42nd INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

SOLUTIONS OF PROBLEMS
Spring 2021

Junior O-Level Paper

1 [4]. Is it possible that a product of 9 consecutive positive integers is equal to a sum of 9 consecutive (not necessarily the same) positive integers?

Boris Frenkin

Answer: yes, it is possible.
Solution. For example, $(8!-4)+(8!-3)+\ldots+8!+\ldots+(8!+4)=9\cdot 8!=9!$.
Note. See also problem 3 in Senior O-Level.

2 [4]. Let AX and BZ be altitudes of the triangle ABC. Let AY and BT be its angle bisectors. It is given that angles XAY and ZBT are equal. Does this necessarily imply that ABC is isosceles?

Jury

Answer: no, not necessarily.
Solution. For example, it is easy to show that in a triangle with angles $\angle A=40^\circ$, $\angle B=80^\circ$, $\angle C=60^\circ$ both indicated angles are equal to 10°, and in a triangle with angles $\angle A=30^\circ$, $\angle B=90^\circ$, $\angle C=60^\circ$ both indicated angles are equal to 15°.
Note. Any triangle with angle C equal to 60° fits the requirements.

3 [4]. Maria has a balance scale that can indicate which of its pans is heavier or whether they have equal weight. She also has 4 weights that look the same but have masses of 1001, 1002, 1004 and 1005 g. Can Maria determine the mass of each weight in 4 weighings? The weights for a new weighing may be picked when the result of the previous ones is known.

Jury

Answer: yes, she can.
Solution 1. The first three weighings are as follows. Split the weights into two pairs in three different ways and in each case compare the pairs on the balance scale. We will obtain the equality for pairs 1001, 1005 and 1002, 1004. Moreover, only the weight 1001 in two other weighings was in the "light" pair and only the weight 1005 in two other weighings was in the "heavy" pair, that is how we can find them. The remaining two weights 1002 and 1004 can be distinguished by the fourth weighing.
Solution 2. Firstly, put two weights on each pan.

1) One of the pans is heavier than the other. Then the weights are divided into two pairs, light and heavy. Here we have two options: the light pair consists of weights 1001, 1002 and the heavy pair consists of weights 1004, 1005, or the light pair consists of weights 1001, 1004 and the heavy pair consists of weights 1002, 1005. By the next two weighings we can range the weights in each pair by their mass. By the fourth weighing we compare the heaviest weight of the light pair and the lightest weight of the heavy pair, thus we can distinguish between our two options.

2) The pans have equal masses. Then the weights are divided into two pairs of equal mass: 1001, 1005 and 1002, 1004. By the second and third weighings we range the weights in each pair by their mass. By the fourth weighing we compare the lightest weights of the two pairs, thus we can find out which pair is which.

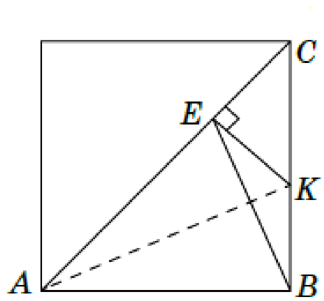
4 a) [3]. Is it possible to split a square into 4 isosceles triangles such that no two are congruent?

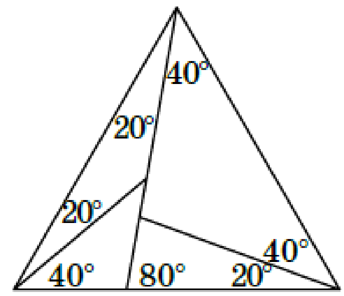
b) [3]. Is it possible to split an equilateral triangle into 4 isosceles triangles such that no two are congruent?

Vladimir Rastorguev

Answer: yes, it is possible both in a) and b).

Solution. See figures. In the left figure, firstly, draw bisector AK of angle BAC, and then reflect point B through AK obtaining point E.





5. There are several dominoes on a board such that each domino occupies two adjacent cells and none of the dominoes are adjacent by side or vertex. The bottom left and top right cells of the board are free. A token starts at the bottom left cell and can move to a cell adjacent by side: one step to the right or upwards at each turn. Is it always possible to move from the bottom left to the top right cell without passing through dominoes if the size of the board is

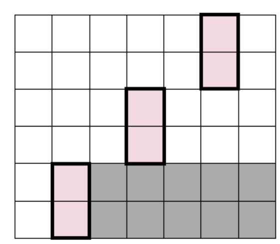
a) [2] 100×101 cells;

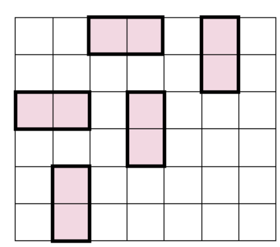
b) [4] 100×100 cells?

Nikolay Chernyatiev

a) Answer: no, not always.

Solution. In the upper right figure it is shown how to place dominoes on the board 6×7 in such a way that the token cannot move from the bottom left to the top right cell. Indeed, it is impossible to get to the (grey) area to the right of the lowest domino, because to get there, we need to go higher than this domino at first, getting above the grey area (and we cannot move downwards). Then it is impossible to get to the similar grey area to the right of the next domino, etc. This pattern can be generalized for any board $2n \times (2n+1)$.





Note. For a simpler proof one could add horizontally placed dominoes above the vertically placed ones, so the only way for the token would be blocked by the last vertically placed domino (see the lower right figure).

b) Answer: yes, always.

Solution. The first and the last cells are on the main diagonal of the board, and their “coordinates” are $(1,1)$ and $(100,100)$. Let us prove that we can get to any free cell on this diagonal.

Indeed, suppose we have got to the cell (n,n). If the cell $(n+1,n+1)$ is free, then one of the cells $(n,n+1)$ and $(n+1,n)$ is also free and we can use it to get to the cell $(n+1,n+1)$.

If, otherwise, the cell $(n+1,n+1)$ is occupied then exactly one of the cells, adjacent to it by side, is occupied, therefore one of two ways from (n,n) to $(n+2,n+2)$ is not blocked.

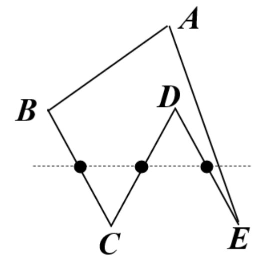
Senior O-Level Paper

1. a) [2] A convex pentagon is partitioned into three triangles by nonintersecting diagonals. Is it possible for centroids of these triangles to lie on a common straight line?
b) [2] The same question for a non-convex pentagon.

Alexandr Gribalko

Answers: a) yes, it is possible; **b)** no, it is not possible.

Solution. Obviously, there are exactly two diagonals drawn and they have a common vertex (suppose it is vertex A). Then the centroids can be obtained by homothety of the midpoints of sides BC, CD and DE with centre A and ratio $2/3$.



a) These midpoints cannot lie on a common straight line, because a straight line that does not contain any side of a convex polygon can intersect its boundary in at most two points.

b) These midpoints can lie on a common straight line as it is shown in the figure.

