7, 2).

4.5. Solve the equation (6, 2).

4.6. Let x be a solution of the equation (k, m). Prove that the equality

$$(x+i_1)\ldots(x+i_{m-1}) = (x+j_1)\ldots(x+j_{m-1}), \quad \text{where} \quad 0 \leqslant i_1 \leqslant \cdots \leqslant i_{m-1} \leqslant k-1;\ 0 \leqslant j_1 \leqslant \cdots \leqslant j_{m-1} \leqslant k-1$$

is possible only if the sets of indices coincide.

4.7. Prove that all the integers a_i are different.

4.8. Let $m = 3$. Prove that all the pairwise products $a_i a_j$, $0 \leqslant i \leqslant j \leqslant k - 1$, are different.

4.9. Let $m = 3$. Prove that the fractions of the form $\dfrac{a_i a_j}{a_r a_s}$, where $0 \leqslant i \leqslant j \leqslant k - 1$, $0 \leqslant r \leqslant s \leqslant k - 1$, are not equal to the cube of rational number (excluding 1).

4.10. Let $m = 3$, $k = 75$. Prove that at least 20 integers a_i have no prime factors greater than 10.

4.11. Prove that the equation $(75, 3)$ has no solutions.

4.12. Denote by $\pi(k)$ the number of primes which are less or equal to k. Prove that almost all integers a_i are "small" in the following sense: we can choose $k - \pi(k)$ of them, such that the product of chosen numbers divides $k!$. (You may consider the cases $m = 2$, 3 only.)

4.13. Let $m = 2$, $B_x(k) = a_0 a_1 \ldots a_{k-1}$. Prove that $B_x(k) > \left(\frac{4}{3}\right)^k k!$ for an arbitrary large k.

Solutions

The equation under consideration has no solutions at all. So the answer “There are no solutions” will not be repeated in each solution.

1 *Some partial cases*

1.1. Follows from the next problem.

1.2. Numbers x and $x+1$ are relatively prime, thus both must be perfect m-th power. However this is impossible.

1.3. Follows from the next problem.

1.4. Let us denote that numbers $x+1$ and $x(x+2) = (x+1)^2 - 1$ are relatively prime. So each of them is a perfect m-th power. Let $x+1 = u^m$, then $x(x+2) = (u^2)^m - 1 = v^m$. This is impossible because two perfect m-th powers differ by more than 1.

1.5. Let us apply an identity $x(x+1)(x+2)(x+3)+1 = (x^2+3x+1)^2$.

1.6. Note that $x < y < x+k-1$. Then the left hand side of the equation contains a factor $y+1$, which is relatively prime with the right hand side.

1.7. Let us multiply factors pairwise (first with last, second with last but one etc.). We get

$$x(x+1)(x+2)\dots(x+7) = (x^2+7x)(x^2+7x+6)(x^2+7x+10)(x^2+7x+12)\,.$$

Let $a = x^2+7x$, we obtain the equation

$$a(a+6)(a+10)(a+12) = y^4\,. \qquad (2)$$

Direct calculation ensures us that if $a > 10$

$$(a+6)^4 < a(a+6)(a+10)(a+12) < (a+7)^4$$

(right inequality also follows from Cauchy inequality).

1.8. Let us transform the product of eight consecutive integers to the form (2). Note that $a = x^2+7x$ is even number, set $a = 2b$, $y = 2y_1$ and cancel both sides of the equation by 2. We obtain

$$b(b+3)(b+5)(b+6) = y_1^2\,.$$

Direct calculations ensure us that

$$(b^2+7b+6)^2 < b(b+3)(b+5)(b+6) < (b^2+7b+7)^2\,.$$

It is easy to check because these inequalities are quadratic.

1.9.

1.10. Remark that greatest common divisor of each pair of these numbers is not more than 4. Thus all large prime factors of y must be contained in the decomposition of $x, \dots, x+4$ in even powers. Hence each factor in the left hand side of the equation must be of one of the forms n^2, $2n^2$, $3n^2$ or $6n^2$. Since left hand side of the equation contains five factors, some two of them have the same form. But the difference between two numbers of the same form could not be small, hence the equation has no solution.

