43rd INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior A-Level Paper, Fall 2021

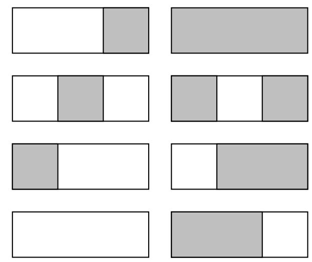
(The result is computed from the three problems with the highest scores.)

points problems

5 1. Alice wrote a sequence of $n > 2$ nonzero nonequal numbers such that each is greater than the previous one by the same amount. Bob wrote the inverses of those n numbers in some order. It so happened that each number in his row also is greater than the previous one by the same amount, possibly not the same as in Alice's sequence. What are the possible values of n?

Alexey Zaslavsky

6 2. On the table there are all 8 possible horizontal bars 1×3 such that each 1×1 square is either white or gray (see figure). It is allowed to move them in any direction by any (not necessarily integer) distance. We may not rotate them or turn them over. Is it possible to move the bars so that they do not overlap, all the white points form a polygon bounded by a closed non-self-intersecting broken line and the same is true for all the gray points?



Mikhail Ilyinsky

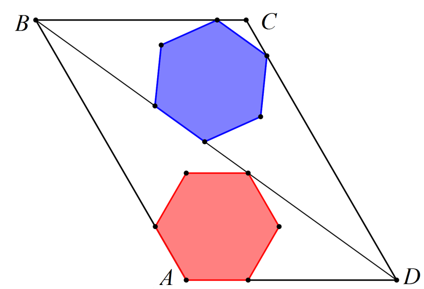
7 3. The hypotenuse of a right triangle has length 1. Consider the line passing through the points of tangency of the incircle with the legs of the triangle. The circumcircle of the triangle cuts out a segment of this line. What is the possible length of this segment?

Maxim Volchkevich

8 4. The number 7 is written on a board. Alice and Bob in turn (Alice begins) write an additional digit in the number on the board: it is allowed to write the digit at the beginning (provided the digit is nonzero), between any two digits or at the end. If after someone's turn the number on the board is a perfect square then this person wins. Is it possible for some of the players to guarantee the win?

Alexandr Gribalko

9 5. A parallelogram $ABCD$ is split by the diagonal BD into two equal triangles. A regular hexagon is inscribed into the triangle ABD so that two of its consecutive sides lie on AB and AD and one of its vertices lies on BD. Another regular hexagon is inscribed into the triangle CBD so that two of its consecutive vertices lie on CB and CD and one of its sides lies on BD. Which of the hexagons is bigger?



Konstantin Knop

9 6. Let $\lfloor x \rfloor$ denote the integer part of the number x, that is, the largest integer that is not greater than x. Prove that for any positive integers $a_1, a_2, \ldots, a_n$ the following inequality holds true:

$$\left\lfloor \frac{a_1^2}{a_2} \right\rfloor + \left\lfloor \frac{a_2^2}{a_3} \right\rfloor + \ldots + \left\lfloor \frac{a_n^2}{a_1} \right\rfloor \geq a_1 + a_2 + \ldots + a_n.$$

Maxim Didin

12 7. There are 20 buns with jam and 20 buns with treacle arranged in a row in random order. Alice and Bob take in turn a bun from any end of the row. Alice starts, and wants to finally obtain 10 buns of each type; Bob tries to prevent this. Is it true for any order of the buns that Alice can win no matter what are the actions of Bob?

Alexandr Gribalko

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points	problems	
5	1.	The wizards A, B, C, D know that the integers $1, 2, \ldots, 12$ are written on 12 cards, one integer on each card, and that each wizard will get three cards and will see only his own cards. Having received the cards, the wizards made several statements in the following order. A: "One of my cards contains the number 8". B: "All my numbers are prime". C: "All my numbers are composite and they all have a common prime divisor". D: "Now I know all the cards of each wizard". What were the cards of A if everyone was right? *Mikhail Evdokimov*
7	2.	There was a rook at some square of a 10×10 chessboard. At each turn it moved to a square adjacent by side. It visited each square exactly once. Prove that for each main diagonal (the diagonal between the corners of the board) the following statement is true: in the rook's path there were two consecutive steps at which the rook first stepped away from the diagonal and then returned back to the diagonal. *Alexandr Gribalko*
7	3.	Grasshopper Gerald and his 2020 friends play leapfrog on a plane as follows. At each turn Gerald jumps over a friend so that his original point and his resulting point are symmetric with respect to this friend. Gerald wants to perform a series of jumps such that he jumps over each friend exactly once. Let us say that a point is *achievable* if Gerald can finish the 2020th jump in it. What is the maximum number N such that for some initial placement of the grasshoppers there are just N achievable points? *Mikhail Svyatlovskiy*
7	4.	What is the minimum k for which among any three nonzero real numbers there are two numbers a and b such that either $\lvert a-b\rvert \le k$ or $\lvert \frac{1}{a} - \frac{1}{b}\rvert \le k$? *Maxim Didin*
9	5.	Let $ABCD$ be a parallelogram and let P be a point inside it such that $\angle PDA = \angle PBA$. Let ω_1 be the excircle of PAB opposite to the vertex A. Let ω_2 be the incircle of the triangle PCD. Prove that one of the common tangents of ω_1 and ω_2 is parallel to AD. *Ivan Frolov*
10	6.	There are 20 buns with jam and 20 buns with treacle arranged in a row in random order. Alice and Bob take in turn a bun from any end of the row. Alice starts, and wants to finally obtain 10 buns of each type; Bob tries to prevent this. Is it true for any order of the buns that Alice can win no matter what are the actions of Bob? *Alexandr Gribalko*
	7.	A checkered square of size 2×2 is covered by two triangles. Is it necessarily true that:
6	a)	at least one of its four cells is fully covered by one of the triangles;
6	b)	some square of size 1×1 can be placed into one of these triangles? *Alexandr Shapovalov*