43rd INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior A-Level Paper, Spring 2022

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points problems

4 1. Find the largest positive integer n such that for each prime p with $2 < p < n$ the difference $n - p$ is also prime.

Igor Akulich

7 2. Prove that for any convex quadrilateral it is always possible to cut out three smaller quadrilaterals similar to the original one with the scale factor equal to $1/2$. (The angles of a smaller quadrilateral are equal to the corresponding original angles and the sides are twice smaller then the corresponding sides of the original quadrilateral.)

Alexandr Yuran

7 3. For each of the nine positive integers n, $2n$, $3n$, $\ldots$, $9n$ Alice takes the first decimal digit (from the left) and writes it onto a blackboard. She selected n so that among the nine digits on the blackboard there is the least possible number of different digits. What is this number of different digits equal to?

Alexey Tolpygo

4. Consider a white 100×100 square. Several cells (not necessarily neighbouring) were painted black. In each row or column that contains some black cells their number is odd. Hence we may consider the *middle* black cell for this row or column (with equal numbers of black cells in both opposite directions). It so happened that all the middle black cells of such rows lie in different columns and all the middle black cells of the columns lie in different rows.

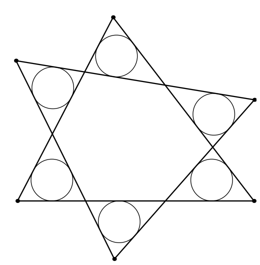
5 a) Prove that there exists a cell that is both the middle black cell of its row and the middle black cell of its column.

5 b) Is it true that any middle black cell of a row is also a middle black cell of its column?

Boris Frenkin

10 5. The intersection of two triangles is a hexagon. If this hexagon is removed, six small triangles remain. Those six triangles have the same radii of the incircles. Prove that the radii of the incircles of the two original triangles are also equal.

Andrey Kushnir



10 6. There were made 7 golden, 7 silver and 7 bronze medals for a tournament. All the medals of the same material should weigh the same and the medals of different material should have different weight. However, it so happened that exactly one medal had a wrong weight. If this medal is golden, it is lighter than a standard golden medal; if it is bronze, it is heavier than a standard bronze one; if it is silver it may be lighter or heavier than a standard silver one. Is it possible to find the nonstandard medal for sure, using three weighings on a balance scale with no weights?

Alexandr Gribalko

12 7. Let p be a prime number and let M be a convex polygon. Suppose that there are precisely p ways to tile M with equilateral triangles with side 1 and squares with side 1. Show that there is some side of M of length $p - 1$.

Nikolay Belukhov

43rd INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Senior A-Level Paper, Spring 2022

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points problems

5 1. For each of the nine positive integers n, $2n$, $3n$, $\ldots$, $9n$ Alice takes the first decimal digit (from the left) and writes it onto a blackboard. She selected n so that among the nine digits on the blackboard there is the least possible number of different digits. What is this number of different digits equal to?

Alexey Tolpygo

2. On a blank paper there were drawn two perpendicular axes x and y with the same scale. The graph of a function $y = f(x)$ was drawn in this coordinate system. Then the y axis and all the scale marks on the x axis were erased. Provide a way how to draw again the y axis using pencil, ruler and compass if:

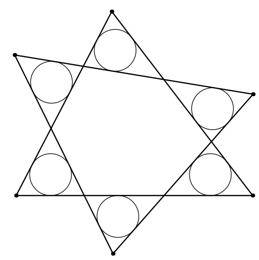
4 a) $f(x) = 3^x$;

4 b) $f(x) = \log_a x$, where $a > 1$ is an unknown number.

Mikhail Evdokimov

8 3. The intersection of two triangles is a hexagon. If this hexagon is removed, six small triangles remain. Those six triangles have the same radii of the incircles. Prove that the radii of the incircles of the two original triangles are also equal.

Andrey Kushnir



8 4. A rook travelled through an $n \times n$ board, stepping at each turn to the cell neighbouring the previous one by a side, so that each cell was visited once. Bob has put the integer numbers from 1 to n^2 into the cells, corresponding to the order in which the rook has passed them. Let M be the greatest difference of the numbers in neighbouring by side cells. What is the minimal possible value of M?

Boris Frenkin

8 5. What is the maximal possible number of roots on the interval $(0, 1)$ for a polynomial of degree 2022 with integer coefficients and with the leading coefficient equal to 1?

Alexey Kanel-Belov

8 6. The king assembled 300 wizards and gave them the following challenge. For this challenge, 25 colors can be used, and they are known to the wizards. Each of the wizards receives a hat of one of those 25 colors. If for each color the number of used hats would be written down then all these numbers would be different, and the wizards know this. Each wizard sees what hat was given to each other wizard but does not see his own hat. Simultaneously each wizard reports the color of his own hat. Is it possible for the wizards to coordinate their actions beforehand so that at least 150 of them would report correctly?

Alexandr Gribalko

7. A starship is located in a halfspace at the distance a from its boundary. The crew knows this but does not know which direction to move to reach the boundary plane. The starship may travel through the space by any path, may measure the way it has already travelled and has a sensor that signals when the boundary is reached. Is it possible to reach the boundary for sure, having passed no more than:

6 a) $14a$;

6 b) $13a$?

Mikhail Evdokimov