2 *Part 2*

2.1. This solution is from [4]. Denote by $f(x)$ the polynomial in the right hand side of equation (1).

Assume that for $m = 2$, $k = 2n$ equation (1) has infinitely many solutions (x_i, y_i), where $x_i \to +\infty$ and $f(x_i) = y_i^2$. Remark that $f(x)$ is not a square of a polynomial, because all its roots have multiplicities 1. Find a polynomial $a(x)$ of degree n such that $\deg(f - a^2) \leqslant n-1$. Let $r = f - a^2$, $a(x_i) = z_i$. Then $z_i \sim x_i^n$ for $i \to +\infty$ (the reader not familiar with the notion of limit could read this sentence as: $z_i > 0.99x_i^n$ for large i).

Moreover $y_i^2 - z_i^2 = r(x_i) \neq 0$ for large i and at the same time $|r(x_i)| \leqslant \mathrm{const} \cdot x_i^{n-1}$. But on the other hand $|r(x_i)| = |y_i^2 - z_i^2| = (y_i + z_i)|y_i - z_i| \geqslant z_i \sim x_i^n$, which contradicts to the estimation just obtained.

2.2. Answer: Yes, it is possible. $2 \cdot 3 \cdot 4 \cdot 6 = 12^2$.

2.3. First solution. Let $x = kd$. Then $k(k+1)(k+2)d^3 = y^2$. Put $d = k(k+1)(k+2)$.

Second solution. It uses Pythagorean triples. Let $\bar{x} = x + d$. Then the equation will be written in a form $\bar{x}^2(\bar{x}^2 - d^2) = y^2$. We could take $d = 2ab(a^2 - b^2)$, $\bar{x} = (a^2 + b^2)^2$ as a solution.

Third solution. Note that if (x, y, d) — a solution then for each k the triple (k^2x, k^3y, k^2d) is also a solution. So to solve the problem it is enough to find one partial solution, for example $(1, 35, 24)$.

2.4. The proof is taken from [4]. Let us denote $P_{k,c}(x) = x(x+1)(x+2)\dots(x+k-1) + c$. Suppose that $P_{k,c}(x) = a(x)^2$, $k = 2n$. Then

$$P_{k,c}(x+1) - P_{k,c}(x) = k(x+1)(x+2)\dots(x+k-1) = a(x+1)^2 - a(x)^2\,.$$

Hence

$$\big(a(x+1) - a(x)\big)\big(a(x+1) + a(x)\big) = k(x+1)(x+2)\dots(x+k-1)\,.$$

Since the graph of the polynomial $y = a(x+1)$ could be obtained by translation to the left by 1 from the graph $y = a(x)$, each of $n-1$ solutions of the equation $a(x+1) = a(x)$ lies between a pair of roots of the polynomial $a(x) + a(x+1)$ (which have n roots). Hence

$$a(x+1) - a(x) = n(x+2)(x+4)\dots(x+2n-2)\,,$$
$$a(x+1) + a(x) = 2(x+1)(x+3)\dots(x+2n-1)\,.$$

Adding these expressions we get

$$2a(x+1) = 2(x+1)(x+3)\dots(x+2n-1) + n(x+2)(x+4)\dots(x+2n-2)\,.$$

And substituting the same changing x by $x+1$ we obtain

$$2a(x+1) = 2(x+2)(x+4)\dots(x+2n) - n(x+3)(x+5)\dots(x+2n-1)\,.$$

Two obtained expressions contradict to each other. To be ensure this put $x = 0$ to both and subtract one from another. We get

$$(n+2)\big(1 \cdot 3 \cdots (2n-1)\big) = 3n\big(2 \cdot 4 \cdots (2n-2)\big)\,,$$

Here the right hand contains two as a factor with more power than left hand side.

