Invariants of polygons

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ANNOUNCEMENT

I. DISSECTIONS OF A TRIANGLE.

Definition. Two similar triangles in a plane are called *oriented oppositely* if one of them includes angles α, β, γ clockwise, and another one counterclockwise (Fig. 1, angles α, β, γ are supposed to be mutually unequal).

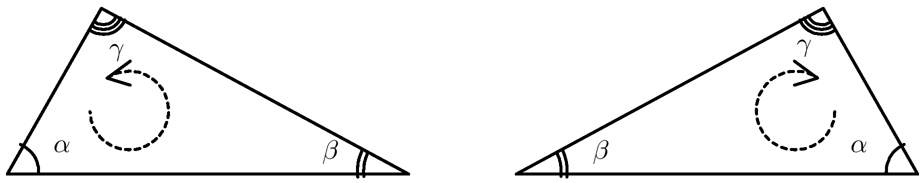

Fig. 1.

Problem A. A cake is of triangular shape. The box for the cake has shape of a triangle equal to the given one but oriented oppositely. Is it always possible to dissect the cake into two parts such that they can be put into the box without flipping?

Problem B. Is it valid that any triangle can be dissected into triangles similar to it but oriented oppositely?

Denote the angles of the triangles in the above problems as α, β and γ. The most interesting in these problems are the specific instances of dissections. For example, in the cases $\alpha = 90°$, $\alpha = 2\beta$ or $\alpha = 3\beta$ the cake in Problem A can be dissected. (Dissect it!).

Call the numbers α, β and γ *commensurable* if $k\alpha + l\beta + m\gamma = 0$ for some integers k, l and m, not vanishing simultaneously. The main purpose in the first part of the project is to prove the following statement:

Statement I. If α, β and γ are incommensurable then the cake in Problem A and the triangle in Problem B cannot be dissected.

II. DISSECTIONS OF A RECTANGLE.

Problem C. (HILBERT'S THIRD PROBLEM) Prove that a regular tetrahedron cannot be dissected into a finite number of polyhedrons which can be combined into a cube.

Problem D. A room is of rectangular shape with side ratio x. The floor of the room is covered by rectangular tiles with the same side ratio. Furthermore, some tile is oriented crosswise, not lengthwise (Fig. 2). Prove that x is a root of a polynomial with integer coefficients.

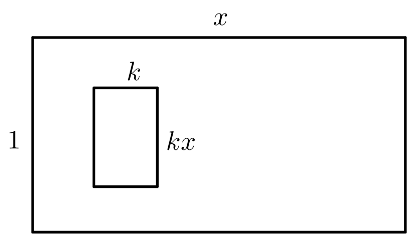

Fig. 2.

Surprisingly, Hilbert's third problem can be solved by considering dissections of rectangles only, not of polyhedrons! In the problems following the preliminary finish, a new version of elementary solution of Hilbert's third problem is proposed which is based on this idea.

As regards Problem D, it occurs to be possible to solve it by means of... physical interpretation! Specifically, to every dissection of a rectangle we attach a circuit formed of resistors.

All the problems concerned admit a common approach based on using of *invariants of polygons*.

I. DISSECTIONS OF A TRIANGLE.

Constructions

1. Dissect the cakes of the indicated form (Fig.3) into two parts in the described mode:
(a) $\alpha = 90°$; **(b)** $\alpha = 3\beta$; **(c)** $\alpha = 2\beta < 90°$; **(d)** $\alpha = 2\beta > 90°$; **(e)*** $\alpha = 30°$, $\beta = 20°$, $\gamma = 130°$;
(f)* $\alpha = \frac{n+1}{n}\beta$, n integer. **(g)** Dissect an arbitrary cake into 3 parts in the described mode.

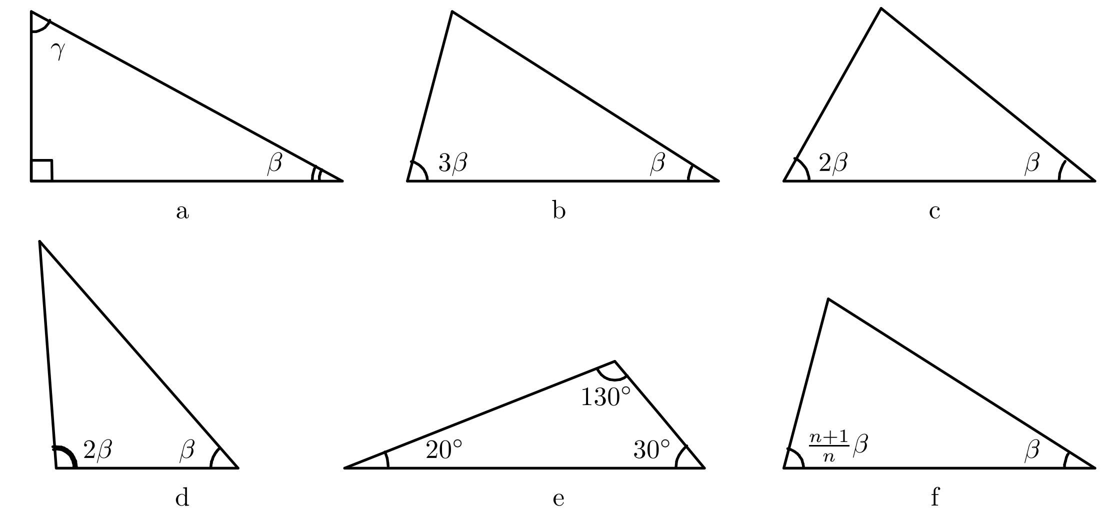

Fig. 3.

2. Dissect a triangle having angles α, β, γ into n triangles similar to it:
(a) $\alpha = 90°$, $n = 2$; **(b)** $\alpha = 30°$, $\beta = 30°$, $\gamma = 120°$, $n = 5$; **(c)** α, β, γ arbitrary, $n \geq 4$, $n \neq 5$.

3. Let α, β, γ be distinct and different from $90°$. Then the triangle from Problem B cannot be dissected into:
(a) 2 parts; **(b)** 3 parts; **(c)** 4 parts.

Invariants.

Let M be an arbitrary polygon. At each its side, let an arrow mark the direction such that points close to this side from the *left* belong to the polygon, and those from the *right* do not (Fig. 4). Now choose some *directed line* l, that is, a line and a direction on it marked by an arrow.

Let $J_l(M)$ denote the algebraic sum of lengths for all sides of M parallel to l, such that those sides having *the same* direction as l (sides AB, DE and FG at Fig. 5), are taken with + sign, and those having the *opposite* direction (side KL at Fig. 5) are taken with − sign. If M has no sides parallel to l, we set $J_l(M) = 0$. The number $J_l(M)$ is called *the additive invariant.*

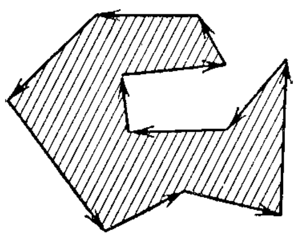

Fig. 4.

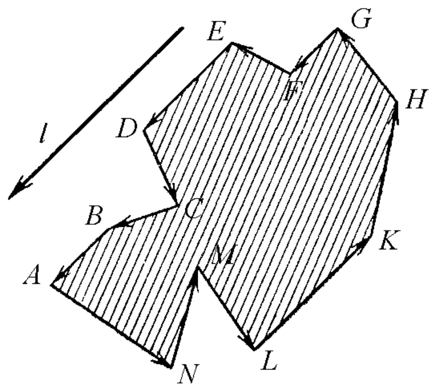

Fig. 5.

4. **(a)** Describe all convex polygons M such that $J_l(M) = 0$ for any directed line l.
(b) A polygon M is dissected into several polygons $M_1, \dots, M_k$. Then $J_l(M) = J_l(M_1) + \dots + J_l(M_k)$.
(c) A polygon M is dissected into several polygons, and they were combined to form a new polygon M' using only parallel translations of the parts. Then $J_l(M) = J_l(M')$.
(d) A convex polygon M is dissected into several polygons, and they were combined to form a square using only parallel translations of the parts. Then M is central symmetrical.

Let ϕ be some angle. Let $J_{l,\phi}(M)$ denote the sum of $J_{l'}(M)$ for all distinct lines l' obtained from some directed line l by rotations through angles divisible by ϕ (divisibility is up to 2π). The expression $J_{l,\phi}(M)$ makes sense since only a finite number of terms does not equal zero in this sum.

