44th INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior A-Level Paper, Spring 2023

Grades 8 – 9 (ages 13-15)

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

points	problems
4	1. A right-angled triangle has an angle equal to 30°. Prove that one of the bisectors of the triangle is twice shorter than another one. *Egor Bakaev*
5	2. There is a bacterium in one of the cells of the 10×10 checkered board. At the first move, the bacterium shifts to a cell adjacent by side to the original one, and divides into two bacteria (both stay in the same cell). Then again, one of the bacteria on the board shifts to a cell adjacent by side and divides into two bacteria, and so on. Is it possible that after some number of such moves the number of bacteria in each cell of the board is the same? *Alexandr Gribalko*
7	3. Let us call a positive integer *pedestrian* if all its decimal digits are equal to 0 or 1. Suppose that the product of some two pedestrian integers also is pedestrian. Is it necessary in this case that the sum of digits of the product equals the product of the sums of digits of the factors? *Viktor Kleptsyn, Konstantin Knop*
8	4. The sides of the regular triangle ABC are also sides of triangles $AB'C$, $CA'B$, $BC'A$ constructed outside it. In the resulting hexagon $AB'CA'BC'$ each of the angles $A'BC'$, $C'AB'$, $B'CA'$ is greater than $120°$, and the sides satisfy the equalities $AB' = AC'$, $BC' = BA'$, $CA' = CB'$. Prove that the segments AB', BC', CA' can form a triangle. *David Brodsky*
8	5. The positive integers from 1 to 100 are painted into three colors: 50 integers are red, 25 integers are yellow and 25 integers are green. The red and yellow integers can be divided into 25 triples such that each triple includes two red integers and one yellow integer which is greater than one of the red integers and smaller than another one. The similar assertion is valid for the red and green integers. Is it necessarily possible to divide all the 100 integers into 25 quadruples so that each quadruple includes two red integers, one yellow integer and one green integer such that the yellow and the green integer lie between the red ones? *Alexandr Gribalko*
	6. Let X be a set of integers which can be partitioned into N disjoint increasing arithmetic progressions (infinite in both directions), and cannot be partitioned into a smaller number of such progressions. Is such partition into N progressions unique for every such X if
4	a) $N = 2$;
4	b) $N = 3$? (An increasing arithmetic progression is a sequence of numbers such that each number exceeds its left neighbor by the same quantity.) *Viktor Kleptsyn*
10	7. A regular 100-gon is dissected into some number of parallelograms and two triangles. Prove that these triangles are equal. *Alexandr Yuran*

44th INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS
Senior A-Level Paper, Spring 2023
Grades 10 – 11 (ages 15 and older)
(The result is computed from the three problems with the highest scores.)

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points	problems
4	1. Given are two sequences of letters A and B, each sequence contains 100 letters. At each step it is possible either to insert an arbitrary number of identical letters into a sequence at any position (maybe at the beginning or at the end), or remove from a sequence an arbitrary number of consecutive identical letters. Prove that it is possible to transform the first sequence into the second one in at most 100 steps. *Vladislav Novikov*
5	2. The perimeter of triangle ABC equals 1. The circle ω touches the side BC, the extension of side AB at point P and the extension of side AC at point Q. The line containing the midpoints of sides AB and AC meets the circumcircle of triangle APQ at points X and Y. Determine the length of segment XY. *David Brodsky*
6	3. Let $P(x)$ be a polynomial of degree $n > 5$ with integer coefficients and with n distinct integer roots. Prove that the polynomial $P(x) + 3$ has n distinct real roots. *Mikhail Malkin*
8	4. A regular 100-gon is dissected into some number of parallelograms and two triangles. Prove that these triangles are equal. *Alexandr Yuran*
8	5. Given an integer $h > 1$. Let us call a positive ordinary fraction (not necessarily irreducible) *good* if the sum of its numerator and denominator equals h. Let us call the integer h *remarkable* if every positive ordinary fraction with the denominator smaller than h can be expressed through good fractions (not necessarily distinct) using only addition and subtraction. Prove that h is remarkable if and only if it is prime. (Remind that an ordinary fraction has an integer numerator and a positive integer denominator.) *Tatiana Kazitsyna*
10	6. The midpoints of all altitudes of some tetrahedron lie on the sphere inscribed in it. Is this tetrahedron necessarily regular? *Mikhail Evdokimov*
12	7. At an island, there are chameleons of five colors. If a chameleon bites another one, the color of the bitten chameleon changes into one of these 5 colors according to some rule, and the result depends only on the colors of the biting and the bitten chameleon. It is known that 2023 red chameleons can agree on a sequence of bites such that all of them will eventually become blue. What is the least k such that we can guarantee that k red chameleons can become blue, biting each other only? (For instance, the following rules are possible: if a red chameleon bites a green one then the bitten one becomes blue; if a green chameleon bites a red one then the bitten one remains red, so „changes its color to red“; if a red chameleon bites a red one then the bitten one becomes yellow, and so on. Other rules are possible as well.) *Mikhail Raskin*