2. a) [2] Maria has a balance scale that can indicate which of its pans is heavier or whether they have equal weight. She also has 4 weights that look the same but have masses of 1000, 1002, 1004 and 1005 g. Can Maria determine the mass of each weight in 4 weighings? The weights for a new weighing may be picked when the result of the previous ones is known.
b) [2] The same question when the left pan of the scale is lighter by 1 g than the right one, so the scale indicates equality when the mass on the left pan is heavier by 1 g than the mass on the right pan.

Alexey Tolpygo

a) Answer: no, she cannot.

Solution. Consider various *orderings* of the weights. Let us call the arrangement of the weights by their mass a *permutation.* No matter how Maria puts the weights on the scale, she will never obtain equality. Therefore, each weighing divides a set of possible orderings of weights into no more than two parts. In the beginning there were 24 possible orderings, after the first weighing in case of "unsuccessful" outcome, there are at least 12 of them left, after the second one there are at least 6 of them left, ..., after the fourth weighing there are at least 2 of them left.

b) Answer: yes, she can.

Solution 1. At first, put two weights on each pan. As a result, all weights are divided into two pairs: the light one and the heavy one (if pans have equal weights, then we know that the more heavy pair is on the left pan). There are three options: the light pair consists of weights 1000, 1002 and the heavy pair consists of weights 1004, 1005, the light pair consists of weights 1000, 1004 and the heavy pair consists of 1002, 1005, the light pair consists of weights 1000, 1005 and the heavy pair consists of weights 1002, 1004. By the next two weighings we range the weights in each pair by their mass. By the fourth weighing we compare more heavy weights of both pairs by putting the weight from the heavy pair on the left pan. If the first option holds, then the left pan will be heavier, if the third option holds, then it will be the right pan, and for the second option the pans will be of equal mass (with the weight 1005 on the left pan and 1004 of the right pan).

Solution 2. Put weight A on the left pan and weight B on the right pan. If there is no equilibrium then put the more heavy weight on the left pan and by the second weighing compare it to weight C. If there is no equilibrium once more, then again put the more heavy weight on the left pan and by the third weighing compare it to weight D. We have one more weighing left.

If the equilibrium occurred at least once, then the more heavy weight in this weighing has mass of 1005 g, the other one has mass of 1004 g, and the two other weights can be distinguished by one more weighing.

If there was no equilibrium, then we have found the heaviest weight (1005 g). Let us consider different cases and show that in each case we already know where the weight 1004 g is (and then the other two weights can be distinguished by the fourth weighing).

1) Suppose it is A. Then for all three weighings it was on the left pan, thus the scale had to be in equilibrium once, and this case was considered above.

2) Suppose it is B. It was twice on the left pan, therefore only A can have mass of 1004 g.

3) Suppose it is C. Then the weight that has mass of 1004 g is the one that was lighter than C in the second weighing (since it is heavier than A and B, and D cannot have mass 1004 g).

4) Suppose it is D. Then the weight that has mass of 1004 g is the heaviest amongst three other weights (it was found in the second weighing).

3. [5] For which n is it possible that a product of n consecutive positive integers is equal to a sum of n consecutive (not necessarily the same) positive integers?

Boris Frenkin

Answer: for any odd n.

Solution. A product of n consecutive positive integers is divisible by n, therefore, the sum of integers that has to equal it is divisible by n. Thus the arithmetic mean of this numbers is integer, therefore, n is odd. Here is an example for $n = 2m+1$: $((2m)! - m) + ((2m)! - m + 1) + \ldots + ((2m)! + m) = (2m+1) \cdot (2m)! = n!$.

4. [5] It is well-known that a quadratic equation has no more than 2 roots. Is it possible for the equation $\lfloor x^2 \rfloor + px + q = 0$ with $p \neq 0$ to have more than 100 roots? (By $\lfloor x^2 \rfloor$ we denote the largest integer not greater than x^2.)

Alexey Tolpygo

Answer: yes, it is possible.

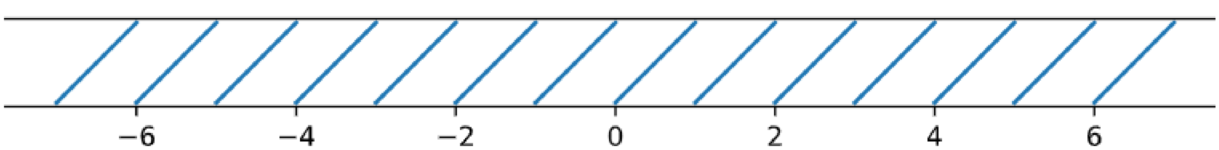
Solution. Consider, for example, the equation $[x^2] - 100x + 2500 = 0$. It has 199 roots in the form $50 + \frac{k}{100}$ (where $k = -99, -98, \ldots, 99$). Indeed,

$$\left[\left(50 + \frac{k}{100}\right)^2\right] = \left[2500 + k + \left(\frac{k}{100}\right)^2\right] = 2500 + k = 100 \cdot \left(50 + \frac{k}{100}\right) - 2500.$$

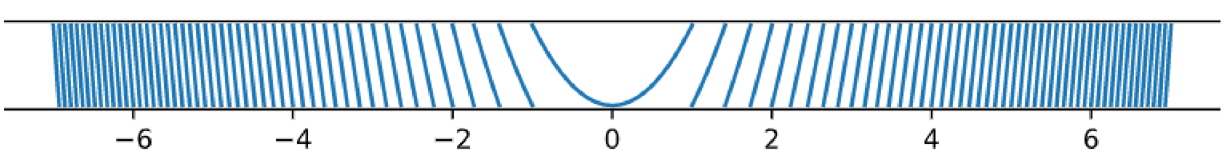
Main idea. The straight line $y = 100x - 2500$ is tangent to the parabola $y = x^2$ at the point $(50, 2500)$.

Note. Let us explain informally how one could come up with the solution.