5.(a) A polygon M is dissected into several polygons, and theys are combined to form a polygon M' by rotating each part through some angle divisible by ϕ. Then $J_{l,\phi}(M) = J_{l,\phi}(M')$.
(b) Given l and ϕ, describe all triangles M such that $J_{l,\phi}(M) = 0$.
(c) Let an angle ϕ be incommensurable with π. Let M and M' be two equal non-isosceles triangles such that for any directed line l we have $J_{l,\phi}(M) = J_{l,\phi}(M')$. Then the sides of these triangles can be numbered so that the angles between sides with equal numbers are divisible by ϕ.
(d) Let the cake from Problem A be dissected into two parts which are packed into a box, being rotated: one part through some angle ϕ, and another part through some angle ψ. Suppose the angle $\phi - \psi$ is incommensurable with π. Prove that the angles $2(\alpha - \beta)$, $2(\beta - \gamma)$ and $2(\gamma - \alpha)$ are divisible by $\phi - \psi$.
(e)* Prove Statement I for Problem A.

To every directed line XY in the plane, attach a number $f(XY)$ so that it changes sign when the direction of the line is changed: $f(XY) = -f(YX)$. Let $M = X_1X_2 \dots X_n$ be an arbitrary polygon whose vertices are numbered counterclockwise. Set

$$J_f(M) = f(X_1X_2)|X_1X_2| + f(X_2X_3)|X_2X_3| + \dots + f(X_nX_1)|X_nX_1|,$$

where $|X_1X_2|, |X_2X_3|, \dots, |X_nX_1|$ are side lengths of the polygon, and $X_1X_2, X_2X_3, \dots, X_nX_1$ are the corresponding directed lines.

6. (a) A polygon M is dissected into several polygons $M_1, \dots, M_n$. Then $J_f(M) = J_f(M_1) + \dots + J_f(M_n)$.
(b) Let triangle ABC be dissected into triangles $A_iB_iC_i$ similar to it and oriented oppositely. Prove that for any i the angle between directed lines A_iB_i and AB can be represented in the form $k\alpha + l\beta + m\gamma$, where k, l, m are integers.
(c) Suppose the angles of triangle ABC are incommensurable. Construct a function $f(XY)$ such that $J_f(ABC) \neq 0$ but $J_f(A_iB_iC_i) = 0$ for any triangle $A_iB_iC_i$ similar to ABC and oriented oppositely.
(d) Prove Statement I for Problem B.
(e)* Does there exist a non-right non-isosceles triangle which can be dissected into triangles similar to it but oriented oppositely?

7. Is it possible to dissect a circle into a finite number parts by segments and arcs so that the parts can form a square?

II. DISSECTIONS OF A RECTANGLE.

Constructions.

8. Dissect a cube into 6 equal tetrahedrons.

9. Tile as required in Problem D the rooms with the following side ratio:
(a) $x = \sqrt{2}$; **(b)** $x = \sqrt{p/q}$, p and q integer; **(c)** $x = \sqrt[4]{2}$; **(d)*** $x = \sqrt{r}$ where r is a periodic continued fraction;
(e)* $x = \sqrt{s}$ where s is a root of a cubic polynomial with integer coefficients having no rational roots;
(It is asked to construct such a tiling only for a single value of s, not for any s satisfying this property.)
(f) Tile an arbitrary room by n bars oriened lengthwise, for $n \geq 4$, $n \neq 5$.

Hilbert's Third Problem: reduction to a planimetric problem.

Let M be a polyhedron. Let $l_1, l_2, \dots, l_n$ be lengths of its edges, $\alpha_1, \alpha_2, \dots, \alpha_n$ be dihedral angles between the corresponding edges. Attach to M a set of rectangles $l_i \times \alpha_i$ in a plane, such that sides l_i are horizontal and sides α_i are vertical (Fig. 6).

Call two such sets *rectangular-scissor-congruent (□-scissor-congruent)* if each rectangular from some set can be dissected into several rectangles which can be combined to form the other set using only parallel translations of parts (Fig. 7). Two polyhedrons are *scissor-congruent* if some of them can be dissected into several polyhedrons which can be combined to form the other polyhedron.

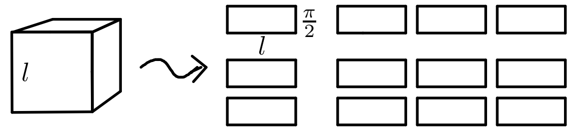

Fig. 6.

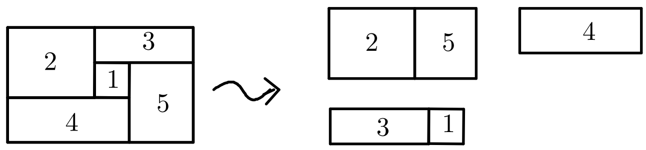

Fig. 7.

Lemma I. If two polyhedrons are scissor-congruent then the corresponding sets of rectangles become □-scissor-congruent after adding appropriate rectangles of the form $l \times \pi$.

The proof of this lemma is contained in Problem 10.

Suppose a convex polyhedron M is dissected into polyhedrons $M_1, M_2, \dots, M_k$.

10. (a) Let e be an edge of a plyhedrons M, l is its length, and α is the dihedral angle at this edge. Denote by $l_1, l_2, \dots l_n$ the lengths of the edges in M_i, belonging to e, and by $\alpha_1, \alpha_2, \dots, \alpha_n$ the dihedral angles at the corresponding edges. Then the rectangle $l \times \alpha$ can be dissected into n rectangles $l_1 \times \alpha_1, \dots, l_n \times \alpha_n$.
(b) Let ℓ be a line in the space not containing the edges of a polyhedron M. Denote by $l_1, l_2, \dots l_n$ the lengths of the edges in polyhedrons M_i, belonging to ℓ, and by $\alpha_1, \alpha_2, \dots, \alpha_n$ the dihedral angles at the corresponding edges. Then the set of n rectangles $l_1 \times \alpha_1, \dots, l_n \times \alpha_n$ is □-scissor-congruent to some rectangle of the form $l \times \pi$.
(c) Prove Lemma I.
(d) Prove that the dihedral angle θ at an edge of the regular tetrahedron is incommensurable with π.

Hilbert's Third Problem: solution of the planimetric problem.

Lemma II. If θ and π are incommensurable, then rectangles $a \times \theta$ and $b \times \pi$ are not □-scissor-congruent for any a and b. Moreover they remain not □-scissor-congruent after adding any rectangles of the form $l \times \pi$.

The proof of this lemma is contained in Problem 11.

Let some set of rectangles be given. We may obtain a new set by dissecting one of the given rectangles into two rectangles. This operation is called an *elementary transformation* of the set.

11. (a) Two sets of rectangles are □-scissor-congruent then one of them can be obtained from the other one by a sequence of elementary and inverse to them transformations.

Assume that θ and π are incommensurable. Suppose that the rectangle $b \times \pi$ is obtained from the rectangle $a \times \theta$ by a sequence of elementary and inverse to them transformations. Let $\theta, \pi, y_1, y_2, y_3, \dots, y_N$ be the lengths of vertical sides of all the rectangles occuring in this sequence of transfornations. Set $Y = \{\theta, \pi, y_1, \dots, y_N\}$.

(b) There exist numbers $y_1', y_2', \dots, y_n' \subset Y$ such that any number $y \in Y$ is uniquely represented in the form $y = p\theta + q\pi + p_1 y_1' + p_2 y_2' + \dots + p_n y_n'$ with rational $p, q, p_1, p_2, \dots, p_n$.

Fix some set of such numbers $y_1', \dots, y_n'$. For $y \in Y$ set $f(y) = p$, where p is the coefficient in the representation $y = p\theta + q\pi + p_1 y_1' + \dots + p_n y_n'$. If M is the set of rectangles $x_1 \times y_1, x_2 \times y_2, \dots, x_n \times y_n$ where all $y_i \in Y$ then put

$$J(M) = x_1 f(y_1) + x_2 f(y_2) + \dots + x_n f(y_n).$$

(c) The value of $J(M)$ is invariant under an elementary transformation of the set M.
(d) Prove Lemma II.
(e) Prove Dehn's theorem: regular tetrahedron and cube are not scissor-congruent.

12. (a) Prove another Dehn's theorem: if a rectangle $a \times b$ is dissected into squares then $\frac{a}{b}$ is rational.
(b) Prove that a regular tetrahedron cannot be dissected into several (more than 1) regular tetrahedrons.

Dissections of a rectangle and electrical circuits.

To a dissection of a rectangle into rectangles we can attach an electrical circuit as shown at Fig. 8. To every rectangle there corresponds a resistor, and to every vertical line in the dissection (as well as to every vertical side of the original rectangle) there corresponds a node where several resistors connect. The resistance of every resistor equals the ratio of the horizontal side of the corresponding rectangle and the vertical one. It can be shown that the total resistance of the circuit equals the ratio of sides of the initial triangle.