3 *Convenient numbers*

3.1. Let an integer $x+i$ be relatevely prime with all other factors from the left hand side. Then $x + i = u^m$ and

$$(u^m - k + 1)(u^m - k + 2)\dots u^m \leqslant x(x+1)\dots(x+k-1) \leqslant u^m(u^m+1)\dots(u^m+k-1)\,.$$

Let us check that if u is large

$$(u^k - 1)^m < (u^m - k + 1)(u^m - k + 2)\dots u^m \leqslant u^m(u^m+1)\dots(u^m+k-1) < (u^k+1)^m\,. \tag{3}$$

If this is true then for large u the inequality $u^k - 1 < y < u^k + 1$ fulfils. Obviously that $y = u^k$ is not a solution of the equation, hence the inequality has no solutions if u is large. Thus the equation has only finitely many solutions.

For checking left inequality (3) note than

$$(u^k + 1)^m > u^{km} + mu^{km-k}\,.$$

On the other hand

$$u^m(u^m+1)\dots(u^m+k-1) < u^{mk} + \frac{k(k-1)}{2}u^{km-m}$$

Hence if $m > k$ and u is large we have inequality

$$u^m(u^m+1)\dots(u^m+k-1) < u^{mk} + \frac{k(k-1)}{2}u^{km-m} \overset{*}{<} u^{km} + mu^{km-k} < (u^k+1)^m\,. \tag{4}$$

Similarly for left inequality (3) we obtain

$$(u^m-k+1)(u^m-k+2)\dots u^m-(u^k-1)^m = mu^{km-k}-\frac{k(k-1)}{2}u^{km-m}+\dots.$$

Here the right hand side is a polynomial of u, missed summands have less degree of u, and leading term mu^{km-k} is positive. Hence this polynomial is positive if u is large. So the inequality we need is obtained.

3.2. Follows from the next point.

3.3. It is enough to prove that if $m \geqslant 2k$ the inequality (3) holds. Right inequality (3) is obvious because even if $m \geqslant k+1$ the medium inequality (4) mark by asterisk is true.

Let us prove the left one (3). Since

$$(u^m-k+1)(u^m-k+2)\dots u^m > (u^m-k+1)^k\,,$$

then it is enought to prove

$$(u^m-k+1)^k > (u^k-1)^m\,. \tag{5}$$

for $m \geqslant 2k$.

We would prove the inequality (5) by induction by m. Base, $m=2k$

$$(u^{2k}-k+1)^k > (u^k-1)^{2k}\,.$$

Taking the root of degree k-th and expanding we get the obvious inequality $2u^k > k$. Step of induction. It is enought to check

$$(u^m-k+1)^k(u^k-1) < (u^{m+1}-k+1)^k\,.$$

Let us write this inequality in the form

$$u^k-1 < \left(\frac{u^{m+1}-k+1}{u^m-k+1}\right)^k\,.$$

This inequality is obvious because the fraction in parenthesis in the right hand side is not less than u

This inequality is obvious since the fraction in parenthesis in the right hand side is not less than u.

3.4. Let an integer $z=x+i$ be relatively prime with all other factors of the right hand side of the equations. Then $z=x+i=u^m$ and

$$(u^m-k+1)^k < (z-k+1)\dots(z-1)z \leqslant x(x+1)\dots(x+k-1) \leqslant z(z+1)\dots(z+k-1) < (u^m+k-1)^k\,.$$

Let us prove that for $m \geqslant k+2\log_2 k$ and $u \geqslant 2$ we have inequalities

$$(u^m+k-1)^k < (u^k+1)^m \tag{6}$$

$$(u^m-k+1)^k > (u^k-1)^m \tag{7}$$

Proof of inequality (6). Let us apply Bernoully inequality for the quotient of right and left hand sides

$$\frac{(u^k+1)^m}{u^{km}}\cdot\frac{u^{km}}{(u^m+k-1)^k} = \left(1+\frac{1}{u^k}\right)^m\left(1-\frac{k-1}{u^m+k-1}\right)^k \geqslant 1+\frac{m}{u^k}-\frac{k(k-1)}{u^m+k-1}-\frac{mk(k-1)}{u^k(u^m+k-1)}\,.$$

We want prove that the last expression is greater than 1. It is sufficient to establish that the sum of the three last expressions is positive, i. e.

$$\frac{m}{u^k} > \frac{k(k-1)}{u^m+k-1}+\frac{mk(k-1)}{u^k(u^m+k-1)}\,.$$

Multiply by the denominators

$$mu^m > k(k-1)u^k+(k-1)^2m\,.$$

Since $m > k+2\log_2 k \geqslant k+2\log_u k$, then $u^m > k^2u^k$. Replacement of the expression u^m at the left hand side by k^2u^k, and factors $k-1$ at the right hand side by k makes the inequality stronger:

$$mk^2u^k > k^2u^k+k^2m\,.$$

We obtained to the correct inequality $ab > a+b$.