Since $[x^2] = x^2 - \{x^2\}$, we can rewrite our equation in the form $x^2 + px + q = \{x^2\}$. Let us solve it graphically: we will search for intersections of graphs of the parabola and the fractional part of x^2. The plot of the fractional part $y = \{x\}$ is a row of equally spaced inclined half-opened intervals:



Similarly, the plot of $y = \{x^2\}$ consists of segments of the parabola: imagine that we divide the parabola $y = x^2$ by horizontal lines $y = n$, where $n = 0, 1, 2, \ldots$, and then we move each segment down to the horizontal axis parallel to the vertical axis. But these segments are not equally spaced: the further we go from the origin, the denser they become (because as x approaches infinity, y-axis coordinate increases by 1 while x-coordinate increases by smaller and smaller number (that approaches 0)):

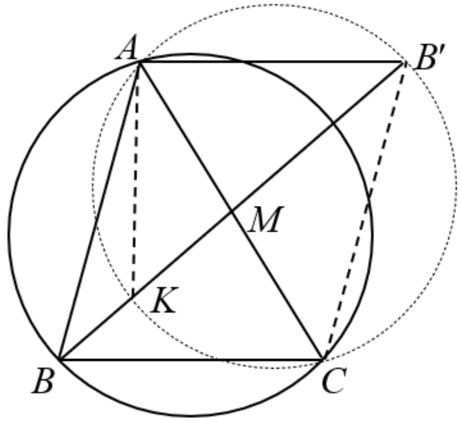


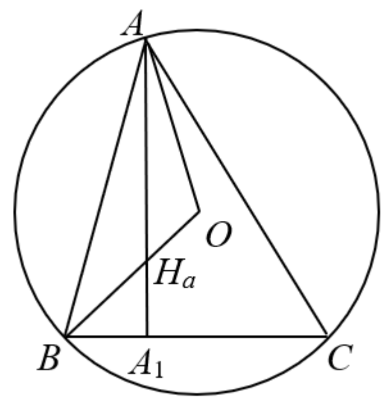
But then any equation of the form $(x-a)^2 = \{x^2\}$ with sufficiently large a suits our needs: in a neighbourhood of its vertex parabola $y = (x-a)^2$ intersects a lot of segments of the plot $y = \{x^2\}$.

5. [6] Let O be the circumcenter of an acute triangle ABC. Let M be the midpoint of AC. The straight line BO intersects the altitudes AA_1 and CC_1 at the points H_a and H_c respectively. The circumcircles of the triangles BH_aA and BH_cC have a second point of intersection K. Prove that K lies on the straight line BM.

Mikhail Evdokimov

Solution 1. Suppose B' is a reflection of the point B through M, and suppose the escribed circle of the triangle ACB' intersects median BM at the point K. Then exterior angle AKB' of triangle AKB is equal to $\angle ACB' = \angle A$ (see further the left figure). But exterior angle BH_aA_1 of triangle AH_aB is also equal to $\angle BAA_1 + \angle ABO = 90° - \angle B + 90° - \angle C = \angle A$ (see further the right figure). Therefore, $\angle AKB = \angle AH_aB$, that is, point K is on the excircle of triangle BH_aA. Similarly, it is on the excircle of triangle BH_cC.

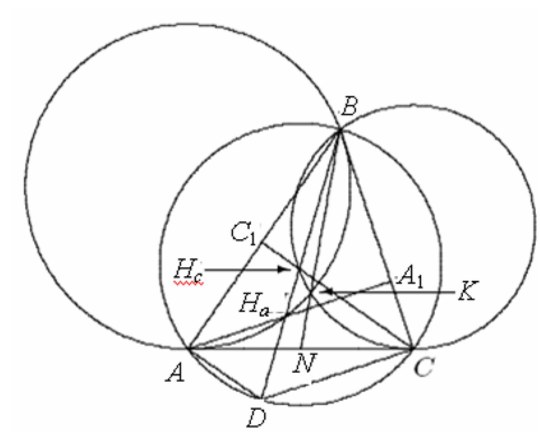




Solution 2. Suppose BD is a diameter of the excircle of triangle ABC. Since $\angle ADB = \angle C$, we have:

$$\angle CAH_a = \angle CAA_1 = 90° - \angle C = 90° - \angle ADB = \angle ABH_a.$$

Thus side AC is tangent to the excircle of triangle BH_aA. Similarly, it is tangent to the excircle of triangle BH_cC. As is known, the radical axis BK of these two circles passes through the midpoint M of segment AC of their common tangent.



Junior A-Level Paper

1. [4] The number $2021 = 43 \cdot 47$ is composite. Prove that if we insert any number of digits "8" between 20 and 21 then the number remains composite.

Mikhail Evdokimov

Solution. Consider two numbers such that the number of digits "8" in them differs by 1. Their difference is of the form of $1880\dots0$. But $188 = 47 \cdot 4$, i.e. it is divisible by 47 and 2021. Therefore, by adding digits "8" one by one, we obtain numbers divisible by 47.

2. [5] In a room there are several children and a pile of 1000 sweets. The children come to the pile one after another in some order. Upon reaching the pile each of them divides the current number of sweets in the pile by the number of children in the room, rounds the result if it is not integer, takes the resulting number of sweets from the pile and leaves the room. All the boys round upwards and all the girls round downwards. The process continues until everyone leaves the room. Prove that the total number of sweets received by the boys does not depend on the order in which the children reach the pile.

Maxim Didin

Solution. Let us imagine (Euclidean) division with remainder of the pile of sweets for k children. We divide the sweets into k smaller piles in such a way that these smaller piles are either of the same size (if the reminder is 0), or in some of them the number of sweets is 1 greater than in the others (and the number of such piles is equal to the remainder).

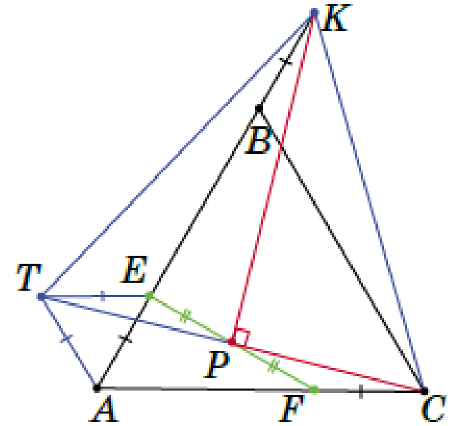
Let the first child rearrange the pile of sweets in a way described above, such that the new piles are sorted in an ascending order. We may assume that a boy will take the rightmost pile and a girl will take the leftmost pile.

When the next child comes to the sweets, they are already rearranged into smaller piles as if they were arranged by the child (because the numbers of children and piles are reduced by 1). And again, a boy will take the rightmost pile and a girl will take the leftmost pile, and so on. As a result, the boys will take the number of rightmost piles equal to the number of boys, and it does not depend on the order.

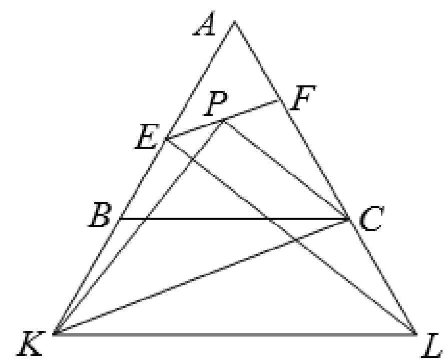
3. [6] There is an equilateral triangle ABC. Let E, F and K be points such that E lies on side AB, F lies on side AC, K lies on the extension of side AB and $AE = CF = BK$. Let P be the midpoint of segment EF. Prove that the angle KPC is right.

Vladimir Rastorguev

Solution 1. Take point T on the extension of CP beyond P such that $CP = PT$. Then $FCET$ is a parallelogram, and therefore TE is equal and parallel to FC. But then triangle TEK is equal to triangle KBC, because their obtuse angles equal $120°$ and corresponding sides of these included angles are the same. Therefore, triangle TKC is isosceles, and its median KP is also an altitude.