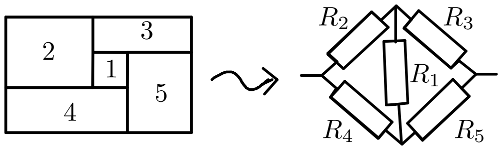

Fig. 8.

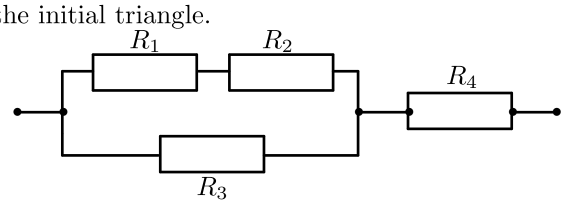

Fig. 9.

Let us show how to find the total resistance of an electrical circuit.

Consider an electrical circuit consisting of resistors. Let the *resistance* R_k be given for each resistor. Fix the beginning and the end of the circuit, and a real number $U > 0$ (voltage of the circuit). To each node we are going to assign a real number U_i called *the voltage* at the node as follows. For the beginning of the circuit we define the voltage to be equal to zero, and for the end of the circuit we define it to be equal to U. Choose the voltages at the remaining nodes in such a way that the sum of the values $\frac{(\Delta U_k)^2}{R_k}$ over all the resistors is minimal, where ΔU_k is the difference between the voltages at the ends of the k-th resistor. Denote this sum by P, it is called *the total calorification.*

The total resistance is $R = \frac{U^2}{P}$.

Further it is allowed to use that the distribution of the voltages with the minimal calorification exists.

Example 1. Consider a circuit consisting of two parallel resistors R_1 and R_2. By definition $P = \frac{U^2}{R_1} + \frac{U^2}{R_2}$ and the total resistance is $R = \frac{U^2}{P} = \frac{R_1 \cdot R_2}{R_1 + R_2}$.

Example 2. Consider a circuit consisting of two subsequent resistors R_1 and R_2. Let U_1 be the voltage at their common node. The value $\frac{U_1^2}{R_1} + \frac{(U-U_1)^2}{R_2}$ should be minimal possible. This is a quadratic function with respect to U_1. Evaluating $U_1 = \frac{U}{R_2(\frac{1}{R_1} + \frac{1}{R_2})}$ we get $R = R_1 + R_2$.

An elementary transformation of an electrical circuit is one of the following operations:

1) replacing a resistor of resistance $\frac{R_1 \cdot R_2}{R_1 + R_2}$ by two parallel resistors of resistances R_1 and R_2;

2) replacing a resistor of resistance $R_1 + R_2$ by two subsequent resistors of resistances R_1 and R_2;

3) joining two nodes with the same voltage.

13. Find the total resistance and corresponding rectangle dissections for the following circuits:
(a) the circuit in Fig. 8 for $R_1 = R_2 = R_3 = R_4 = R_5$; **(b)** the circuit in Fig. 9.

14. **(a)** Suppose a square is dissected into squares and rectangles with the ratio of the horizontal and the vertical sides equal to R. Then the corresponding electrical circuit consists of resistors with resistance 1 and R and has total resistance 1.

(b)* A circuit consists of resistors with resistance 1 or R. Prove that the total resistance of the circuit can be expressed as $\frac{P(R)}{Q(R)}$ where $P(x)$ and $Q(x)$ are polynomials with integer coefficients.

(c) Suppose that the voltages at the two nodes connected with a resistor of resistance R are distinct. Prove that the total resistance of the circuit increases if R increases.

(d) Solve Problem D.

Remark.

The current strength at a resistor $I_k = \frac{\Delta U_k}{R_k}$, where ΔU_k is the difference between the voltages of the two nodes connected with the resistor. Let us show that the sum of the current strengths at all the resistors having a common node (distinct from the beginning and the end of the circuit) equals zero. Fix such a node. Renumber the nodes so that this node is first, and the resistances of the resistors connected with this node are $R_1, R_2, \dots, R_n$. Let us see how the total resistance of the circuit depends on U_1. The total calorification is $\sum\limits_{i=1}^{n} \frac{(U_i - U_1)^2}{R_i} + C$, where C is a constant not depending on U_1. The minimum attends at the vertex of the parabola, so $U_1 = \frac{\sum\limits_{i=1}^{n} \frac{U_i}{R_i}}{\sum\limits_{i=1}^{n} \frac{1}{R_i}}$, which is equivalent to $\sum\limits_{i=1}^{n} \frac{U_i - U_1}{R_i} = 0$.

Our definition imply also *the Kirchgoff laws*:

1) the sum of current strengths at all the resistors having a common node (distinct from the beginning and the end of the circuit) equals zero;

2) $I_1R_1 + I_2R_2 + \dots + I_nR_n = U$ for any path $1, 2, \dots, n$ going from the beginning of the circuit to the end, where U is the total voltage not depending on the path.

Vice versa, the Kirchgoff laws imply that the current strength distribute so that the total calorification is minimal.

SOLUTIONS: PART I.

1a. Dissect the triangle by the median from the vertex of the right angle (Fig. 10a).
1b. Dissect the triangle by the line which divides the angle α as $2:1$ (Fig. 10b).
1c. Dissect the triangle by the line which separates angle of size β from the angle γ (Fig. 10c).
1d. Dissect the triangle by the line symmetrical to the side opposite to γ relative the bisector of this angle (Fig. 10d).
1e. *First method.* Take an open polygon $ABCDE$ with 4 equal edges and equal angles 130° between them. Extend the edges AB, BC and DE. Denote by BFG the triangle formed by these lines. Then the angles of triangle BFG are equal to 30°, 20° and 130°. Thus we constructed the required dissection of the triangle: dissect triangle BFG by open polygon BCD.
Second method. Let $\delta = 10^\circ$. Take an open polygon $ABCDEF$ with 5 equal edges and equal angles $180^\circ - \delta$ between them. Connecting its endpoints A and F we obtain a symmetrical hexagon with angles $A = F = 2\delta$. Construct triangle AFG with angles $GAF = 2\delta$ and $GFA = 3\delta$ such that edge AB lies on AG. Hexagon $BCDEFG$ is symmetrical as well because angles B and F are equal to δ. Thus we have obtained the required dissection since the angles of triangle AFG are equal to 30°, 20° and 130° respectively: dissect triangle AFG by open polygon $ABCDEF$.
Third method. (Fig. 10e) Dissect triangle ABC by open polygon $KLMN$ with 3 edges where $K \in BC$, $N \in AB$, $BK = KL = LM = MN = NA$, $\angle BKL = \angle LMN = \pi - \alpha$, $\angle KLM = \angle MNA = \pi - \beta$.
1f. Dissection is constructed similarly to the second or third method in the solution of Problem 1e.
1g. Dissect the triangle by three perpendiculars from the incenter to the sides of the triangle.

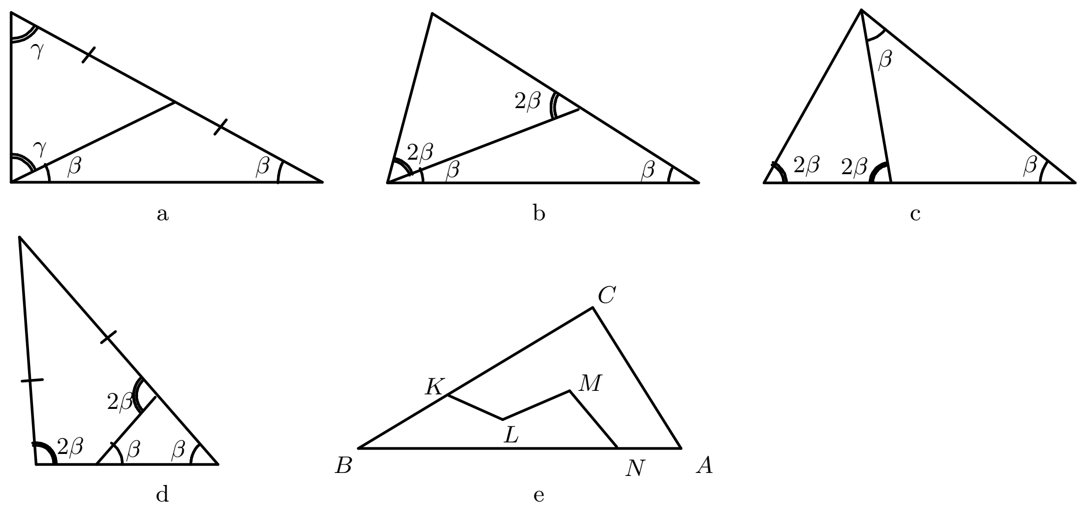

Fig. 10.