Proof of inequality (7). Consider the quotient $\frac{(u^m-k+1)^k}{(u^k-1)^m} = \frac{(u^m-k+1)^k}{u^{km}} \frac{u^{km}}{(u^k-1)^m}$ and prove that it is greater then 1. Indeed due to Bernoully inequality we have

$$\left(1-\frac{k-1}{u^m}\right)^k \left(\frac{u^k}{u^k-1}\right)^m > \left(1+\frac{m}{u^k-1}\right)\left(1-\frac{k(k-1)}{u^m}\right).$$

Besides $1-\frac{k(k-1)}{u^m} > 1-\frac{k^2}{u^m}$. Hence it is sufficient to prove that

$$1 < \left(1+\frac{m}{u^k-1}\right)\left(1-\frac{k^2}{u^m}\right) = 1+\frac{m}{u^k-1}-\frac{k^2}{u^m}-\frac{mk^2}{u^m(u^k-1)} = 1+\frac{mu^m-k^2u^k+k^2-mk^2}{u^m(u^k-1)}.$$

But $u^m \geqslant k^2u^k$, therefore,

$$mu^m-k^2u^k+k^2-mk^2 > mk^2u^k-k^2u^k+k^2-mk^2 = k^2(m-1)(u^k-1) > 0,$$

QED.

3.5. Like in previous problems it is sufficient to prove two inequalities

$$u^7(u^7+1)(u^7+2)(u^7+3)(u^7+4) < (u^5+1)^7, \quad (8)$$
$$u^7(u^7-1)(u^7-2)(u^7-3)(u^7-4) > (u^5-1)^7. \quad (9)$$

To prove (8) expand parenthesis

$$u^{35}+10u^{28}+35u^{21}+50u^{14}+24u^7 <$$
$$< u^{35}+7u^{30}+21u^{25}+35u^{20}+35u^{15}+21u^{10}+7u^5+1.$$

We see that if $u \geqslant 2$ then each summand of upper row less than corresponding one of the lower row.

To prove (9) expand parenthesis

$$u^{35}-10u^{28}+35u^{21}-50u^{14}+24u^7 > u^{35}-7u^{30}+21u^{25}-35u^{20}+35u^{15}-21u^{10}+7u^5-1.$$

Collect together summands of the same sign.

$$6u^{30}+\quad u^{30}+35u^{21}+35u^{20}+24u^7+21u^{10}+1 >$$
$$> 10u^{28}+21u^{25}+35u^{15}+50u^{14}+7u^5$$

We see that if $u \geqslant 2$ then each summand of lower row less than corresponding one of the upper row.

3.6. This problem and next two ones are taken from [1].

3.7. As an example one can take a set of 17 integers starting from 2184.

3.8.

4 *Общие свойства решений уравнения*

4.1. If $x \leqslant k$ all integers between $x+k$ and $\frac{1}{2}(x+k)$ are factors of the left hand side of the equation. According to Bertrand postulate one of them is a prime number p. Obviously that the left hand side is not divisible by p^2. Therefore it could not be an m-th power.

4.2. According to Sylvester theorem there is a factor $x+i$ that is divisible by a prime $p > k$. Since other factors at the left hand side are not divisible by p the product could be an m-th power only if $x+i$ is divisible by p^m. Then $x+i \geqslant p^m \geqslant (k+1)^m$. If at the same time $x \leqslant k^p$, then $k^p+i \geqslant x+i \geqslant (k+1)^p$. Hence $i > pk$, which is wrong.

