POINTED GRAPHS IN THE SPHERE AND IN THE PLANE
SOLUTIONS

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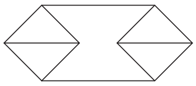
(1) Not every. If a rule is knotted and its ends are sufficiently long, it is not straightable (Fig. 1).

(2) No, it doesn't, because a polygon is contained in the convex hull of its vertices.

(3) The above argument for the sphere is wrong since there is no convex hull for spherical polygons. An example (Fig. 2) can be constructed as follows. Take two antipodal points on the sphere and connect them by two different great semicircles. We get a polygon with two vertices and two edges. Such a polygon can be arbitrary narrow (this fact will be used later). Breaking somewhat its edges, we get a 4-gon with 2 convex angles. Note that the positions of the original two vertices change and the polygon gets longer.

(4) For instance,  .

(5) Place on the sphere two examples of the graph from Fig. 3. The result can be altered in the following way. Add two new vertices and a new edge as is depicted in Fig. 9. The pointed property is preserved.
Repeat this trick sufficiently many times and get the required.

(6) To construct such a graph, put 12 points in the plane (in a non-convex position). Start adding edges maintaining the pointed property. Sooner or later

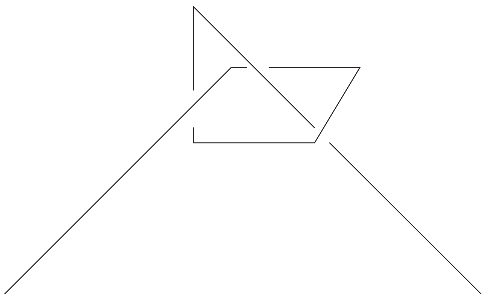


Figure 1.

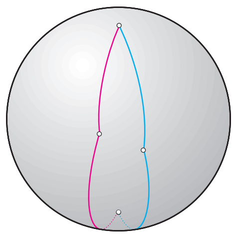


Figure 2.

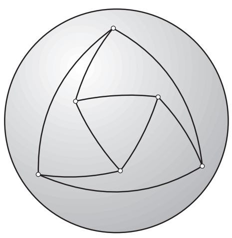


Figure 3.

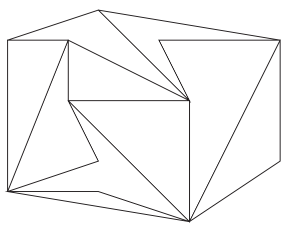


Figure 4.

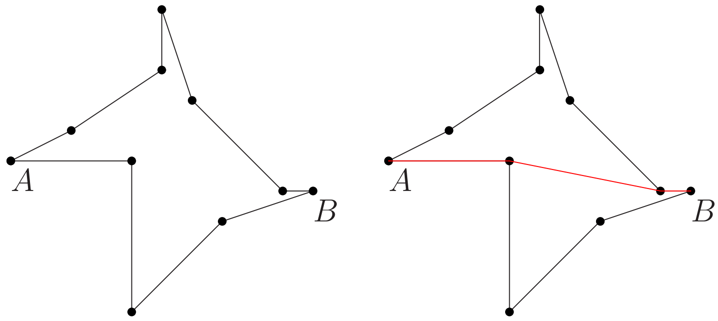


Figure 5.

this becomes impossible. This means that we get a maximal pointed graph. For instance, see Fig. 4.

(7) Prove 1.

Suppose the contrary. Consider the convex hull C of the set of all vertices of the graph and an edge of C which is not an edge of the graph. It can be added to the graph preserving the pointed property. A contradiction to maximality.

Prove 2.

Let some tile K have more than 3 convex angles. Take two convex vertices such that they are disjoint on the boundary of K by some other convex vertices. Denote them by A and B. Take the shortest path from A to B lying in the polygon K. It is a broken line containing some segment which doesn't lie on the boundary of K (Fig. 5). It can be added to the graph preserving the pointed property. A contradiction.

(8) Use the previous problem.

Let V be the number of vertices of the graph and E be the number of its edges. The Euler formula yields that the number of tiles $F = 2-V+E$. Count the total number C of convex angles (for all the tiles) in two different ways.

On the one hand, each bounded tile gives exactly 3 convex angles. Therefore $C = 3(F-1) = 3(1-V+E)$.

On the other hand, each vertex gives one non-convex angle. The total number of angles (both convex and non-convex) equals $2E$. Therefore $C = 2E - V$.