2a. Draw the altitude from the vertex of the right angle.
2b. Draw slits from the vertex of the angle equal to 120°, having angle 30° with its sides. Then dissect the obtained regular triangle connecting its center with its vertices.
2c. For $n \geq 4$ even, divide some side of the triangle (demote it by a) into $n/2$ equal parts. Through the dividing points, draw various segments parallel to the remaining sides of the triangle, up to the nearest meet with itself or with another side of the triangle. It is easily seen that all obtained points of meet belong to the same line parallel to a. Draw this line to obtain the required dissection.
For $n \geq 7$ odd, dissect the triangle by the above method into $n-3$ parts, then dissect one of the obtained triangles into 4 parts by the same method (i.e., draw midlines).
Remark. A natural question arises: what is the situation for $n = 5$? It turns out that examples in Problems 2a and 2b exhaust triangles allowing dissection into 5 triangles similar to the original one. The proof of this nice fact will be published in one of the next few issues of the journal "Kvant".
3a. A slit has to connect some vertex of the triangle with a point at the opposite side. If the angles to this point are not equal then their sum is $< \pi$, since they are equal to two angles of the original triangle. Hence the angles are equal and thus are right. By the way, we have established a simple but useful fact which will be used in the rest of this problem.
Fact 1. If some node of the dissection has only two adjacent small triangles then the original triangle is in fact right. If such a situation occurs, we say for brevity that "we have obtained rectangularity".
3b. Let a triangle be dissected into three triangles similar to it. The smallest of its angles cannot be dissected. The small triangle including this angle has all its vertices at the sides of the original triangle. Either one or two of these vertices coincide with the vertices of the original triangle. In the first case, the remaining part is a convex quadrangle, and in the second case, a triangle. In both cases, an easy examination of cases shows that the subsequent dissection into two triangles yields rectangularity.
We have established a new useful fact.
Fact 2. Let the original triangle have angles $\alpha \leq \beta \leq \gamma$. Then the angle α may not be dissected. Each vertex of the small triangle containing it belongs to the boundary of the original triangle. We will call this small triangle α-triangle.
3c. Suppose a triangle is dissected into 4 triangles similar to it but oriented oppositely. Consider the segment separated by the α-triangle.
Suppose this segment dissects the triangle into the α-triangle and a quadrangle. Then the endpoints of this segment belong to some other slits, for otherwise we have rectangularity. To obtain just 4 small triangles, we need 2 such slits having a common point at the side opposite to the angle α. Note that in the obtained layout for dissection, angles of all the small triangles are uniquely determined. Indeed, the condition of opposite orientation determines all angles of three triangles containing the angles

of the original triangle. Then the angles of the "central" triangle are uniquely determined: they equal to $\pi - 2\alpha$, $\pi - 2\beta$ and $\pi - 2\gamma$. Order the angles of the original triangle: $\alpha \le \beta \le \gamma$. Then $\pi - 2\alpha \ge \pi - 2\beta \ge \pi - 2\gamma$. Thus $\pi - 2\alpha = \gamma$. Hence $\beta = \pi - \alpha - \gamma = \alpha$. Thus we see that the original triangle is isosceles, contrary to our assumption.
Now assume that the remaining part is a triangle. Order the angles of the original triangle: $\alpha \le \beta \le \gamma$, denote by A, B, C the corresponding vertices. Denote by D the vertex of the angle γ in the α-triangle. If some angle of the triangle BCD is not dissected then we obviously obtain rectangularity. So all three its vertices are connected with some point O inside the triangle. Clearly, $\angle ABC > \angle OBC$ and thus $\angle OBC = \alpha$. Now the orientation condition implies $\angle OCB = \gamma = \angle ACB$, a contradiction.
4a. Polygon M has to be central symmetrical. In fact, a convex polygon can have not more than two sides parallel to the direction l. Thus if $J_l(M) = 0$ for any directed line l then the sides of M divide into pairs of equal and parallel sides. Suppose $M = A_1A_2 \dots A_{2n}$. Its convexity implies that the only possible case is $A_1A_2 \parallel A_{n+1}A_{n+2}, A_2A_3 \parallel A_{n+2}A_{n+3}, \dots, A_nA_{n+1} \parallel A_{2n}A_1$. Hence $\overrightarrow{A_1A_k} = -\overrightarrow{A_{n+1}A_{n+k}}$ for each $k = 2, 3, \dots, n$. Thus the midpoint of the segment A_1A_{n+1} is the center of symmetry for M.
4b. We present the proof from [1].
Consider all segments which are sides of $M, M_1, M_2, \dots, M_k$. Mark all points on them which are vertices of $M, M_1, M_2, \dots, M_k$. Then we obtain a finite number of smaller segments which we will call links. Each side of each polygon $M, M_1, M_2, \dots, M_k$ consists of one or more links. Fig. 11 shows a dissection of a polygon into smaller parts. Side consists of three links AM, MN, NB; side NP of the shaded polygon in the figure consists of three links as well.
Note that in calculating the invariant $J_l(M)$ of the polygon M (or of any polygon $M_1, M_2, \dots, M_k$) we may use the algebraic sum of links parallel to l, instead of sides, since the length of each side equals the sum of length of links contained in it. So for calculating the sum in the right side of the relation in Problem 4b, we have to form the algebraic sum of lengths of all links parallel to l and counted over all polygons $M_1, M_2, \dots, M_k$.
Consider a link which is entirely (excluding endpoints, possibly) situated inside M (link EF at Fig. 11). Then it is adjacent for two polygons among $M_1, M_2, \dots, M_k$ which adjoin the link from opposite sides (right and left). So in calculating the invariant of one of these polygons, the link will take plus sign, and for the other polygon it will take minus sign, so that in the total algebraic sum these two links cancel. We see that in calculating of the right-hand side in the relation from Problem 4b, we may ignore links situated inside M.
Now consider a link which belongs to the outline of M and is parallel to the line l (link AM at Fig. 11). To this link, there adjoins only one of polygons $M_1, M_2, \dots, M_k$, and from the same side as M. Hence this link has the same sign in the sum $J_l(M_1) + J_l(M_2) + \dots + J_l(M_k)$ as in the invariant $J_l(M)$.
Thus the right-hand side of the relation in Problem 4b equals $J_l(M)$, and our assertion is proven.

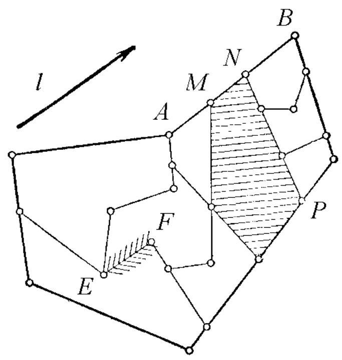

Fig. 11.

4c. Suppose a polygon M' is dissected into polygons $M_1, M_2, \dots, M_k$ which are combined into a polygon M' using only parallel shifts of parts. Note that the values of $J_l(M_i)$ are invariant under shifts of M_i. Thus the result of Problem 4b implies

$$J_l(M) = J_l(M_1) + \dots + J_l(M_k) = J_l(M),$$

as required.
4d. For any square M' and any directed line l we have $J_l(M') = 0$. Hence by Problem 4c, $J_l(M) = J_l(M') = 0$. Then by Problem 4a the polygon M is central symmetrical.
5a. Follows from 4b.
5b. If the angle $n\pi$ is divisible by ϕ for some odd n then obviously $J_{l,\phi}(M) = 0$. In the sequel suppose that $n\pi$ is not divisible by ϕ for any odd n. Clearly, if no angle between l and the sides of triangle M is divisible by ϕ then $J_{l,\phi}(M) = 0$. Suppose now that $J_{l,\phi}(M) = 0$, and the angle between some side AB and the line l is divisible by ϕ. Then side AB has nonzero contribution to $J_{l,\phi}(M)$. Hence its contribution has to cancel with contributions of the remaining sides. Consequently, the triangle has another side, say BC, whose angle with l is divisible by ϕ. All three sides cannot be in use because $AB \pm BC \pm CA \ne 0$ by triangle inequality. Hence the triangle is isosceles, $AB = BC$, and $\angle B = n\phi$ for some integer n. Conversely, for any such triangle we have $J_{l,\phi}(M) = 0$, provided the angle between AB and l is divisible by ϕ and the angle between AC and l is not. These are all the possible cases.
5c. Consider a directed line l containing some side s. Then $J_{l,\phi}(M) \ne 0$ by Problem 5b. Thus $J_{l,\phi}(M') \ne 0$. Hence there exists a side s' of triangle M' such that the angle between s and s' is divisible by ϕ. This implies what is required.
5d. We may assume $\psi = 0$. The result of Problem 5a implies that two triangles M and M', the cake and the box, must have the same invariant $J_{l,\phi}(M)$ for any directed line l. From Problem 5c we see that the sides of triangles M and M' can be enumerated so that the angle between sides with the same numbers are divisible by ϕ. For instance, let the corresponding sides of triangles M and M' have the same number. Denote by α_1, α_2, α_3, α_1', α_2', α_3' the angle between these sides and some fixed

line. Then $\alpha_i - \alpha_i' = k_i\phi$ for some integer k_i and $i = 1, 2, 3$. On the other hand, $\alpha_i - \alpha_{i+1} = \alpha_{i+1}' - \alpha_i'$ for $i = 1, 2, 3$, where we set $\alpha_4 = \alpha_1$ by definition. The resulting system of 6 linear equations easily leads to the required consequence. In particular, there exist integers k, l, m such that some of them is not zero and $k(\alpha_1 - \alpha_2) + l(\alpha_2 - \alpha_3) + m(\alpha_3 - \alpha_1) = 0$.