4.3. If a number a_i has a prime divisor $p > k$, then other integers among x, $x+1$, …, $x+k-1$ are not divisible by p. Then $x+i$ is divisible by p^{dm}. Since $x+i = a_i z_i^m$, then p is relatively prime with a_i. We got a contradiction.

4.4. Note, if we find at least 5 integers a_i which are divisible by primes 2 and 3 only, then we deduce that the equation has no solutions similarly to the problem 1.10. Such 5 integers really exist, because we have 7 integers a_i with divisors 2, 3 and 5 only, and at most two of them are divisible by 5.

4.5. Similarly to the previous problem we would find 5 integers divisible by 2 and 3 only. The integers a_i have prime factors 2, 3, and 5 only (each factor in the power 0 or 1). The product of all a_i is a perfect square. At most 2 of initial consecutive numbers are divisible by 5, hence at most 2 of a_i are divisible by 5. It is sufficient to consider a case when we have exactly 2 integers a_i which are divisible by 5. This case is possible only if a_0 and a_5 are divisible by 5.

Consider 4 integers $x+1$, $x+2$, $x+3$, $x+4$. We know from the problem 1.5 that their product is not a perfect square. Hence the total power of divisor 2 or the total power of divisor 3 in the product $a_1a_2a_3a_4$ is odd. It is possible only in two cases:

1) There is only one integer among a_1, a_2, a_3, a_4 divisible by 2;
2) There is only one integer among a_1, a_2, a_3, a_4 divisible by 3;

Besides that we have at most 2 integers a_i divisible by 2; the same is true for 3. Then it is easy to see that in each case we have among integers $x+1$, $x+2$, $x+3$, $x+4$ two integers either of the form t^2 or of the form $2t^2$, or of the form $3t^2$. But this is impossible.

4.6. This statement is a part of lemma 1 of [3]. Cancel equal factors. Since $\mathrm{GCD}(n+i, n+j) < k$ and $n > k^m$, we see that any factor at the left hand side does not divide the product at the right hand side.

4.7. Let $a_i = a_j$, where $0 \leqslant j < i < k+1$. Since $n+i = a_i z_i^m > n+j = a_j z_j^m$, then $z_i \geqslant z_j + 1$. Therefore

$$k > a_j z_i^m - a_j z_j^m = a_j\big((z_j+1)^m - z_j^m\big) > a_j m z_j^{m-1} \geqslant a_j^{(m-1)/m} z_j^{m-1} = (a_j z_j^m)^{(m-1)/m} = (x+j)^{(m-1)/m} > x^{1/m}\,,$$

that contradicts to the statement of the problem 4.2.

4.8. It follows from the next solution. One can put $u = v = 1$.

4.9. This statement is lemma 1 from [3]. Assume that

$$a_{i_1} a_{i_2} = a_{j_1} a_{j_2} t^3\,.$$

Let us check that $t = 1$ and sets of indices coincide. WLOG $(x+i_1)(x+i_2) > (x+j_1)(x+j_2)$ (the equality is impossible due to problem 4.6).

Let $t = u/v$ ($\mathrm{GCD}(u,v) = 1$). Then $a_{i_1}a_{i_2}/u^3 = a_{j_1}a_{j_2}/v^3$ and both sides are integers. Let $A = a_{i_1}a_{i_2}/u^3 = a_{j_1}a_{j_2}/v^3$.

By the definition of a_i we have $x+i = a_i z_i^3$, then

$$(x+i_1)(x+i_2) = a_{i_1}a_{i_2} \cdot \frac{s^3}{u^3}\,,$$
$$(x+j_1)(x+j_2) = a_{j_1}a_{j_2} \cdot \frac{r^3}{v^3}\,,$$

where $s = u z_{i_1} z_{i_2}$, $r = v z_{j_1} z_{j_2}$. Then $As^3 > Ar^3$, so $s \geqslant r+1$. Thus

$$(x+i_1)(x+i_2) - (x+j_1)(x+j_2) \geqslant A\big((r+1)^3 - r^3\big) > 3Ar^2\,.$$

Note that $Ar^3 = (x+j_1)(x+j_2) > x^2$, then the last inequality could be rewritten as

$$(x+i_1)(x+i_2) - (x+j_1)(x+j_2) \geqslant 3A \cdot \left(\frac{x^2}{A}\right)^{2/3} \geqslant 3x^{4/3}\,.$$

On the other hand

$$(x+i_1)(x+i_2) - (x+j_1)(x+j_2) < (x+k)^2 - x^2 < 3kx\,.$$

Obtained estimations contradict to each other, because due to problem 4.2 we know that $x > k^3$, and therefore $3kx < 3x^{4/3}$.

