44th INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior O-Level Paper, Spring 2023

Grades 8 – 9 (ages 13-15)

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points problems

1. There are N mess-loving clerks in the office. Each of them has some rubbish on the desk. The mess-loving clerks leave the office for lunch one at a time (after return of the preceding one). At that moment all those remaining put half of rubbish from their desks on the desk of the one who left. Can it so happen that after all of them have had lunch the amount of rubbish at the desk of each one will be the same as before lunch if:

1 a) $N = 2$;

3 b) $N = 10$?

Alexey Zaslavsky

2. Medians BK and CN of triangle ABC intersect at point M. Consider

4 quadrilateral $ANMK$ and find the maximum possible number of its sides having length 1.

Egor Bakaev

3. There are 2023 dice on the table. For 1 dollar, one can pick any dice and put it back on any of its four (other than top or bottom) side faces. How many

5 dollars at a minimum will guarantee that all the dice have been repositioned to show equal number of dots on top faces? (The number of dots on faces of each cube dice equals 1, 2, 3, 4, 5 and 6. The total number of dots on two opposite faces always equals 7.)

Egor Bakaev

4. Consider the sum

5
$$Q(x) = \lfloor x \rfloor + \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{4} \right\rfloor + \ldots + \left\lfloor \frac{x}{10000} \right\rfloor$$

for an arbitrary x. Find the difference $Q(2023) - Q(2022)$. (Here $\lfloor x \rfloor$ denotes the integer part of x, that is the maximum integer not exceeding x.)

Alexey Tolpygo

5. There is a single coin on each square of a 5×5 board. All the coins look the same. Two of them are fakes and have equal weight. Genuine coins are heavier than fake ones and also weigh the same. The fake coins are on the squares sharing just one vertice. Is it possible to determine for sure:

2 a) 13 genuine coins;

3 b) 15 genuine coins;

2 c) 17 genuine coins

in a single weighing on a balance with no unit weights?

Rustem Zhenodarov, Alexandr Gribalko, Sergey Tokarev

44th INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS
Senior O-Level Paper, Spring 2023
Grades 10 – 11 (ages 15 and older)
(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points problems

1. There are 2023 dice on the table. For 1 dollar, one can pick any dice and put it back on any of its four (other than top or bottom) side faces. How many dollars at a minimum will guarantee that all the dice have been repositioned to show equal number of dots on top faces? (The number of dots on faces of each cube dice equals 1, 2, 3, 4, 5 and 6. The total number of dots on two opposite faces always equals 7.)

4

Egor Bakaev

2. A positive integer n is given. Consider the sum

$$Q(x) = \lfloor x \rfloor + \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{4} \right\rfloor + \ldots + \left\lfloor \frac{x}{10^n} \right\rfloor$$

4

for an arbitrary x. Find the difference $Q(10^n) - Q(10^n - 1)$. (Here $\lfloor x \rfloor$ denotes the integer part of x, that is the maximum integer not exceeding x.)

Alexey Tolpygo

3. Let I be the incenter of triangle ABC. Let N be the foot of the bisector of angle B. The tangent line to the circumcircle of triangle AIN at vertice A and the tangent line to the circumcircle of triangle CIN at vertice C intersect at point D. Prove that lines AC and DI are perpendicular.

5

Mikhail Evdokimov

4. Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be infinite increasing arithmetic progressions. Their terms are positive numbers. It is known that the ratio a_k/b_k is an integer for all k. Is it true that this ratio does not depend on k?

5

Boris Frenkin

5. The distance between any two of the five given points exceeds 2. Is it true that the distance between some two of these points exceeds 3 if these five points are in:

3 a) the plane;

3 b) three-dimensional space?

Alexey Tolpygo