5e. Proof of Statement I for Problem A. It suffices to consider the case of a non-isosceles triangle. We may assume that while packing the parts into the box, one of them remains fixed, and the other one is joined to it being rotated through an angle ϕ around some point O. Two cases are possible.
(1) The angle ϕ is incommensurable with π. Then the result of Problem 5 implies our assertion.
(2) The angle ϕ is commensurable with π.
For any directed line l, denote by L the set of lines obtained from l by rotations *around point* O through angles divisible by ϕ. On the set of directed lines, introduce the following function f: $f(XY) = 1$ if $XY \in L$, $f(XY) = -1$ if $YX \in L$, and $f(XY) = 0$ otherwise.
First let the point O be distinct from the vertices of the triangle M. Then there exist two sides of M, say AB and BC, not containing the point O. For the line l containing the side AB, $J_f(M) = \pm|AB|$ (otherwise one of the angles of the triangle M immediately is divisible by ϕ, thus commensurable with π. Let M' be the triangle obtained from M after moving the parts. Arguing similarly to Problem 5a we obtain that $J_f(M') = J_f(M) = \pm|AB|$. Thus the side of length $|AB|$ in the triangle M' and the side AB of M form an angle divisible by ϕ. Similarly, the side of length $|BC|$ in M' and the side BC of M form an angle divisible by ϕ. Then using commensurability of ϕ and π, and arguing as in Problem 5d we obtain $k\alpha + l\beta + m\gamma = 0$ for some integers k, l, m, not vanishing simultaneously.
It remains to consider the case when the point O coincides with one of the vertices of the triangle M. Let us introduce one more invariant. Let OX be some *ray* starting at O. For any directed segment AB, denote by $J_{OX}(AB)$ the length of the intersection $AB \cap OX$ taken with $+$ sign if AB has the same direction as OX, and with $-$ sign otherwise. For any polygon P denote by $J_{OX,\phi}(P)$ the sum of values $J_{OY}(AB)$, where AB runs over all directed sides of P, and OY runs over all rays obtained from OX by rotations through angles divisible by ϕ. Then the given invariant is not zero for the sides of the triangle M containing O. Arguing as in Problem 5d we get the required relation $k\alpha + l\beta + m\gamma = 0$ for some integers k, l, m, not vanishing simultaneously.

6a. The proof is similar to the solution of Problem 4b.
6b. Any triangle $A_iB_iC_i$ can be connected with side AB by a chain of triangles $A_1B_1C_1, A_2B_2C_2, \dots, A_iB_iC_i$ such that consecutive triangles have a common part of the boundary, and A_1B_1 is contained in AB. Hence it suffices to prove that if some side of triangle $A_jB_jC_j$ and the line AB form an angle of the form $k\alpha + l\beta + m\gamma$ then this is true for two other sides. The latter statement is obvious.
6c. Problem 6b suggests that the *revealing* function f is to be some function in k, l and m.
Thus let $f(XY) = f(k, l, m)$ where integers k, l, m are such that the angle between directed lines XY and AB equals $k\alpha+l\beta+m\gamma$. By definition, the angle between directed lines XY and AB is the angle of the rotation which maps AB into XY as directed lines. The angle between directed lines is determined up to 2π. Hence integers $k + 2$, $l + 2$, $m + 2$ define the same angle as k, l, m. Thus we obtain a condition for our function: $f(k + 2, l + 2, m + 2) = f(k, l, m)$. Since α, β, γ are incommensurable, integers k, l, m are uniquely determined by directed line XY up to substitution $k, l, m \to k+2, l+2, m+2$. Hence any function subject to $f(k+2, l+2, m+2) = f(k, l, m)$ correctly defines a function on the set of directed lines. (We set $f(XY) = 0$ if XY is distinct from dissecting lines.)
Now we determine the other conditions on $f(k, l, m)$. First, $f(XY) = -f(YX)$, hence

$$f(k + 1, l + 1, m + 1) = -f(k, l, m). \qquad (1)$$

This condition also implies $f(k + 2, l + 2, m + 2) = f(k, l, m)$.
Consider now the condition $J_f(A_iB_iC_i) = 0$. Suppose the vertices of triangle ABC are situated in the above order clockwise, and the vertices of triangle $A_iB_iC_i$ counterclockwise. Let the angle between directed lines A_iB_i and AB be equal to $k\alpha + l\beta + m\gamma$. Then the angle between lines A_iC_i and AB equals $(k - 1)\alpha + l\beta + m\gamma$, and the angle between lines C_iB_i AB equals $k\alpha + (l + 1)\beta + m\gamma$. Hence

$$J_f(A_iB_iC_i) = f(k, l, m)|A_iB_i| - f(k, l + 1, m)|B_iC_i| - f(k - 1, l, m)|C_iA_i|.$$

Since triangles $A_iB_iC_i$ and ABC are similar, the condition $J_f(A_iB_iC_i) = 0$ may be rewritten in the form

$$f(k, l, m)|AB| - f(k, l + 1, m)|BC| - f(k - 1, l, m)|CA| = 0. \qquad (2)$$

The same argument shows that the condition $J_f(ABC) \neq 0$ may be rewritten in the form

$$f(0, 0, 0)|AB| - f(0, -1, 0)|BC| - f(1, 0, 0)|CA| \neq 0. \qquad (3)$$

Thus it suffices to find a function $f(k, l, m)$ satisfying (1)–(3). Note that the second relation determines some restrictions on $f(k, l, m)$ for m fixed. Thus it suffices to define the function for $m = 0$, and then the first relation will determine it for all other m. In view of utter arbitrariness, put $f(k, 0, 0) = 1$ for all k. Then $f(k, 1, 0) = (|CA| - |AB|)/|BC|$ for all k. Furthermore we may assume $f(k, l, 0) = ((|CA| - |AB|)/|BC|)^l$, and correction for the first relation gives

$$f(k, l, m) = (-1)^m \cdot \left(\frac{|CA| - |AB|}{|BC|}\right)^{l-m}.$$

For this function $f(k, l, m)$ relations (1)–(3) are verified immediately.

6d. Proof of Statement I for Problem B. Straightforward from Problems 6a and 6c.
6e. The answer is unknown to the authors.
7. To any curvilinear polygon M, attach an integer $J(M)$ equal to the length sum of boundary arcs such that the polygon adjoins them from the "concave" side, minus the length sum of arcs such that the polygon adjoins them from the "convex"

side. It is easy to check that $J(M)$ has equal values for two polygons such that one of them is obtained from the other one but dissection by segments and arcs and combining the obtained parts. On the other hand, a circle has $J(M) \neq 0$, and a square has $J(M) = 0$.

SOLUTIONS: PART II.