Solution 2. Construct equilateral triangle AKL. It is clear that PC is a midsegment of triangle EFL. Triangle EKL is equal to CAK ($KL = AK$, $EK = AC$, $\angle EKL = \angle CAK$). Therefore, $CK = EL = 2PC$. Triangle EAL is equal to CLK, thus $\angle ELA = \angle CKL$. Then $KCP = 60^\circ - \angle PCA + \angle BCK = 60^\circ - \angle ELA + \angle CKL = 60^\circ$ (we used the fact that $PC \parallel EL$ and $BC \parallel KL$). But then KPC is a half of an equilateral triangle, therefore angle KPC is right.



4. [7] A traveller arrived to an island where 50 natives lived. All the natives stood in a circle and each announced firstly the age of his left neighbour, then the age of his right neighbour. Each native is either a knight who told both numbers correctly or a knave who increased one of the numbers by 1 and decreased the other by 1 (on his choice). Is it always possible after that to establish which of the natives are knights and which are knaves?

Alexandr Gribalko

Answer: yes, always.

Solution 1. Let us choose any native (let us call him Bob) and show how to find his age. Imagine putting a hat on every second native starting with Bob. Then enumerate natives without a hat standing after Bob in a clockwise order: 1, 2, ..., 24, 25.

Note that every native announces correctly the sum of ages of his/her neighbours (if one adds up the numbers announced by that native). Let us add up the numbers announced by the 1-st, 3-rd, ..., 25-th native without a hat — this is the sum of ages of all natives in a hat *plus* the age of Bob. Let us sum up the numbers announced by the 2-nd, 4-th, ..., 24-th native without a hat — this is the sum of ages of all natives *minus* the age of Bob. Then we can obtain Bob's age by subtracting the second sum from the first one and dividing the result by 2.

When knowing the age of any native it is easy to find out who of his/her neighbours are, based on their answers.

Solution 2. For the sake of convenience suppose that there are 25 men and 25 women in the circle alternating. Let us show how to find out which man is a knight or a knave (the reasoning for women is similar). Note that two announcements about the age of a woman differ by 1 if and only if her neighbours are a knight and a knave. So it is sufficient to know the 'type' only for one man, since the rest of them will be easily defined.

Consider two cases.

1) For one of the women announcements about her age differ by 2. Then both of her neighbours are knaves and we can define all the men.

2) There are no such women. All the men are divided into groups: in each of the groups both neighbours of each women tell the same thing about her age. But it is still unknown, which group consists of knaves and which group consists of knights.

By adding all numbers announced by men we get double the sum of ages of all women. Now we add the first numbers announced by men. If the sum we get has the same parity as the sum of ages of all women, then there is an even number of knaves amongst men, and there is an odd number otherwise. Since the number of knaves is even only in one of the cases, the traveller can establish which case holds.

5. In the center of each cell of a checkered rectangle M there is a pointlike light bulb. All the light bulbs are initially switched off. In one turn it is allowed to choose a straight line not intersecting any light bulbs such that on one side of it all the bulbs are switched off, and to switch all of them on. In each turn at least one bulb should be switched on. The task is to switch on all the light bulbs using the largest possible number of turns. What is the maximum number of turns if:
a) [4] M is a square of size 21×21;
b) [4] M is a rectangle of size 20×21?

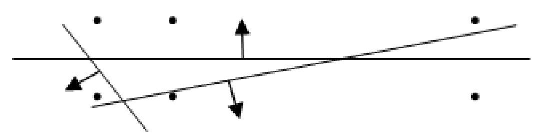
Alexandr Shapovalov

Answers: a) 3 turns; **b)** 4 turns.

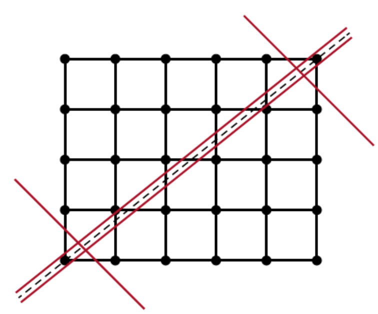
Solution. Instead of the given rectangle consider rectangle N with the vertices in the corner light bulbs.

Estimates. Note that with each turn at least one of the four corner light bulbs is switched on, therefore the number of turns is not greater than 4. Also note that in a) we have to switch on the bulb in the center at some turn. At least two corner bulbs will be on the same side with this central bulb for some chosen line (because the line parallel to the chosen one and passing through the centre divides square N into parts, symmetrical through the centre).

Examples. **a)** Firstly, switch all the bulbs on except the lowest row of light bulbs. Then switch all the remaining bulbs on except a corner bulb. Finally, switch the corner bulb on. (In the figure there are two lower rows of bulbs, and arrows show on what side of a line bulbs are switched on.)



b) Rectangle N is of the size 19×20. There are no other light bulbs on its diagonals except corner ones because 19 and 20 are coprimes. Let us choose the first line parallel to and slightly lower than a diagonal in such a way that these two light bulbs are above the line, and all other bulbs are on the same side as it was for the diagonal. Then let us switch on all the bulbs below the line. Similarly, draw the second line parallel to and slightly higher than the diagonal, and switch on all the bulbs above it as shown in the figure. (In our example N is of size 4×5, that is less than the given size, but also with coprime sides.) The remaining two corner bulbs can be switched on in two turns, by the lines that cut off other bulbs.



6. [10] 100 tourists arrive to a hotel at night. They know that in the hotel there are single rooms numbered as $1, 2, \ldots, n$, and among them k (the tourists do not know which) are under repair, the other rooms are free. The tourists, one after another, check the rooms in any order (maybe different for different tourists), and the first room not under repair is taken by the tourist. The tourists don't know whether a room is occupied until they check it. However it is forbidden to check an occupied room, and the tourists may coordinate their strategy beforehand to avoid this situation. For each k find the smallest n for which the tourists may select their rooms for sure.

Fyodor Ivlev

Answer: $n = 100(m+1)$ for $k = 2m$ and $n = 100(m+1)+1$ for $k = 2m+1$.

Solution. Suppose $k = 2m$ or $k = 2m+1$.

Algorithm. Imagine we divide the rooms into 100 groups, each one consisting of $m+1$ rooms, and if k is odd, then the room that was left we call the spare room. Let the i-th tourist check at first all the rooms in the i-th group moving from left to right, then go to the spare room (if it exists), and then check the rooms in the $(i+1)$-th group but moving from right to left (if $i = 100$ then check the 1-st group). No two tourists can get into the same room, because in the two groups (including the spare room, if it exists) there are $k+2$ rooms all together.

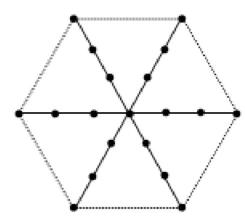
Estimate. Each of the 100 tourists needs a list of $k+1$ different rooms, which he or she will check to select the room not under repair for sure. We may assume that these lists stay the same during the process of room checking by other tourists (because one cannot obtain any new information about the others).