Combining these equalities, we get the required.

(9) This statement for the sphere is wrong since there exist spherical polygons with just two convex angles (see Problem 2). The number of such exceptional polygons rules the value of the expression $E - 2V$.

It looks like this. Place on the sphere D disjoint 4-gons from the problem 2 (Fig. 6 depicts four 4-gons). Add edges to get a maximal pointed graph. The vertices-edges count shows that $E = 2V + D - 6$.

(10) (a)Yes. Take a convex 18-gon and draw necessary diagonals.

(b) Yes. A required graph can be obtained from a complete graph with 3 vertices (vividly speaking, from a triangle) by a consecutive applying of the following operation. Each time we add to the graph a new vertex and connect it with edges to two old vertices. This operation can be applied preserving the pointed property. One should start with a triangle and add new vertices on it.

(c) No. This graph contains (as a subgraph) a complete graph with 4 vertices (i.e., a graph with 4 vertices such that each pair is connected by an edge). It is impossible to draw such a graph in the plane (see Problem 8).

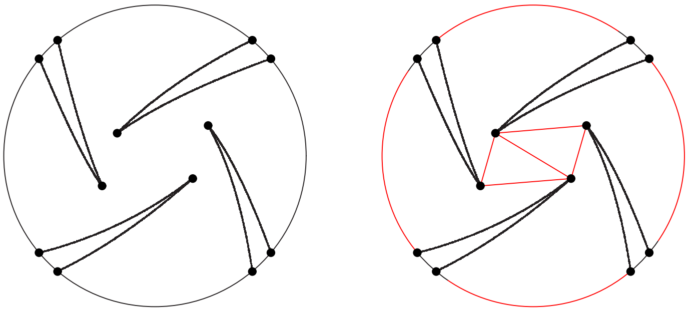


Figure 6.

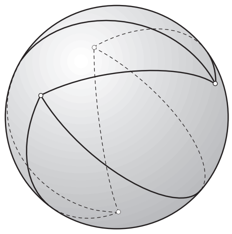


Figure 7.

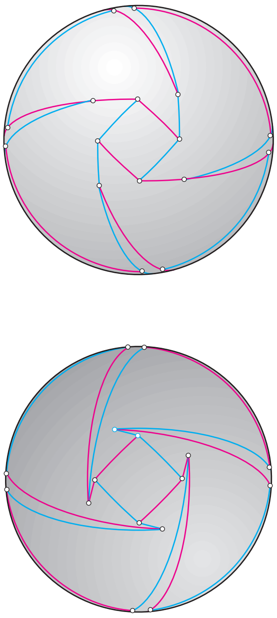


Figure 8.

(11) No. The reasons are similar to those of solution 14.

(12) Yes.

a) See Fig. 7.

b) Fig. 8 depicts the upper and lower hemispheres.

c) Describe the following refining operation. Suppose two points lying on red edges can be connected by a segment avoiding intersections with edges of the graph. Break somewhat the edges and add a new blue edge as is shown in Fig. 9. Analogously, red edges connecting blue points can be added.

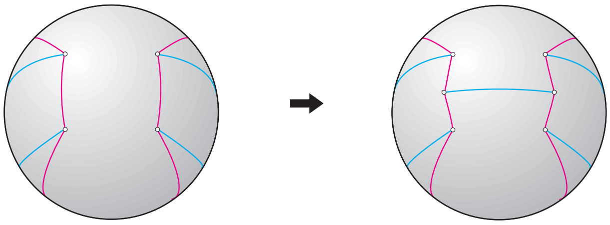


Figure 9.

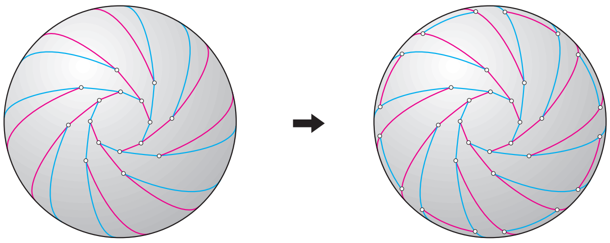


Figure 10. Depicted is the upper hemisphere. The lower one looks similarly.

Iterative repetition of this operation allows to refine the graph from the previous item. It is remarkable that the graph from Fig. 7 is not refinable in this way.

