Hints and solutions to the problems from sections A and B

♦ **A 1.** a) $a-2x=0$
b) $a-2x-1=0$
c) $a-x^2=0$
d) $a-x^3=0$

♦ **A 2.** $\begin{cases} \mathfrak{D}_1(x_1,\dots,x_n)=0 \\ \dots \\ \mathfrak{D}_m(x_1,\dots,x_n)=0 \end{cases} \iff \mathfrak{D}_1(x_1,\dots,x_n)^2+\dots+\mathfrak{D}_m(x_1,\dots,x_n)^2=0$

♦ **A 3.** If the sets $\mathbb{A}=\{\mathfrak{F}(x_1,\dots,x_n)=0\}$ and $\mathbb{B}=\{\mathfrak{G}(x_1,\dots,x_n)=0\}$ then $\mathbb{A}\cap\mathbb{B}=\{\mathfrak{F}(x_1,\dots,x_n)\cdot\mathfrak{G}(x_1,\dots,x_n)=0$ and $\mathbb{A}\cup\mathbb{B}=\mathfrak{F}(x_1,\dots,x_n)^2+\mathfrak{G}(x_1,\dots,x_n)^2=0$.

♦ **A 4.**

♦ **A 5.** a) $a=b+x+1$
b) $a=b\cdot x$
c) $a=c\cdot x+b$ and $b<c$
d) $\left[\begin{array}{l} \begin{cases} a=c\cdot x+b \\ b\leqslant c-b \end{cases} \\ \begin{cases} a=c\cdot x-b \\ b<c-b \end{cases} \end{array}\right.$
e) $b=c\cdot a+x$ and $x<c$

♦ **A 6.** $a(x_1-x_2)+b(y_1-y_2)=1$ if and only if then a and b co-prime.

$a=kt$, $b=lt$, $GCD(k,l)=1$ if and only if then $t=GCD(a,b)$.

$LCM(a,b)=\frac{ab}{GCD(a,b)}$

♦ **A 7.** a) $x^2<a<(x+1)^2$

♦ **A 8.** a) $d=k^2\Rightarrow(x-ky)(x+ky)=1\Rightarrow x-dy=x+dy\Rightarrow y=0\Rightarrow x=\pm1$

b) Since $u_3-v_3\sqrt{d}=(u_1-v_1\sqrt{d})\cdot(u_2-v_2\sqrt{d})$, then $u_3^2-v_3^2d=(u_1^2-v_1^2d)(u_2^2-v_2^2d)=1$

c) $|x'|=\sqrt{1+y'^2d}\Rightarrow|x'|+|y'|=\sqrt{1+y'^2d}+|y'|$ is monotonic by $|y'|$. Let $(x+y\sqrt{d})^n<x'+y'\sqrt{d}<(x+y\sqrt{d})^{n+1}$ $(x>0,\ y>0,\ x'>0,\ y'>0)$. Multiplying it by $(x-y\sqrt{d})^n>0$ we get $1<a+b\sqrt{d}<x+y\sqrt{d}$, such that $a^2-b^2d=1$. Since $0<a+b\sqrt{d}$, then $a-b\sqrt{d}>0$. Since $a-b\sqrt{d}<1<a+b\sqrt{d}$, then $b>0$. Since $a-b\sqrt{d}>0$, then $a>0$. So, since while $|x|+|y|$ is increasing, $|y|$ is increasing, $|x|+|y|\sqrt{d}$ is increasing, then (x,y) is not a minimal solution.

♦ **A 9.** a) See the previous solution.

b) This statement (and that is $x_n \equiv 1 \pmod{k-1}$) can be proved by induction on n.

c) It can be solved as the particular case of the next problem: $1 = x^2 - (\frac{b^2}{4} - - 1)y^2 = (x + \frac{b}{2}y)^2 - b(x + \frac{b}{2}y)y + y^2$

⧫ **A 10.** Consider the minimal solution and prove that every other solution can be constructed as the iteration of

⧫ **A 11.** This can be proved by induction of n.

⧫ **A 12.** This can be proved by induction of l.

⧫ **A 13.** This can be proved by using the previous problem (since $\alpha_{km}(b)$ divides on $\alpha_m(b)$, $\alpha_{4m-1}(b) = \alpha_{2m-1}(b)(\alpha_{2m}(b) - \alpha_{2m-2}(b))$ and $\alpha_{2m}(b) = \alpha_m(b)(\alpha_{m+1}(b) - \alpha_{m-1}(b)))$.

⧫ **A 14.** This can be proved by induction on n.

⧫ **A 15.** No comments

⧫ **A 16.** It is obvious that $\alpha_n(b)(\operatorname{amod} v) = \alpha_n(w)(\operatorname{amod} v)$ and $n(\operatorname{amod} u) = \alpha_n(w)(\operatorname{amod} u)$. Therefore, since $v > 2\alpha_k(b) > 2\alpha_n(b)$ and $u > 2k > 2n$, then we get the statement we need

⧫ **A 17.** No comments

⧫ **A 18.** The inequality $k \leqslant \alpha_k(b)$ can be proved by induction on k

⧫ **A 19.** No comments

⧫ **A 20.** This can be proved by induction on k

⧫ **A 21.** This can be proved by induction on n

⧫ **A 22.** $\frac{(bn+4)^c}{(n-1)^c} \geqslant \frac{\alpha_{c+1}(bn+4)}{\alpha_{c+1}(n)} \geqslant \frac{(bn+3)^c}{n^c}$. The left and right parts of this inequality tend to b^c.

⧫ **A 23.** No comments

⧫ **B 1.** $c = \left[\frac{a}{b^{k-1}}\right] \pmod{b^k}$

⧫ **B 2.** a) $b = 2^n$, $a = (2^n + 1)^n$
b)

⧫ **B 3.** p is prime if and only if then $GCD(p, (p-1)!) = 1$

⧫ **B 4.** $(x_0 + 1)(1 - D(x_0, x_1, \ldots, x_m)^2) - 1 \geqslant 0$ if and only if then $1 - D(x_0, x_1, \ldots, x_m)^2 > 0$, i.e. $D(x_0, x_1, \ldots, x_m) = 0$. So, $a = (x_0+1)(1-0)-1 = x_0$.

⧫ **B 5.** It follows from the two previous problems

♦ **B 6.** a) The first addendum in this sum equals to the number of the factors divisible by p, the second — by p^2 etc.

b) $\sum(\left[\frac{m+n}{p^k}\right] - \left[\frac{m}{p^k}\right] - \left[\frac{n}{p^k}\right])$ is equal to the demanded number.

♦ **B 7.** $\binom{x_n p^{n-1}+\ldots+x_1 p+x_0}{y_n p^{n-1}+\ldots+y_1 p+y_0}$ divides on p if and only if then there exists i, such that $x_i < y_i$.

♦ **B 8.** $a < e \cdot \frac{p^n-1}{p-1}$

♦ **B 9.** $(y_n b_1^{n-1} + \ldots + y_1 b_1 + y_0) - (y_n b_2^{n-1} + \ldots + y_1 b_2 + y_0) \equiv 0 \pmod{b_2 - b_1}$ (since $b_1^k - b_2^k = (b_1 - b_2)(b_1^{k-1} + b_1^{k-2} b_2 + \ldots + b_2^{k-1})$)

♦ **B 10.** $a_1 = a_2 \pmod{b_2 - b_1}$ (since $a_1 < b_1^n < b_2 - b_1$)