Consider for each tourist the first $m+1$ rooms from his/ her list. All the $100(m+1)$ of these numbers are different, otherwise two tourists with the same number can both get into that room (if the previous rooms for them are all under repair, the total number of such rooms is not greater that $m+m = 2m$). Therefore, $n \geqslant 100(m+1)$.

For an even k this estimate is sufficient. For an odd k, if for one of the tourists (let us call her Alice) the $(m+2)$-th room is the same as some of the $100(m+1)$ "first" rooms for some other tourist (let us call him Bob), then for Alice the first $m+1$ of her rooms are under repair, and for Bob, all the rooms before the same one with Alice (it is no more than m such rooms) are under repair, and they both will get into the same room. Therefore, all of the $(m+2)$-th rooms differ from the $100(m+1)$ first rooms (although, they might be not different from each other), i.e. $n \geqslant 100(m+1)+1$.

Note. For an even number of tourists (and we have 100) the algorithm can be formulated a little bit different.

For an even k imagine hotel plan as 50 hallways, and in each of them there are doors to $k+2$ rooms along one of the walls. Assign one hallway for each pair of tourists and let them go along it starting from the opposite ends. That pair can find no more than k rooms under repair in total, therefore there will be two free rooms for them left.



For an odd k imagine these hallways as diagonals of 100-sided regular polygon: on each diagonal there are $k+2$ rooms with one room common for all hallways (you can see in the figure the same scheme for 6 tourists and $k = 5$). Each pair of tourists go along their hallway from the opposite ends. Note that if some tourist got to the central room, then he or she found $\frac{k+1}{2}$ rooms under repair, therefore no other tourist can get to the central room.

7. [12] Let p and q be two coprime positive integers. A frog hops along the integer line so that on every hop it moves either p units to the right or q units to the left. Eventually, the frog returns to the initial point. Prove that for every positive integer d with $d < p+q$ there are two numbers visited by the frog which differ just by d.

Nikolay Belukhov

Solution 1. Case for $p = q = 1$ is obvious. Otherwise p and q are different, suppose $p < q$. The frog travels a path of a total length divisible by both p and q, and therefore, divisible by pq, because p and q are coprimes. Then the length of the rout is equal to kpq for some natural k, and the frog did kq "short" hops to the right and kp "long" hops to the left.

As is known, for coprime p and q, we can express d as $d = ap - bq$, where a and b are integers. Obviously, this equality remains valid if one simultaneously increases (or decreases) a by q and b by p. Therefore, we can take for a a natural number not greater than q. In this case b is not negative (otherwise $d \geqslant p+q$), and since $a \leqslant q$, we have $b < p$ (because $d > 0$). Thus $a + b < p + q \leqslant k(p+q)$.

Let us call each sequence of $a+b$ consequential hops of the frog a *window*. We may assume for our purpose that include several last hops and several first hops. Then there are exactly $k(p+q)$ windows total.

It is required to find such window that the frog took exactly a short hops (and b long hops), then it will move by d during these $a+b$ hops. Such window exists if there exists a window with not less than a short hops and a window with not more than a short hops: one can move the first window in a circle until getting to the second window. The number of short hops in our window changes each time at most by 1, therefore at some point this number becomes equal to a.

Summing up the number of short hops in all windows we get $kq(a+b)$, because each hop is counted $a+b$ times. There are $k(p+q)$ windows, and on average there are $\frac{kq(a+b)}{k(p+q)}$ short hops for each window. This number is equal to

$$\frac{kq(a+b)}{k(p+q)} = \frac{qa+qb}{p+q} = \frac{pa+qa-d}{p+q} = a - \frac{d}{p+q},$$

that is greater than $a-1$ and less than a. Thus there exists a window with not less than a short hops, and a window with not more than a hops.

Solution 2. Let us call the given frog *old*. We may assume that it takes its sequence of hops infinitely many times in cycles. Let us put a *new* frog on a line at point d and make it take the same sequence of hops as the old one (also in an infinite cycle).

The set of numbers visited by the new frog can be obtained from the set of numbers visited by the old frog by shifting it by d. If at least one of the numbers from the new set equals a number from the old set, then the required pair of numbers is obtained by the back shift. Suppose that is not the case.

As in the previous solution we express d as $ap - bq$ for some non-negative a and b. Then we make the old frog take $a+b$ hops in its cycle. That will bring it to point $e = xp - yq$, where $x + y = a + b$. Since $a - x = y - b$, the difference between coordinates of the new and the old frog is divisible by $p+q$, and we get $d - e = (a-x)p - (b-y)q = (a-x)(p+q)$.

Now we make the frogs take hops simultaneously, the old one continues its initial trajectory, and the new one continues the shifted trajectory. At each move the difference between their coordinates either stays the same (if they hop in the same direction), or changes by $p+q$ (if one of them hops $+p$ and the other, $-q$). Therefore the difference is always divisible by $p+q$; since by assumption it cannot be equal to zero, the sign will stay the same.

Suppose the frogs passed full cycles and came back (the new one to d and the old one to e). Denote the number of moves in the cycle by T, the sum of all numbers visited by the new frog (not including the starting point) by S_1, and the sum of all numbers visited by the old frog by S. On the one hand, numbers in the corresponding moves differ by at least $p+q$ with the difference always be of the same sign, therefore $|S_1 - S| \geq T(p+q)$. On the other hand, the set of numbers visited by the new frog during the cycle differs from the similar set for the old frog by the shift in d, thus $|S_1 - S| = Td$ (note that these sets can contain some numbers several times if during the cycle a frog visited them repeatedly). By substituting and reducing by T we get $d \geq p+q$, and that contradicts problem's statement.

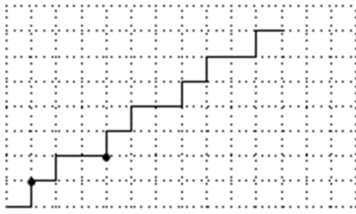
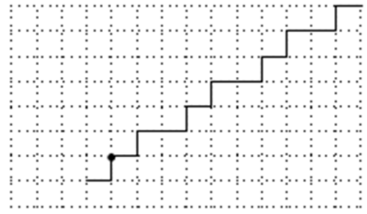
Solution 3. As in solution 2 we assume that the frog hops in an infinite cycle. Also we use the expression $d = ap - bq$ for non-negative a and b, the sum $a+b$ is denoted by r.