8. *Geometric solution.* The cube $ABCDA'B'C'D'$ can be dissected into 6 tetrahedrons $AC'BB'$, $AC'B'A'$, $AC'A'D'$, $AC'D'D$, $AC'DC$, $AC'CB$ by six planes passing through the pair of the opposite vertices A, C' and one of the remaining vertices of the cube. The congruence of the tetrahedrons follows from symmetry (for instance, tetrahedron $AC'BB'$ maps onto tetrahedron $AC'A'D'$ under rotation of the cube through 120° around the line AC').
Algebraic solution. The cube $0 \leqslant x, y, z \leqslant 1$ can be dissected into 6 tetrahedrons $0 \leqslant x \leqslant y \leqslant z \leqslant 1$, $0 \leqslant x \leqslant z \leqslant y \leqslant 1$, $0 \leqslant y \leqslant x \leqslant z \leqslant 1$, $0 \leqslant y \leqslant z \leqslant x \leqslant 1$, $0 \leqslant z \leqslant x \leqslant y \leqslant 1$, $0 \leqslant z \leqslant y \leqslant x \leqslant 1$.
9a. Draw the line joining the midpoints of two long sides of the rectangle.
9b. Divide two long sides of the rectangle into q equal parts, and two short sides into p equal parts. Through the corresponding division points, draw lines parallel to the sides of the rectangle.
9c. Let a rectangle $1 \times x$, $x = \sqrt[4]{2}$ be given. Cut off a rectangle $1 \times \frac{1}{x}$. From the obtained strip $1 \times (x - \frac{1}{x})$, cut off two rectangles $(x^2 - 1) \times (x - \frac{1}{x})$. From the strip $(3 - 2x^2) \times (x - \frac{1}{x})$, cut off a rectangle $(3 - 2x^2) \times (\frac{3}{x} - 2x)$. The obtained rectangle $(3 - 2x^2) \times (3x - \frac{4}{x})$ has side ratio equal to x as well since $x^4 = 2$.
9d. Let

$$r = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}$$

is a periodic continuous fraction. Since the sequence a_k is periodic, for some n we have

$$r = a_1 + \cfrac{1}{a_2 + \dots \cfrac{1}{a_{2n} + \cfrac{1}{r}}}\ .$$

Starting from this equality, we easily construct a dissection of the rectangle $1 \times r$ into several squares and a single rectangle with side ratio r. In fact, cut off a_1 squares 1×1 first. We obtain the strip $1 \times (r - a_1)$ with side ratio

$$\frac{1}{r - a_1} = a_2 + \cfrac{1}{a_3 + \dots \cfrac{1}{a_{2n} + \cfrac{1}{r}}}\ .$$

Now cut off a_2 squares $(r - a_1) \times (r - a_1)$ etc. Proceeding in such a way, we obtain a rectangle with side ratio r.
The constructed dissection of the rectangle $1 \times r$ leads easily to the required dissection of the rectangle $1 \times \sqrt{r}$: contract the rectangle $1 \times r$ in $\sqrt{r}$ times along side r.
9e. The rectangle with side ratio a is dissected into 3 vertical strips. First strip includes top-down rectangles with side ratio $a, \frac{1}{a}$; the second one, respectively: $a, a, \frac{1}{a}$; the third one: $a, \frac{1}{a}, \frac{1}{a}$. In fact,
$\frac{1}{a+\frac{1}{a}} + \frac{1}{\frac{1}{a}+\frac{1}{a}+a} + \frac{1}{\frac{1}{a}+a+a} = a$;
$(a^2 + 1)(2a^2 + 1)(a^2 + 2) = (a^2 + 1)(a^2 + 2) + (a^2 + 1)(2a^2 + 1) + (2a^2 + 1)(a^2 + 2)$;
$2a^6 + 2a^4 - 4a^2 - 3 = 0$.
The polynomial $2x^3 + 2x^2 - 4x - 3$ has no rational roots. Indeed, if $\frac{p}{q}$ is a fraction in its lowest terms then p divides the intercept, and q divides the leading coefficient. The examination of cases shows easily that $\pm 3, \pm 1, \pm \frac{3}{2}, \pm \frac{1}{2}$ are not roots of the given polynomial.
9f. It is easy to dissect the rectangle into 4, 6 and 8 parts. Given the dissection into n parts it is easy to construct the dissection into $n + 3$ parts.
10a. Let e_i be the corresponding edge of some polyhedron M_j. Consider a cylinder C having axis e and radius 1. The dihedral angle at edge e cuts off in the cylinder surface a band L having length α and width l. In the surface of the sub-cylinder C_i having axis e_i and radius 1, the dihedral angle at edge e_i cuts off a band L_i having α_i and length l_i. Since the dissection polyhedrons are disjoint and cover$M$, the band L is dissected into bands $L_1, L_2, \dots, L_n$. It remains to establish the natural correspondence between points of L and of the rectangle $l \times \alpha$ to obtain its dissection into rectangles $l_i \times \alpha_i$.
10b. Any common point of the line ℓ and of polyhedron M is either an internal point of some polyhedron M_i or belongs to the boundary of several dissection polyhedrons. Let $e_1, e_2, \dots, e_n$ be the edges of dissection polyhedrons belonging to the line ℓ (and having length $l_1, l_2, \dots, l_n$ resp.). Let $f_1, f_2, \dots, f_m$ be the lengths of various intersections of ℓ with faces of M_i, not coinciding with edges. Thus edges $e_1, e_2, \dots, e_n$ form a family of segments on ℓ. Without loss of generality, $e_1, e_2, \dots, e_s$ is the set of edges belonging to such a segment I. We will prove that the set of rectangles $e_1 \times \alpha_1, e_2 \times \alpha_2, \dots, e_s \times \alpha_s$ $\square$ is scissor-congruent to the rectangle $l \times \pi$. Having proved this for every such segment and joining together the obtained rectangles of width π, we get the assertion of the problem.
Let C be the surface of a cylinder having axis I and radius 1 without heads. The dihedral angles at $e_1, e_2, \dots, e_n$ cut off from C bands $l_i \times \alpha_i$(having line length l_i and circle width α_i). Since the polyhedrons are disjoint and cover the whole polyhedron M, C is dissected into bands $l_i \times \alpha_i$ and $f_i \times \pi$. Extend all circular slits. Then C is dissected into rings. Remove from rings all bands having width π (parts of redundant bands having line length f_i), and dissect not changed rings into 2 bands having circular width π. All the remaining can be combined into a band of circular width π which corresponds to the rectangle of width π dissected into parts of rectangles $l_1 \times \alpha_1, l_2 \times \alpha_2, \dots, l_s \times \alpha_s$, obtained by shifts, vertical and horizontal slits.

10c. The set of rectangles corresponding to the first polyhedron, being combined with some set of rectangles having width π, is by 10a and 10b $\Box$-scissor-congruent to the join of sets of rectangles corresponding to the dissection polyhedrons. The same is true for the second polyhedron. But obviously the $\Box$-scissors-congruence relation is transitive and symmetrical, and we are done.

10d. Let M be the midpoint of CD. Since AM and BM are perpendicular to CD, the value of $\angle AMB$ equals the value of the dihedral angle at edge CD of the tetrahedron. Suppose the length of the edge of the tetrahedron equals a, then the formula for the altitude of the regular triangle gives $AM = BM = \frac{\sqrt{3}}{2}a$. By cosine theorem applied to triangle AMB, $cos\theta = \frac{AM^2+BM^2-AB^2}{2AMBM} = \frac{1}{3}$.

Let us prove by induction that $cosn\theta = \frac{a_n}{3^n}$ where a_n is an integer not divisible by 3. For initial values $n = 0$ and $n = 1$, the fact is obvious.

The induction step. Formula for the sum of cosines for $n \geqslant 1$: $cos(n+1)\theta + cos(n-1)\theta = 2cosn\theta cos\theta$, so $cos(n+1)\theta = 2cosn\theta cos\theta - cos(n-1)\theta = \frac{2a_n - 3a_{n-1}}{3^{n+1}}$. Indeed, $2a_n - 3a_{n-1}$ is not divisible by 3.

Hence $cosk\theta \neq 1$, then $k\theta \neq 2\pi n$, i.e. $\theta \neq \frac{p}{q}\pi$.

11a. Suppose the first set is dissected into rectangles which are shifted are combined into the second set.Extend all the vertical slits in the dissection of the first set and horizontal slits in the dissection of the second one. The obtained dissection can be fulfilled by elementary dissections: first dissect the first set through all vertical slits, then dissect each vertical strip by horizontal slits. Collect all horizontal strips in the dissection of the the second set and join them.

