POINTED GRAPHS IN THE SPHERE AND IN THE PLANE

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As it was recently discovered, sometimes it makes sense to consider objects that are *as non-convex as possible.* For instance, to embed planar graphs in the plane such that the tiles do not look like convex polygons.

This helps to solve a series of problems which seemed to be independent: A. D. Alexandrov's problem on uniqueness of convex surfaces, carpenter's rule problem, some algorithmic problems on graphs embeddings...

Many problems are still to be solved. For instance, the problem N 17. Any progress in this direction would be of a great interest.

Here is the most popular problem.

Carpenter's rule problem. It was formulated in 1970 by the leading authority in the rigidity theory R. Connelly.

A carpenter's rule is a non-crossing broken line in the plane with a finite number of edges. It should be considered as a so-called *linkage or a bar-and-join mechanism*, i.e., a collection of rigid bars (line segments) that are hinged together at the endpoints (vertices) so that they can rotate with respect to one another.

An *isotopy* of a carpenter's rule is its continuous movement in the plane which avoids self-intersections and preserves the edges lengths.

Is each carpenter's rule straightable in the plane (the self-intersections should be avoided during the straightening)? (See Fig. 1)

Although the carpenter's rule problem looks like an olympiad one, it has no elementary solution (the problem has been open for 30 years).

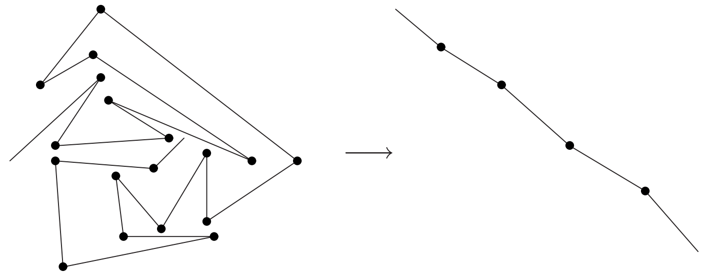

Figure 1. Straightening of a carpenter's rule

(1) Carpenter's rule problem in the 3-dimensional space.
Is each non-crossing broken line straightable in $\mathbb{R}^3$?
(Mostly probably you will find a solution, but a detailed proof would need some advanced knowledge.)

Polygons in the plane and in the sphere. *A great semicircle* in the sphere is the intersection of the sphere with a plane passing through its center.

A polygon is a non-crossing closed broken line (its edges are line segments in the plane or the segments of great circles in the sphere) taken together with a bounded part of the plane (or the sphere) bounded by the broken line.

A polygon can be non-convex. Some of its angles can be greater than π. Such angles are called non-convex. The angles smaller than π are called convex.

(2) Does there exist a polygon in the plane with exactly two convex angles?
(3) (A simple but a very important problem. To be used in the sequel.) Draw a 4-gon in the sphere with exactly two convex angles.

Pointed graphs. A graph in the plane or in the sphere is called *pointed* if

- all its edges are line segments (for a graph in the plane) and segments of great circles (for a graph in the sphere);
- it is non-crossing (its edges have no intersections);
- (pointed property) every vertex is incident to an angle greater than π (Fig. 2.);
- its vertices are in generic position.

A graph is called *trivalent* if each its vertex has exactly three adjacent edges.

(4) Draw a pointed trivalent graph in the plane.
(5) Does there exist a pointed graph in the sphere of radius 1 such that the area of each tile is smaller than 1/10 and all its edges are shorter than 1/10?

Maximal pointed graphs. A graph in the plane or in the sphere is called *maximal pointed* if it is impossible to add a new edge (without adding new vertices) preserving the pointed property.

(6) Draw a maximal pointed graph with 12 vertices in the plane. The vertices must not lie in convex position (i.e. they shouldn't serve as vertices of some convex polygon).
(7) Prove the following two properties of a maximal pointed graph in the plane.
1. There is a closed broken line consisting of some edges of the graph and bounding a convex polygon M. All vertices of the graph lie in M.
2. Each of the bounded tiles has exactly 3 convex angles.

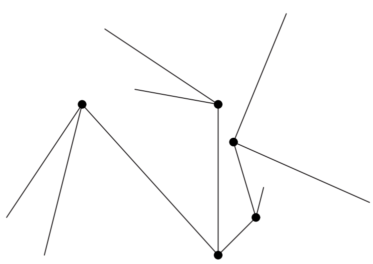

Figure 2. Part of a pointed graph

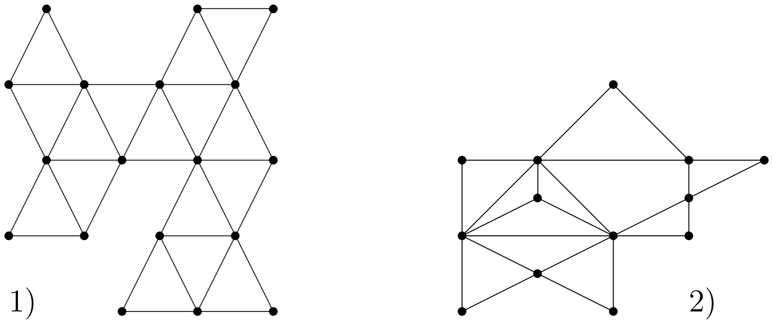

Figure 3.

