45th INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior A-Level Paper, Fall 2023

(The result is computed from the three problems with the highest scores.)

points	problems
4	1. Every square of a 8×8 board is filled with a positive integer, such that the following condition holds: if a chess knight can move from some square to another then the ratio of numbers from these two squares is a prime number. Is it possible that some square is filled with 5, and another one with 6? *Egor Bakaev*
6	2. A unit square paper has a triangle-shaped hole (vertices of the hole are not on the border of the paper). Prove that a triangle with area of 1/6 can be cut from the remaining paper. *Alexandr Yuran*
7	3. Let us call a bi-squared card 2×1 *regular*, if two positive integers are written on it and the number in the upper square is less than the number in the lower square. It is allowed at each move to change both numbers in the following manner: either add the same integer (possibly negative) to both numbers, or multiply each number by the same positive integer, or divide each number by the same positive integer. The card must remain regular after any changes made. What minimal number of moves is sufficient to get any regular card from any other regular card? *Alexey Glebov*
7	4. A triangle ABC with angle A equal to $60°$ is given. Its incircle is tangent to side AB at point D, while its excircle tangent to side AC, is tangent to the extension of side AB at point E. Prove that the perpendicular to side AC, passing through point D, meets the incircle again at a point equidistant from points E and C. (The excircle is the circle tangent to one side of the triangle and to the extensions of two other sides.) *Azamat Mardanov*
9	5. Tom has 13 weight pieces that look equal, however 12 of them weigh the same and the 13th piece is fake and weighs more than the others. He also has two balances: one shows correctly which pan is heavier or that their weights are equal, the other one gives the correct result when the weights on the pans differ, and gives a random result when the weights are equal. (Tom does not know which balance is which). Tom can choose the balance before each weighting. Prove that he can surely determine the fake weight piece in three weighings. *Andrey Arzhantsev*
10	6. The baker has baked a rectangular pancake. He then cut it into n^2 rectangles by making $n-1$ horizontal and $n-1$ vertical cuts. Being rounded to the closest integer, the areas of resulting rectangles equal to all positive integers from 1 to n^2 in some order. For which maximal n could this happen? (Half-integers are rounded upwards.) *Georgy Karavaev*
12	7. There are 100 chess bishops on white squares of a 100×100 chess board. Some of them are white and some of them are black. They can move in any order and capture the bishops of opposing color. What number of moves is sufficient for sure to retain only one bishop on the chess board? *Alexandr Gribalko*

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Senior A-Level Paper, Fall 2023

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points	problems
4	1. For every polynomial of degree 45 with coefficients 1, 2, 3, ..., 46 (in some order) Tom has listed all its distinct real roots. Then he increased each number in the list by 1. What is now greater: the amount of positive numbers or the amount of negative numbers? *Alexey Glebov*
5	2. For which maximal N there exists an N-digit number with the following property: among any sequence of its consecutive decimal digits some digit is present once only? *Alexey Glebov*
	3. A square was split into several rectangles so that the centers of rectangles form a convex polygon.
3	a) Is it true for sure that each rectangle adjoins to a side of the square?
6	b) Can the number of rectangles equal 23? *Alexandr Shapovalov*
9	4. A convex quadrilateral $ABCD$ with area of S is given. Inside each side of the quadrilateral a point is selected. These points are consecutively linked by segments, so that $ABCD$ is split into a smaller quadrilateral and 4 triangles. Prove that the area of at least one triangle does not exceed $S/8$. *Mikhail Malkin*
10	5. Chord DE of the circumcircle of the triangle ABC intersects sides AB and BC in points P and Q respectively. Point P lies between D and Q. Angle bisectors DF and EG are drawn in triangles ADP and QEC. It turned out that the points D, F, G, E are concyclic. Prove that the points A, P, Q, C are concyclic. *Azamat Mardanov*
12	6. A table 2×2024 is filled with positive integers. Specifically, the first row is filled with numbers from the set $\{1, \dots, 2023\}$. It turned out that for any two columns the difference of numbers from the first row is divisible by the difference of numbers from the second row, while all numbers in the second row are pairwise different. Is it true for sure that the numbers in the first row are equal? *Ivan Kukharchuk*
14	7. On the table there are $2n$ coins that look the same. It is known that n of them weigh 9 g. each, while the remaining n weigh 10 g. each. It is required to split the coins into n pairs with total weight of each pair 19 g. Prove that this can be done in less than n weighings using a balance without additional weights (the balance shows which pan is heavier or that their weight is equal). *Alexandr Gribalko*