11b. Introduce the following operation for the set $\theta, \pi, y_{i_1}, y_{i_2}, \dots, y_{i_k}$: remove y_i with the greatest i_s such that $p\theta + q\pi + \mu_1 y_{i_1} + \mu_2 y_{i_2} + \dots + \mu_k y_{i_k} = 0$ where all coefficients are rational and $\mu_s \neq 0$. Apply this operation to the initial set repeatedly until possible. We obtain a set $\theta, \pi, y_{j_1}, y_{j_2}, ...y_{j_r}$. Note that for any $x \in Y$ there exist rational $p, q, \mu_1, \mu_2, \dots, \mu_r$ such that $x = p\theta + q\pi + \mu_1 y_{j_1} + \mu_2 y_{j_2} + \dots + \mu_r y_{j_r}$. Suppose $p_1\theta + q_1\pi + \mu_1 y_{j_1} + \mu_2 y_{j_2} + \dots + \mu_r y_{j_r} = x = p_2\theta + q_2\pi + \xi_1 y_{j_1} + \xi_2 y_{j_2} + \dots + \xi_r y_{j_r}$, then $(p_1 - p_2)\theta + (q_1 - q_2)\pi + (\mu_1 - \xi_1)y_{j_1} + \dots + (\mu_s - \xi_s)y_{j_s} = 0$. If $\mu_t \neq \xi_t$ then the set $\theta, \pi, y_{j_1}, y_{j_2}, ...y_{j_r}$ allows to apply the above operation once more — a contradiction. Hence $\mu_t = \xi_t$ for all t. But θ and π are incommensurable and nonzero, hence $p_1 = p_2$ and $q_1 = q_2$, i.e. for any x from Y there exist unique rational $p, q, \mu_1, \mu_2, \dots, \mu_r$ such that $x = p\theta + q\pi + \mu_1 y_{j_1} + \mu_2 y_{j_2} + \dots + \mu_r y_{j_r}$.

11c. Suppose a new set is obtained by dissecting the rectangle $x \times y$.

The slit is vertical: the invariant becomes $x_1 f(y) + x_2 f(y) - x f(y) = 0$.

The slit is horizontal: the invariant becomes $xf(y_1) + xf(y_2) - xf(y)$.

Suppose $y_1 = f(y_1)\theta + q_1\pi + \mu_1 y_1' + \mu_2 y_2' + \dots + \mu_n y_n'$ $y_2 = f(y_2)\theta + q_2\pi + \xi_1 y_1' + \xi_2 y_2' + \dots + \xi_n y_n'$. Then $y = y_1 + y_2 = (f(y_1) + f(y_2))\theta + (q_1 + q_2)\pi + (\mu_1 + \xi_1)y_1' + (\mu_2 + \xi_2)y_2' + \dots + (\mu_n + \xi_n)y_n'$. That is, $f(y) = f(y_1) + f(y_2)$ and the invariant has not changed.

11d. Since the invariant does not change under elementary transformation of the set (11c), by 11a invariants of $\Box$-scissor-congruentsets are equal. However the invariant of the set $(a \times \theta, l \times \pi)$ equals a, and that of $b \times \pi$ is zero. Hence these sets are not $\Box$-scissor-congruent. **11e.** Suppose the contrary. Then by Lemma I the sets of 6 copies of $a \times \theta$, $l_1 \times \pi$ and of 8 copies of $b \times \frac{\pi}{2}$ are $l_2 \times \pi$ $\Box$ - scissor-congruent. However the first set is $\Box$-scissor-congruent to the set $6a \times \theta$, $l_1 \times \pi$, and the second one to the set $(\frac{b}{2} + l_2) \times \pi$. Hence two latter sets are $\Box$-scissor-congruent, but this is impossible by Lemma II in view of 10d. A contradiction.

12a. *Solution based on reduction to Lemma II.* If the rectangle $a \times b$ can be dissected into squares, then it is $\Box$-scissor-congruent to the rectangle $b \times a$ (since the square maps onto itself under rotation through 90°.) Then by Lemma II the relation $\frac{a}{b}$ is rational.

Straightforward solution. We also show that side ratio of a rectangle is uniquely determined by the arrangement of squares. Fix b. Let $x_1, x_2, \dots, x_n$ be the side lengths of squares. The sides of squares may join into segments which are either sides of a rectangle, and hence $x_{i_1} + x_{i_2} + \dots + x_{i_s} = a$ or $x_{i_1} + x_{i_2} + \dots + x_{i_s} = b$, or from both sides they side with squares whose sides satisfy $a_1 + a_2 + \dots + a_s = b_1 + b_2 + \dots + b_t$, here $a_1, a_2, \dots, a_s$ are sides of squares, say, from the left and $b_1, b_2, \dots, b_t$ from the right.

Write down all these equations. Express the variables subsequently and substitute them in the remaining equations starting from a. Eventually, the variables of the group containing a are expressed through the variables of another group by linear combinations with rational coefficients. Prove that the second group contains only b. Note that if the values of variables in the second group are such that all variables are positive then we obtain the required dissection of the rectangle into squares.

$a = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n + \xi b$

$x_i = \mu_{i1} x_1 + \mu_{i2} x_2 + \dots + \mu_{in} x_n + \mu_i b$

Naturally, the coefficients are nonzero only at variables of the second group. Let x_n be in the second group. Take the original dissection, replace x_n by ε so that each x and a remain positive. We obtain another dissection into squares of sizes $x_1 + \mu_{1n}\varepsilon, x_2 + \mu_{2n}\varepsilon, \dots, x_{n-1} + \mu_{n-1n}\varepsilon, x_n + \varepsilon$ and a rectangle $a + \xi_n\varepsilon \times b$.

Write down the equation for areas: $(a + \xi_n\varepsilon)b = (x_1 + \mu_{1n}\varepsilon)^2 + (x_2 + \mu_{2n}\varepsilon)^2 + \dots + (x_{n-1} + \mu_{n-1n}\varepsilon)^2 + (x_n + \varepsilon)^2 \Rightarrow ({\mu_{1n}}^2 + {\mu_{2n}}^2 + \dots + {\mu_{n-1n}}^2 + 1)\varepsilon^2 + (2x_1\mu_{1n} + 2x_2\mu_{2n} + \dots + 2x_{n-1}\mu_{n-1n} + 2x_n - \xi_n b)\varepsilon = 0$

We see that not more than two ε's satisfy this equation but we could start with any ε from some neighborhood of zero. Thus the second group contains only b and no x. As already mentioned, all variables are expressible as linear combinations of variables from the second group, so $a = pb$ where p is rational.

"Physical" solution. Our assertion also follows from the result of Problem 13b with $R = 1$.

12b. Suppose the contrary. Let $a_1, a_2, \dots, a_n$ be the edges of tetrahedrons in a dissection of a tetrahedron having edge a. Then the result of Problem 10 implies that the set $6a_1 \times \theta, 6a_2 \times \theta, \dots, 6a_n \times \theta$ is $\Box$-scissor-congruent to the set $6a \times \theta$, $l \times \pi$. Hence by Problem 11 we have $a_1 + a_2 + \dots + a_n = a$. The volumes are equal: $a_1^3 + a_2^3 + \dots + a_n^3 = a^3$. Cube the first equation: $a_1^3 + a_2^3 + \dots + a_n^3 + A = a^3$ where $A > 0$ — a contradiction with the second equation.

Sketch of a geometric solution. Take an edge of a smaller tetrahedron completely contained in a face of a larger tetrahedron. Then all the dihedral angles at this edge are equal to θ, but their sum should be π. Since θ and π are incommensurable, we obtain a contradiction.

13a. Let U_1 and U_2 be voltage in the upper and the lower non-boundary nodes. If the calorification at the 2nd and 3nd resistors is greater than at the 4th and 5th ones then replace U_1 by U_2. The total calorification will decrease. If the calorification is equal then we decrease the total calorification by the same way. Hence the minimum calorification is obtained at $U_1 = U_2$. Now the circuit is reduced by obvious elementary transformations to a circuit consisting of a single resistor.

13b. The circuit is reduced by obvious elementary transformations to a circuit consisting of a single resistor.

14a. *Geometrical solution.* By problem 11, the given dissection of the rectangle can be obtained as a sequence of elementary (and inverse to them) transformations. Now note that the total resistance of the circuit does not change under an elementary transformation.

"Physical solution". Suppose that a rectangular plate is made of a homogeneous conductive material. Assume its specific resistance to be equal to 1. Connect the vertical sides of the plate with poles of a direct current source. Then the resistance of the plate equals the ratio of horizontal and vertical sides. Now suppose that the rectangle is dissected into smaller rectangles. Mark all slits on the plate. Note that the sense of current on the plate is horizontal. So if we dissect the plate through all horizontal slits then its resistance does not change.

Now we may dissect the plate through all vertical slits and connect by wires those pairs of vertical sides of small rectangles which did coincide in the original rectangle. Clearly total resistance of the circuit does not change.

Each of small rectangular plates in the circuit obtained is a resistor with resistance equal to the ratio of the horizontal and the vertical sides of the corresponding plate.

Thus we have shown that total resistance of the circuit corresponding to a dissection of a rectangle equals its side ratio. This implies the assertion of Problem 14a.