Euler formula. For a connected graph in the plane or in the sphere, we have

$$V - E + F = 2\,.$$

Here V is the number of vertices, E is the number of edges, F is the number of tiles (for a graph in the plane, the unbounded tile is also taken into account).

(8) Let Γ be a maximal pointed graph in the plane. Prove that

$$E = 2V - 3\,.$$

(9) Draw an example of a pointedgraph in the sphere such that
a) $E = 2V - 2$.
b) $E = 2V + 2007$.

(10) a) Can the graph 1 (Fig. 3) be redrawn in the plane as a pointed graph? ("To redraw" means to construct another graph with another vertices (and possibly with another edge lengthes), but preserving the vertices-edges correspondence.
b) Can the graph 1 (Fig. 3) be redrawn in the plane as a pointed graph such that the unbounded tile is a complement of a triangle?
c) Can the graph 2 (Fig. 3) be redrawn in the plane as a pointed graph?

Problems presented after semifinal

Trivalent pointed graphs. A pointed trivalent graph *admits a proper coloring* if each its edge can be colored either red or blue such that at each of its vertices the graph looks like as is depicted in Fig. 2 (two side edges are of the same color, whereas the middle edge is of the other color).

(11) Does there exist a properly colored trivalent pointed graph in the plane?

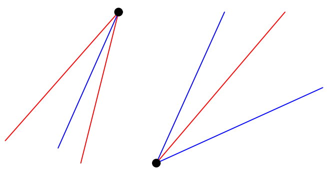

Figure 4. Proper coloring

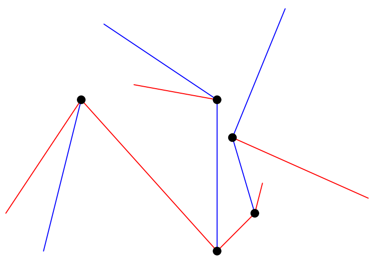

Figure 5. Part of a properly colored graph

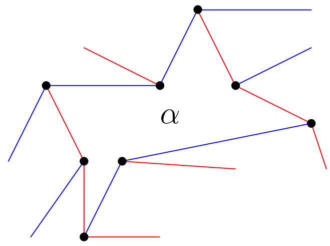

Figure 6.

(12) Does there exist a trivalent properly colored pointed graph in the sphere?
 a) without additional restrictions?
 b) whose edges are shorter than π?
 c) whose edges are shorter than $\pi/100$?
(13) Let Γ be a trivalent properly colored pointed graph in the sphere. For a tile α denote by $n(\alpha)$ the number of color changes when going along the boundary of α (for instance, for the tile in Fig. 6, we have $n(\alpha) = 4$).
 We consider such graphs that $n(\alpha) = 0$ holds for no tile.
 a) (A joke) Does there exist a graph and a tile such that $n(\alpha) = 2007$?
 b) (Not a joke at all) Does there exist a graph and a tile such that $n(\alpha) = 2$?
(14) Let Γ be a trivalent pointed graph in the sphere with a proper coloring. Let $N(\Gamma)$ be the number of tiles such that $n(\alpha) = 2$.
 Prove that $N(\Gamma) \geq 4$.
(15) Draw a trivalent properly colored pointed graph in the sphere such that its edges are shorter than π and $N(\Gamma) = 4, 6, 8, 10...$
(16) Draw a trivalent properly colored pointed graph in the sphere such that its edges are shorter than π and $N(\Gamma) = 5$.
(17) Given a finite set of points in the sphere, under what condition it is the set of vertices of some trivalent pointed properly colored graph?
 (Try to find a non-trivial necessary or sufficient condition)

Non-isotopic linkages. Let a graph be represented by a linkage (a bar-and-join mechanism) in the plane or in the sphere in two different ways Γ_1 and Γ_2.

We say that the position Γ_1 is *isotopic* to Γ_2 if Γ_1 can be pulled to the position Γ_2 without self-intersections. The edges lengths must not change during such a movement.

(18) Draw a linkage with two non-isotopic positions.

(19) A cool example (E. Demain)
 Redraw the graph (Fig. 7) in the plane in two non-isotopic ways. The spider must have equal legs. (I.e., the upper parts of the legs must be of the same length, the parts from knees to the feet must be of the same length as well, and the feet must be equal.)
(20) Find two non-isotopic positions of four great semicircles in the sphere. (In other words, the spherical linkage in question consists of 4 disconnected bars of length π.)

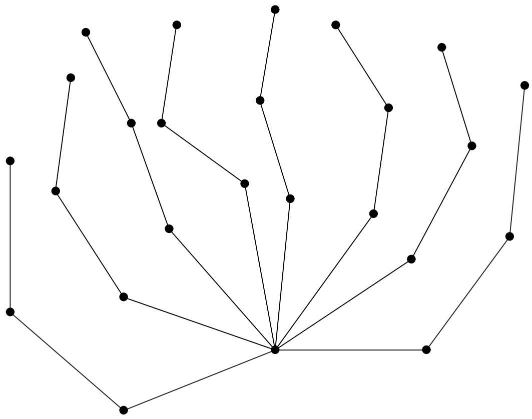

Figure 7.