By δ_i we denote the difference between positions of the frog at the moment $i+r$ (i.e. after $i+r$ moves from the start) and at the moment i. Since the difference between these positions is r moves we have

$$\begin{aligned}\delta_i = xp - (r-x)q &= ap + (x-a)p - bq - (r-x-b)q = \\ &= d + (x-a)p + (x-(r-b))q = d + (x-a)(p+q)\,.\end{aligned}$$

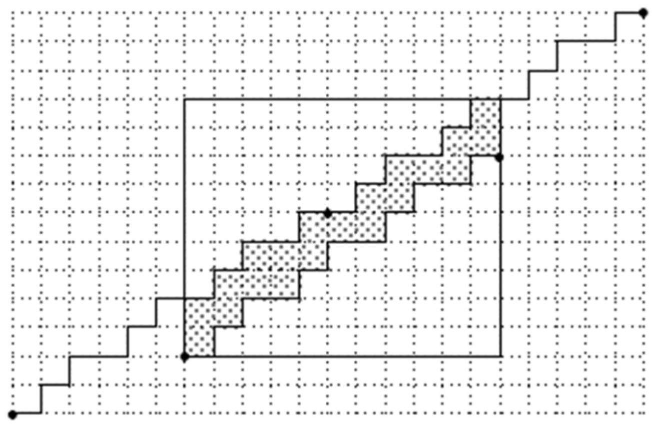
If δ_i is equal to d then we have found the required positions. Suppose by contradiction that $\delta_i \neq d$ for all i. Then all the numbers δ_i can be expressed as $d + (p+q)k_i$ for integer $k_i \neq 0$.

Note that the difference between δ_i and δ_{i+1} is defined by $(i+1)$-th and $(i+r+1)$-th moves. It is easy to conclude considering cases, that the difference equals $\pm(p+q)$ or 0. This means that the numbers δ_i all are either lesser than 0 or greater than 0.

Consider frog's position after rT moves, where T is the number of moves in its cycle. On the one hand, it is equal to the sum $\delta_0 + \delta_r + \delta_{2r} + \ldots + \delta_{r(T-1)}$, which has to be either negative or positive as proven above. On the other hand, after rT   radiction.

Solution 4. Since p a ... nt only by taking kq hops to the right and kp hops ... path in an integer grid in the following way. When ... l when it hops to the left, we will move up by 1. In ... $= 1$ and sequence of hops $7 - 11 + 7 - 11 + 7 + 7 -$ ...

As is known, there exist natural numbers a and b such that $d = pa - qb$. By moving path P by a to the right and by b up, we obtain a new path Q. In the figure above we have path Q obtained from the path in the left figure for $d = 10$, $a = 3$ and $b = 1$ ($10 = 7 \cdot 3 - 11 \cdot 1$). If P and Q have a common point (x, y), then point $(x - a, y - b)$ also lies on P. Corresponding positions of the frog on number line equal $p(x - a) - q(y - b)$ and $px - qy$, but $(px - qy) - (p(x - a) - q(y - b)) = pa - qb = d$, that completes the proof. The same assertion holds if path Q has a common point with extended path $\mathbf{P}$ obtained by adding to path P its copy shifted by (kq, kp), $(2kq, 2kp)$, etc.



Suppose that paths $\mathbf{P}$ and Q have no common points (for example, Q lies beneath $\mathbf{P}$). In the right figure $\mathbf{P}$ consists of two copies of P, and Q is obtained from P by the shift corresponding to $d = 20$, $a = 6$, $b = 2$ ($20 = 7 \cdot 6 - 11 \cdot 2$).

Consider the shaded figure F lying between $\mathbf{P}$ and Q, and the smallest rectangle containing it (rectangle's dimensions are $kq \times (kp + l)$, where l is a natural number). When Q is the same as P, area $S(F)$ equals 0. The shift by a to the right would increase the area by kpa, and the shift by b up would decrease it by kqb. Thus $S(F) = k(pa - qb) = kd$.

Let us estimate the area of F in another way. Figure F can be divided into kq vertical strips of the width of one cell. In each strip there is at least one cell (the bottom one). $kp + l - 1$ horizontal grid lines intersect figure F in interior line segments. The number of cells in each vertical strip is at least 1 greater than the number of grid lines intersecting it, because there is a cell right below each line and a cell above the highest line. Therefore, the total area of F is not less than $kq + kp + l - 1$ cells, that is, not less than $k(p + q)$ cells. This contradicts the inequality $d < p + q$.

Senior A-Level Paper

1. [4] In a room there are several children and a pile of 1000 sweets. The children come to the pile one after another in some order. Upon reaching the pile each of them divides the current number of sweets in the pile by the number of children in the room, rounds the result if it is not integer, takes the resulting number of sweets from the pile and leaves the room. All the boys round upwards and all the girls round downwards. The process continues until everyone leaves the room. Prove that the total number of sweets received by the boys does not depend on the order in which the children reach the pile.

Maxim Didin

Solution. Let us imagine (Euclidean) division with remainder of the pile of sweets for k children. We divide the sweets into k smaller piles in such a way that these smaller piles are either of the same size (if the reminder is 0), or in some of them the number of sweets is 1 greater than in the others (and the number of such piles is equal to the remainder).

Let the first child rearrange the pile of sweets in a way described above, such that the new piles are sorted in an ascending order. We may assume that a boy will take the rightmost pile and a girl will take the leftmost pile.

When the next child comes to the sweets, they are already rearranged into smaller piles as if they were arranged by the child (because the numbers of children and piles are reduced by 1). And again, a boy will take the rightmost pile and a girl will take the leftmost pile, and so on. As a result, the boys will take the number of rightmost piles equal to the number of boys, and it does not depend on the order.

2. [5] Does there exist a positive integer n such that for any real x and y there exist real numbers $a_1, \ldots, a_n$ satisfying

$$x = a_1 + \ldots + a_n \quad \text{and} \quad y = \frac{1}{a_1} + \ldots + \frac{1}{a_n}?$$

Artemiy Sokolov

Answer: Yes, it exists.

Solution 1. Let us prove that $n = 6$ is a suitable positive integer. Firstly, note that any pair $(0, y)$ with a non-zero y can be obtained in the following way: $0 = \frac{3}{2y} + \frac{3}{2y} - \frac{3}{y}$, $y = \frac{2y}{3} + \frac{2y}{3} - \frac{y}{3}$. Any pair $(x, 0)$ with a non-zero x can be obtained similarly. Any pair (x, y) with non-zero x and y can be obtained as a "sum" of the two pairs considered above. Pair $(x, 0)$ can be obtained as the sum of two pairs $(\frac{x}{2}, 0)$, similarly to pair $(0, y)$. Pair $(0, 0)$ can be obtained as $1 + 1 + 1 - 1 - 1 - 1$.

Solution 2. Let us prove that $n = 4$ is a suitable positive integer. Note that if we fix a positive number k and consider all possible pairs of positive numbers a, b with the sum k, then the set of values of $\frac{1}{a} + \frac{1}{b}$ is a ray $[\frac{4}{k}; +\infty)$ (prove it yourself by expressing the sum of the form $\frac{1}{a} + \frac{1}{k-a} = \frac{k}{a(k-a)}$).

Then for given x and y choose positive sums $a + b$ and $c + d$ in such a way that $a + b - c - d = x$ (numbers a, b, c, d are not fixed yet).

