DISTANT CONTEST OF THE TOURNAMENT OF TOWNS 2019-2020
The competition is individual and aimed at students of the last 3 school grades (9-11 in Russian school system).

The Violet problem

A. K. Tolpygo

The range of tasks that we consider is quite wide. However it can be described in a single phrase: *How often does it occur that an integer is a multiple of the sum of its digits?*

All one-digit integers obviously satisfy this condition, so we consider only the integers consisting of two or more (even better three or more) digits.

Definition 1. We call an integer *violet* if it is **a)** greater than 100 and **b)** a multiple of the sum of its digits.

Clearly all the integers with the sum of digits equal to 1, 3 or 9 are violet. But it is not hard to find more examples. For example, integers 224, 605 and 999 999 999 are violet.

Problem 1. Prove that for each $m \in \mathbb{N}$ there exists a violet integer with the sum of digits equal to m.

First of all we consider questions such as:
— how many violet integers can be located successively?
— what is the rate of violet integers in the row from 1 to N, and how does it behave depending on N?

Problem 2. How many violet integers can be located successively (i.e. so that all k integers in a row $m, m+1, \ldots, m+k-1$ are violet)? In particular, is it possible that k **a)** is equal to 5? **b)** is equal to 10? **c)** is infinite?

Problem 3*. (hard). What is the maximum possible k? *If you can't give exact answer then give some lower and upper bound (example: 55 successive integers can't be violet, and for 27 it is possible to find an example).*

Hint: The Generalized Chinese remainder theorem can be useful for solving some of the previous and following problems.

Theorem 1. (Chinese remainder theorem) *Suppose that there are k arbitrary pairwise coprime integers $q_1, q_2, \ldots q_k$ and arbitrary integers $r_1, \ldots, r_k$ such that $0 \leqslant r_i \leqslant q_i$ for all i. Then there exists an integer N with remainder r_i modulo q_i .*

Theorem 2. (Generalized Chinese remainder theorem) *Suppose that there are arbitrary integers $q_1, q_2, \ldots q_k$ and arbitrary integers $r_1, \ldots, r_k$ such that $0 \leqslant r_i \leqslant q_i$ for all i. Suppose that d_{ij} is the greatest common divisor of q_i and q_j. An integer N with remainder r_i modulo q_i exists if and only if*

Problem 4. Complete the statement of the Generalized Chinese remainder theorem and prove it.

Problem 5. How many violet integers can be in one hundred, i.e. among integers from k to $k+99$? (Find all possible answers.)

Problem 6. Estimate the number of violet integers in the range from 1 to 100000. It is necessary to find an upper and a lower bound such that the upper bound is at most 10 times greater than the lower one.

Problem 7. Suppose that $M(N)$ is the number of violet integers among first N integers. Prove that if $N \to \infty$ then $M(N)/N$ has a limit, and find it.

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Now we change our problem. We consider only integers with the given sum of digits. So suppose that an integer S is given. We choose an integer N, and by $K = K(S)$ we denote the set of all the integers not exceeding N with the sum of digits equal to S, and by $\Lambda = \Lambda(S)$ we denote its subset consisting of violet integers (i.e. the multiples of S). Let k and l be the numbers of the elements in K и Λ correspondingly. And finally let $\alpha(S)$ be the limit of the ratio l/k when N increases unlimitedly.

Problem 8. Prove that for each integer S the limit $\lim\limits_{N\to\infty} \alpha(S)$ exists and is positive.

Problem 9. Find $\alpha(S)$ for $S = 1, 2, \ldots, 12$.

Note 1. Pay special attention to $S = 7, S = 11$.
Note 2. If you find this limit for some other S then this will be rewarded. In particular, we recommend to investigate the cases when S is equal to 14, 27, 101.

Problem 10*. (hard) Prove that for each $\varepsilon > 0$ and all prime integers p, except some finite number of these, it is true that $|\alpha(p) - 1/p| < \varepsilon$.

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Our concept of violet integers is naturally related to the decimal notation of an integer: we check whether N is divisible by M, where N certainly doesn't depend on choice of the numeral system but the sum of its digits M depends on it.

Problem 11. Investigate the following question: can a multidigit integer (i.e., consisting of 2 or more digits) have the same sum of digits in different numeral systems? Find corresponding examples. Examine how often such integers can appear.

Let us generalize our notion of the violet integer.

Definition 2. Suppose that given are integers N, q. An integer N is called *q-violet* if it is greater than q^2 and divisible by the sum of its digits in the numeral system with base q.

For example, integer 231 is 11-violet because $231 = 121 + 10 \cdot 11$ and correspondingly is of the form 1[10]0 (3-digit integer consisting of digits 1, 10, 0) in 11-base numeral system.

Problem 12. What is the greatest number of 2-violet integers located successively?

Problem 13. Estimate the possible number of q-violet integers located successively for some arbitrary q.

Problem 14. Is it true that for each integer m there exists such q that in the numeral system with base q there exist m q-violet integers located successively?

You may send papers up to **March 1, 2020.** It is allowed to send solutions by parts:

– scans by e-mail: zktg@turgor.ru *(please write your name and your town in the subject of the letter)*, solutions are accepted in pdf or as zip-archives of .jpg, .png or .gif files.;

– by ordinary mail to:
Russia, 119002, Moscow, Bolshoy Vlasyevskiy per., 11, MCCME, Tournament of Towns
(with note "Distant contest"). ;

You can find all the information about the contest at https://www.turgor.ru/zktg/ .
If you have any questions about the contest please apply to zktg@turgor.ru.