14b. *Analytical solution.* Suppose $U = 1$. Suppose the minimum calorification corresponds to $U_1, U_2, \dots, U_n$. Fix $U_2, U_3, \dots, U_n$ and consider the calorification as a function of U_1. Since this function is a sum of squares of linear functions and is not constant, after grouping coefficients we get a positive coefficient at U_1^2. The minimum of a square function is achieved at the vertex of a parabola, hence $U_1 = a_2(R)U_2 + a_3(R)U_3 + \dots + a_n(R)U_n + a_1(R)$ where $a_i(R)$ is a ratio of two polynomials in R having integer coefficients. Substitute the expression for U_1 into our square function to get a function in $(n-1)$ variables. As a function in U_2, it cannot be constant (consider the behavior of calorification for large U_2). Arguing as above, we obtain $U_n = \frac{P_n(R)}{Q_n(R)}$. Returning, we find $U_i = \frac{P_i(R)}{Q_i(R)}$. Thus the total calorification equals $\frac{U^2}{P} = \frac{P(R)}{Q(R)}$.

Geometrical solution. We will show that the side ratio of the dissected rectangle is the ratio of some polynomials with integer coefficients in side ratios of rectangles. Let $x_1, x_2, \dots, x_n$ be the lengths of vertical sides of rectangles, and $R_1, R_2, \dots, R_n$ be the ratios of their horizontal and vertical sides. The sides of the rectangles may be united into segments. Either such a segment is a side of the initial rectangle, hence $x_{i_1} + x_{i_2} + \dots + x_{i_s} = a$ or $x_{i_1}R_{i_1} + x_{i_2}R_{i_2} + \dots + x_{i_s}R_{i_s} = b$, or this segment is situated between two dissection rectangles whose sides satisfy $x_{i_1} + x_{i_2} + \dots + x_{i_s} = x_{j_1} + x_{j_2} + \dots + x_{j_t}$ or $x_{i_1}R_{i_1} + x_{i_2}R_{i_2} + \dots + x_{i_s}R_{i_s} = x_{j_1}R_{j_1} + x_{j_2}R_{j_2} + \dots + x_{j_s}R_{j_s}$, here $x_{i_1}, x_{i_2}, \dots, x_{i_s}$ are sides of the rectangles from one side, and $x_{j_1}, x_{j_2}, \dots, x_{j_t}$ from the other.

Write down all these equations (in variables x's, a and b). Express the variables except b subsequently and substitute to the remaining equations, starting with a. After using all possibilities we obtain that each variable from the first group (containing a) is expressed through the variables from the second group (containing b) as a linear combination whose coefficients are ratios of polynomials in $R_1, R_2, \dots, R_n$:

$a = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n + \xi b$

$x_i = \mu_{i1}x_1 + \mu_{i2}x_2 + \dots + \mu_{in}x_n + \mu_i b$ ($i = 1, 2, \dots$,n)

The left-hand variables are expressed by equations which have nonzero coefficients only at the variables from the second group (for these variables we add equations $x_i = x_i$).

Let us prove that the second group consists of b only. Suppose the contrary. Let x_n belong to the second group. Note that if the variables from the second group have such values that all equations are valid and the values are positive then we obtain the required dissection of the rectangle (prove this). Take the original dissection, increase x_n by ε so that all x's and a remain positive. We obtain a large rectangle with sides $a + \xi_n\varepsilon$ and b dissected into rectangles with vertical sides $x_1 + \mu_{1n}\varepsilon, x_2 + \mu_{2n}\varepsilon, \dots, x_{n-1} + \mu_{n-1n}\varepsilon, x_n + \varepsilon$.

The equality of areas has the form $(a+\xi_n\varepsilon)b = R_1(x_1+\mu_{1n}\varepsilon)^2 + R_2(x_2+\mu_{2n}\varepsilon)^2 + \dots + R_{n-1}(x_{n-1}+\mu_{n-1n}\varepsilon)^2 + R_n(x_n+\varepsilon)^2 \Rightarrow$ $(R_1\mu_{1n}^2 + R_2\mu_{2n}^2 + \dots + R_{n-1}\mu_{n-1n}^2 + 1)\varepsilon^2 + (2R_1x_1\mu_{1n} + 2R_2x_2\mu_{2n} + \dots + 2R_{n-1}x_{n-1}\mu_{n-1n} + 2R_nx_n - \xi_n b)\varepsilon = 0$

We see that not more than two ε's satisfy this equation, but originally we can take any ε from some neighborhood of zero. Hence the second group in fact does not contain any x but b only. As was already noted above, all variables of the first group can be expressed as linear combinations of the variables from the second group, that is, $a = pb$, where p is a ratio of polynomials in $R_1, R_2, \dots, R_n$.

14c. *Answer*: no, it is impossible. Increase the resistance of this resistor keeping the voltage at all nodes the same. Then the calorification will decrease, and after redistribution it will decrease once more, hence the total resistance will increase.

14d. Suppose a rectangle with side ratio R is dissected into rectangles with side ratio R and $\frac{1}{R}$, and there exists some rectangle of the second form. After dilation with factor R we obtain a square dissected into squares and rectangles having side ratio $\frac{1}{R^2}$. By Problem 14a, we have a circuit with resistance 1, which consists of resistors with resistance 1 and $\frac{1}{R^2}$. By Problem 14b, total resistance is a ratio of polynomials in R with integer coefficients. Assign the value 1 to it.

1) If the ratio of the polynomials is not 1 identically then R is a root of a polynomial with integer coefficients.

2) If both polynomials are equal then under increasing R, total resistance remains equal to 1. Then if some resistor has resistance $\frac{1}{R^2}$ then its resistance decreases. If we decrease these resistance one after another then the solution of Problem 14 shows that total resistance decreases, a contradiction.

SUMMARY: COMPLETE INVARIANTS.

In the present project, we have constructed examples of invariants which make it possible to prove impossibility of some dissections. A natural question arises, what invariants among them are *complete*, that is, in which cases equality of invariants for two polygons implies existence of the required dissection?

It turns out that most of the constructed invariants are in fact complete.

Start with the simplest example, $J_l(M)$ (*Hadwiger invariant*). As shown above, if a polygon M can be dissected into several polygons which can be combined using only parallel shifts of parts to form a new polygon M' then $J_l(M) = J_l(M')$ (Problem 4c). In some sense, the converse turns out to be true as well:

Hadwiger–Gluer Theorem. [1] A polygon M can be dissected into several polygons which can be combined into a polygon M' by parallel shifts of parts only, if and only if the areas of M and M' are equal and any directed line l satisfies $J_l(M) = J_l(M')$.

We do not know whether the similar assertion is valid for the invariant $J_{l,\phi}(M)$ (cf. Problem 5a). It is known to be true in the particular case $\phi = \pi$. Then $J_{l,\phi}(M) \equiv 0$, and another Hadwiger–Gluer theorem states that any two polygons in the plane, having equal area can be dissected into polygons whose corresponding sides are parallel. At first sight, this is improbable: consider, for instance, two congruent triangles in the plane such that one of them is obtained from the other one by rotation through a small angle.

Now consider polyhedrons. The set of rectangles which we attach to polyhedron is called its *Dehn invariant* (this definition is equivalent to the usual algebraic definition [2]). Surprisingly, some converse for Lemma I is true as well:

Sidler theorem [1] If two polyhedrons have equal volume and the corresponding sets of rectangles become $\square$-scissor-congruent after adding appropriate rectangles of the form $l \times \pi$ then two original polyhedrons are scissor-congruent.

The above constructed invariant of $\square$-scissor-congruence of sets of rectangles in the plane is not complete but an analogous procedure leads to a complete invariant (*Kenyon invariant* [4]).

To conclude, let us discuss sufficiency of the obtained conditions for the numbers α, β, γ and x in Problems A, B, D. We don't know whether the number x in Problem D be a root of an *arbitrary* polynomial having integer coefficients. However a similar problem on dissection a square into similar rectangles has negative answer:

Laszkovich–Szekeres–Freiling–Rinne theorem. [3, 5] For any $x > 0$ the following conditions are equivalent:
(1) a square can be dissected into similar rectangles with side ratio x;
(2) the number x is algebraic, and all complex numbers conjugate to it have positive real part;
(3) there exist positive rational numbers $c_1, c_2, \dots, c_n$ such that

$$c_1 x + \cfrac{1}{c_2 x + \cfrac{1}{\dots + \cfrac{1}{c_n x}}} = 1.$$

Thus a square can be dissected into similar rectangles with side ratio $2 + \sqrt{2}$ but cannot be dissected into rectangles with side ratio $1 + \sqrt{2}$.

We don't possess a complete description of angles α, β, γ such that triangles in Problems A and B can be dissected. It is not even known whether a triangle exists which is not right but can be dissected into several triangles similar to it but oriented oppositely.

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