Since expressions $\frac{1}{a} + \frac{1}{b}$ and $\frac{1}{c} + \frac{1}{d}$ can become arbitrarily large as stated above, one can find positive a, b, c, d such that the difference between these expressions equals y.

Solution 3. Let us prove that $n = 4$ is a suitable positive integer. Let us find numbers $a_1, \ldots, a_4$ as roots of a polynomial of the form $P(t) = t^4 - xt^3 - ut^2 - yt + 1$ (according to Vieta's formulas the roots satisfy required inequalities). Since $P(0) = 1$, for polynomial $P(t)$ to have four real roots it is sufficient if numbers $P(1) = 2 - x - u - y$ and $P(-1) = 2 + x - u + y$ are negative. We can obtain that by choosing $u > |x + y| + 2$.

Note. One can prove that $n = 1$, $n = 2$ and $n = 3$ aren't suitable positive integers.

3. [5] Let M be the midpoint of the side BC of the triangle ABC. The circle ω passes through A, touches the line BC at M, intersects the side AB at the point D and the side AC at the point E. Let X and Y be the midpoints of BE and CD respectively. Prove that the circumcircle of the triangle MXY touches ω.

Alexey Doledenok

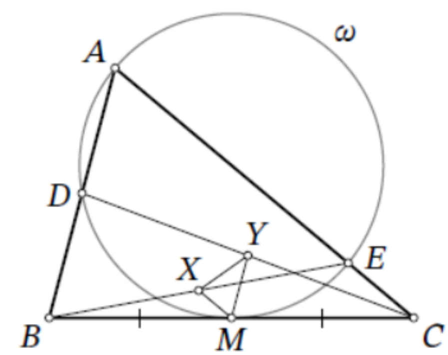
Solution. Note that MX and MY are midsegments of triangles CBE and BCD respectively. By problem's statement

$$BD \cdot BA = BM^2 = CM^2 = CE \cdot CA,$$

thus

$$MX : MY = CE : BD = BA : CA.$$

Since $MX || AC$ and $MY || AB$, triangles MXY and ABC are similar. Therefore $\angle MXY = \angle B = \angle YMC$. By the theorem on angle between a chord and a tangent, side BC is tangent to the excircle of triangle MXY, thus we obtain the problem's statement.



4. [8] There is a row of $100N$ sandwiches with ham. A boy and his cat play a game. In one *action* the boy eats the first sandwich from any end of the row. In one *action* the cat either eats the ham from one sandwich or does nothing. The boy performs 100 actions in each of his turns, and the cat makes only 1 action each turn; the boy starts first. The boy wins if the last sandwich he eats contains ham. Is it true that he can win for any positive integer N no matter how the cat plays?

Ivan Mitrofanov

Solution. Let us prove that for $N = 3^{100}$ the cat wins. For this to happen it is sufficient if at the last boy's turn all the remaining 100 sandwiches are without ham.

Let us number the sandwiches one by one. We divide strategy of the cat into different steps. At first, let us show that it can play so that all the sandwiches with the number equal to 1 modulo 100 are without ham by the moment when there is a third of the sandwiches left.

Mark a sandwich with the number equal to 1 modulo 100 in each hundred of sandwiches. Suppose that during the first 3^{99} turns the cat eats ham from every marked sandwich in the central third of all sandwiches. Since the boy eats $3^{99} \cdot 100$ sandwiches during these turns, none of the sandwiches from the central third are eaten. Then the cat eats the ham from random marked sandwiches for the next 3^{99} turns (and if there are no marked sandwiches left then the cat does nothing). Since the boy eats not more than one marked sandwich in one turn (see note 1 below), after 3^{99} turns all marked sandwiches are without ham.

Next, the cat makes sure that all the sandwiches with the number equal 2 modulo 100 are without ham similarly to the previous step, with the total number of sandwiches three times reduced. At each next step the cat makes sure that the sandwiches with the next number modulo 100 are without ham. After 100 such steps there are 100 sandwiches left and all of them are without ham.

Note 1. At each turn the boy eats sandwiches with different numbers modulo 100 even if he eats them from both ends of the row. It is clear from the fact that before his turn the number of sandwiches with each number modulo 100 is the same since the number of all sandwiches is divisible by 100, and after his turn we have the same layout.

Note 2. The cat's strategy can be specified by showing that for $N = 2^{100}$ it can also win; at each step the number of sandwiches decreases by 2. For this, the cat has to eat a ham from the sandwiches only (with the numbers equal to a given number modulo 100) such that the boy will not get to them at the given step for sure. It can readily be understood that such a sandwich always exists.

And for $N = 2^{100} - 1$ the boy wins. Indeed, by first $2^{99} - 1$ turns he eats any $2^{99} - 1$ hundreds of sandwiches; during these turns the cat can manage to make no more than $2^{99} - 1$ sandwiches without ham. Then if there are 2^k hundreds of sandwiches for the boy to eat and no more than $(100 - k) \cdot 2^k - 1$ of them are without ham, then for $k > 0$ he can eat such half of the row (left or right), where the number of sandwiches without ham is greater. Then there are no more than $(100-k)\cdot 2^{k-1}-1$ sandwiches plus (thanks to the cat) no more than 2^{k-1} new ones in the row, that is no more than $(100-k+1)\cdot 2^{k-1}-1$ sandwiches without ham all together. Playing this way, the boy gets a hundred of sandwiches for $k=0$, and there are no more than 99 of them without ham.

5. [8] 100 tourists arrive to a hotel at night. They know that in the hotel there are single rooms numbered as $1, 2, \ldots, n$, and among them k (the tourists do not know which) are under repair, the other rooms are free. The tourists, one after another, check the rooms in any order (maybe different for different tourists), and the first room not under repair is taken by the tourist. The tourists don't know whether a room is occupied until they check it. However it is forbidden to check an occupied room, and the tourists may coordinate their strategy beforehand to avoid this situation. For each k find the smallest n for which the tourists may select their rooms for sure.

Fyodor Ivlev

Answer: $n = 100(m+1)$ for $k = 2m$ and $n = 100(m+1)+1$ for $k = 2m+1$.

Solution. Suppose $k = 2m$ or $k = 2m+1$.

Algorithm. Imagine we divide the rooms into 100 groups, each one consisting of $m+1$ rooms, and if k is odd, then the room that was left we call the spare room. Let the i-th tourist check at first all the rooms in the i-th group moving from left to right, then go to the spare room (if it exists), and then check the rooms in the $(i+1)$-th group but moving from right to left (if $i = 100$ then check the 1-st group). No two tourists can get into the same room, because in the two groups (including the spare room, if it exists) there are $k+2$ rooms all together.

Estimate. Each of the 100 tourists needs a list of $k+1$ different rooms, which he or she will check to select the room not under repair for sure. We may assume that these lists stay the same during the process of room checking by other tourists (because one cannot obtain any new information about the others).