4.10. We need to find 20 integers a_i, which are divisible by primes 2, 3, 5, 7 only.

We know that integers a_i are divisible by the primes are not greater than 75 only, i.e. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73. The integers a_i are divisors of 75 consecutive integers therefore:

1) for every prime $p \in [41; 73]$ (there are 9 prime numbers in this interval) at most 2 of a_i are divisible by p;
2) for every prime $p \in [29; 37]$ (there are 3 prime numbers in this interval) at most 3 of a_i are divisible by p;
3) at most 4 of a_i are divisible by 23; at most 4 of a_i are divisible by 19;
4) at most 5 of a_i are divisible by 17;
5) at most 6 of a_i are divisible by 13;

6) at most 7 of a_i are divisible by 11;

So we have at most

$$18 + 9 + 8 + 5 + 6 + 7 = 53$$

integers. Hence at least $75 - 23 = 22$ integers have prime divisors 2, 3, 5, 7 only.

4.11. Consider integers a_i which have prime divisors 2, 3, 5, 7 only. By the previous problem we know that at least 20 integers have that property, but for our purposes it is sufficient to take 10 of them. If we have 10 numbers a_i, we can construct $90 = 10 \cdot 9$ different formal quotients of the form a_i/a_j.

On the other hand, due to the statement of problem 4.9 the quotient of two products of the form $a_i a_j$ (including quotients of the form a_i/a_j) does not equal to qube of rational number. Therefore there exist at most $3^4 = 81$ classes for the values of quotients a_i/a_j for our 10 numbers (each prime 2, 3, 5, 7 divides this quotient in power $0 + 3s$, $1 + 3s$, $2 + 3s$). Hence two quotients belong to the same class and we have a relation

$$\frac{a_i}{a_j} = \frac{a_u}{a_v} \cdot t^3 ,$$

which is restricted by the statement of problem 4.9.

4.12. This beautiful solution we take from [3].

For each prime $p_0 < k - 1$ choose a_i for which $x + i$ is divisible by p_0 to the highest power. Then for $j \neq i$ the the power of p_0 dividing $x + j$ is the same as the power of p_0 dividing $(x + i) - (x + j) = j - i$. Thus, if $a_{i_1}, a_{i_2}, \ldots, a_{i_d}$ is a list of nambers that was NOT chosen (then $d \geqslant k - \pi(k)$) and p_0^a is a maximal power of p_0 that divides the product $a_{i_1} a_{i_2} \ldots a_{i_d}$, then p^a divides also the product $(k - i)!(i - 1)!$, and hence p^a divides $(k - 1)!$ (because $\frac{(k-1)!}{(k-i)!(i-1)!}$ is integer). So the the product of integers that was not chosen divides $(k - 1)!$.

4.13. This proof of Erdős we cite by [2].

Since the integers a_i are square free and pairwise different, $B_x(k) > B'(k)$, where $B'(k)$ is a product of first k square free numbers. It is sufficient to prove that $B'(k) > (4/3)^k k!$ when $k \geqslant 24$.

Induction by k. Base, $k = 24$.

$$\frac{26 \cdot 29 \cdot 30 \cdot 31 \cdot 33 \cdot 34 \cdot 35 \cdot 37}{4 \cdot 8 \cdot 9 \cdot 12 \cdot 16 \cdot 18 \cdot 20 \cdot 24} > \left(\frac{4}{3}\right)^{24} .$$

Step of induction. For $r \geqslant 9$ the number of square free integers that does not exceed r is at most $r - \left[\frac{r}{4}\right] - 1 < \frac{3}{4} r$. Therefore for n-th square free integer for $n \geqslant 7$ is at most $\frac{4}{3} n$.

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