(13) a) No. The number $n(\alpha)$ should be even.

b) Note that the convex vertices of the tile are exactly those points where the color changes. Therefore, such polygons in the plane do not exist, for there are no polygons in the plane with exactly 2 convex angles. But this is possible for the sphere. For instance, the graphs in Fig. 10 are properly colored. Each of them gives 8 polygons with 2 color changes. For the graph on the left these polygons are di-gons. The trick from Problem 6 turns them to 4-gons (the graph on the right).

(14) Let V be the number of vertices, E be the number of edges. Trivalence of the graph implies that $E = 3V/2$. Euler formula implies that $F = 2 - V + E = 2 + V/2$.

Count the total number of color changes H taken over all the tiles. Every tile gives either 2 changes or a number greater or equal than 4. Therefore, $H \geq 2N(\Gamma) + 4(F - N(\Gamma)) = 4F - 2N$. This yields $N \geq 2F - H/2$.

On the other hand, the three tiles adjacent to a vertex give exactly two color changes. This gives $H = 2V$.

Combining this equality and the latter inequality, we arrive at $N \geq 2F - H/2 = 4 + V - V = 4$.

(15) The solution is similar to that of Problem 9. Place on the sphere $N(\Gamma)$ disjoint 4-gons with two convex angles. Add necessary edges to get a trivalent graph (see Fig. 10). The graph obtained admits proper coloring. Note that for odd number of 4-gons the graph obtained has no proper coloring, so the next problem is more difficult.

(16) Fig. 11 depicts the upper and the lower hemispheres.

(17) * This is an unsolved problem. Any progress is of a great interest for the author of the problems Gaiane Panina.

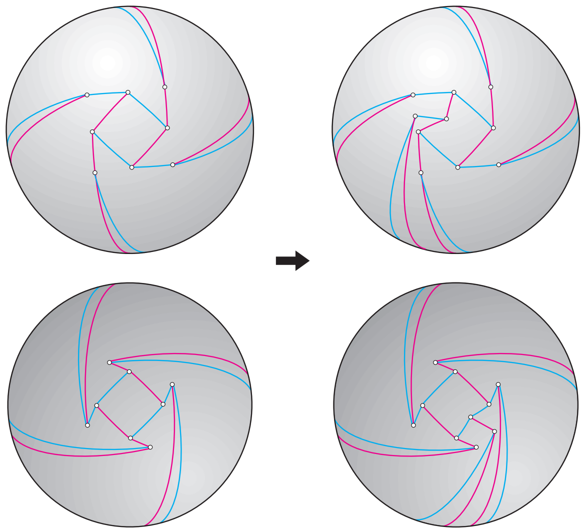


Figure 11.

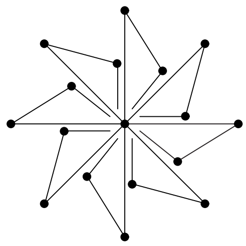


Figure 12.

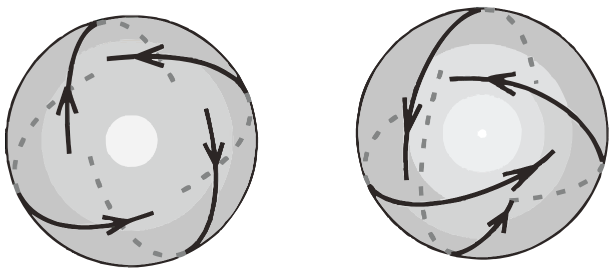
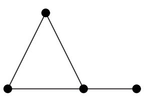
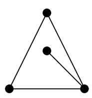
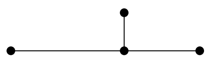
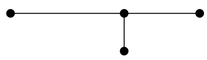


Figure 13.

(18)   Or, say,  

(19) See Fig. 12

(20) See Fig. 13

For every position of semicircles in Fig. 13 construct a graph in the following way. The vertices of the graph correspond to the great semicircles. Two vertices (say, 1 and 2) are connected by an edge if and only if one of the corresponding semicircles (say, 1) can be prolonged such that the prolongation meets first the semicircle 2 (and only after that it meets all the other semicircles). It remains to observe that

— the two positions give different graphs,

— isotopic positions have the same graphs.

P.S. It is worthy to visit the following web sites both for fun and for work.
http://theory.csail.mit.edu/ edemaine/linkage/animations/
http://www.ams.org/featurecolumn/archive/links1.html
http://www.arxiv.org/abs/math/0612672
An interesting book to appear:
http://www.cambridge.org/catalogue/catalogue.asp?isbn=0521857570