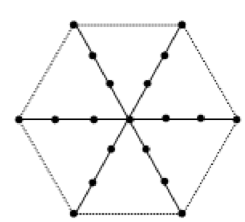
Consider for each tourist the first $m+1$ rooms from his/ her list. All the $100(m+1)$ of these numbers are different, otherwise two tourists with the same number can both get into that room (if the previous rooms for them are all under repair, the total number of such rooms is not greater that $m+m=2m$). Therefore, $n \geqslant 100(m+1)$.

For an even k this estimate is sufficient. For an odd k, if for one of the tourists (let us call her Alice) the $(m+2)$-th room is the same as some of the $100(m+1)$ "first" rooms for some other tourist (let us call him Bob), then for Alice the first $m+1$ of her rooms are under repair, and for Bob, all the rooms before the same one with Alice (it is no more than m such rooms) are under repair, and they both will get into the same room. Therefore, all of the $(m+2)$-th rooms differ from the $100(m+1)$ first rooms (although, they might be not different from each other), i.e. $n \geqslant 100(m+1)+1$.

Note. For an even number of tourists (and we have 100) the algorithm can be formulated a little bit different.

For an even k imagine hotel plan as 50 hallways, and in each of them there are doors to $k+2$ rooms along one of the walls. Assign one hallway for each pair of tourists and let them go along it starting from the opposite ends. That pair can find no more than k rooms under repair in total, therefore there will be two free rooms for them left.

For an odd k imagine these hallways as diagonals of 100-sided regular polygon: on each diagonal there are $k+2$ rooms with one room common for all hallways (you can see in the figure the same scheme for 6 tourists and $k=5$). Each pair of tourists go along their hallway from the opposite ends. Note that if some tourist got to the central room, then he or she found $\frac{k+1}{2}$ rooms under repair, therefore no other tourist can get to the central room.



6. [10] Find at least one real number A such that for any positive integer n the distance between $\lceil A^n \rceil$ and the nearest square of an integer is equal to 2. (By $\lceil x \rceil$ we denote the smallest integer not less than x.)

Dmitry Krekov

Solution. Consider any quadratic equation with integer coefficients where the coefficient of the squared term equals 1, such that the equation has two positive roots whose product equals 1. For example, equation x^2-4x+1 fits the required conditions: its roots are $2+\sqrt{3}$ and $2-\sqrt{3}$. Note that the sum and the product of these roots are integers, and then the sum $(2+\sqrt{3})^n+(2-\sqrt{3})^n$ is integer for any natural n (it can be proved easily by induction or by using expansion: the summands with $\sqrt{3}$ either are in the even power or cancel each other).

Then $((2+\sqrt{3})^n+(2-\sqrt{3})^n)^2$ is a perfect square and it is equal to $(2+\sqrt{3})^{2n}+2+(2-\sqrt{3})^{2n}$ (since the product of the roots equals 1), i.e. it differs by 2 from $(2+\sqrt{3})^{2n}+(2-\sqrt{3})^{2n}$, which is the ceiling of $(2+\sqrt{3})^{2n}$ (because the second root is positive and less than 1).

But then the number $A=(2+\sqrt{3})^2$ is the one we are looking for.

Comment. Obviously, we could take for t any number which is the greatest root of a polynomial of the form $x^2-nx+1=0$ (where n is a natural number not less than 3). Indeed, as in the solution above, the sum of the roots $t^n+\frac{1}{t^n}$ of this polynomial is integer, therefore the problem's statement holds for $A=t^2$.

In this solution it was shown that for the chosen numbers t the difference between t^n and the closest integer approaches zero as t grows. Actually, there are more numbers whose powers approach integers (but the other numbers may be not suitable for the solution of this problem!).

More specifically, suppose $P(x)$ is a monic polynomial with integer coefficients such that all of its roots (including complex ones) except one have absolute values less than 1. Then this one root x_1 is real and the difference between x_1^n and the closest integer approaches 0 as n grows. This follows from the fact that the sum of n-th powers of all roots of the polynomial $P(x)$ can be expressed in integer terms using its coefficients, therefore it is integer. And powers of all the other roots approach 0 because their absolute values are less than 1. This reasoning can be found in the paper "Pisot-Vijayaraghavan numbers" (in Russian) by A. Egorov in the Kvant magazine (numbers 5 and 6, year 2005). See also project "Fractional parts of powers" at the XII Summer Conference of the ToT.

Roots of a monic polynomial such that all the other roots have absolute values less than 1 are called *Pisot-Vijayaraghavan numbers.* These numbers are of interest for problems of Diophantine approximation and were studied in the works of Thue, Hardy, Pisot (see, for example, book "An introduction to Diophantine approximation" by J. W. S. Cassels, Cambridge University Press, 1957 [chapter VIII]).

7. An integer $n>2$ is given. Peter wants to draw n arcs of length α of great circles on a unit sphere so that they do not intersect each other. Prove that

a) [6] for all $\alpha<\pi+\frac{2\pi}{n}$ it is possible;

b) [7] for all $\alpha>\pi+\frac{2\pi}{n}$ it is impossible.

Ilya Bogdanov

Solution. a) Suppose vertical line ℓ goes through the center of the sphere O. Suppose two parallel horizontal planes intercept two equal (not great!) circles γ_+ and γ_- on the sphere. Then there exists great circle Ω_0 tangent to γ_+ and γ_- in (diametrically opposite) points P_0 and M_0 respectively. By rotating Ω_0 by angle $\frac{2\pi k}{n}$ around ℓ, we get great circle Ω_k also tangent to the two circles in points P_k and M_k respectively.

Consider one arc P_0M_0 of circle Ω_0 and also arcs P_kM_k obtained from it by the rotations. All these arcs do not intersect each other because any horizontal plane intersects these arcs in vertices of a regular n-gon. Moreover, each of these arcs P_kM_k can be extended to the closest (such that they do not lie on the arc) points of intersection of Ω_k with other circles Ω_i. Note that points of intersection of Ω_k and Ω_i lie in a (vertical) plane such that the symmetry through that plane swaps these circles (this plane contains, for example, the bisector of angle P_iOP_k). This easily implies that the closest points to our arc are points of intersection with Ω_{k-1} and with Ω_{k+1}, and each arc P_kM_k can be expanded to the arc between these two points (excluding the endpoints).

If now one takes planes intersepting γ_+ and γ_- close to the center of the sphere, then points of intersection of Ω_k and Ω_{k-1} are (because of the symmetry) close to the midpoints of arcs P_kP_{k-1} and M_kM_{k-1}. Therefore the lengths of obtained arcs can be arbitrary close to $\pi + \frac{2\pi}{n}$, which is the required result.

b) Suppose $A_1, A_2, \ldots, A_n$ are pairwise non-intersecting arcs of great circles with lengths $\pi + \alpha_1$, $\pi + \alpha_2, \ldots, \pi + \alpha_n$ for positive α_i. We assume that these arcs include their endpoints. We are going to prove that

$$\sum_{i=1}^{n} \alpha_i \leq 2\pi,$$

which implies the required result.

In the sequel, by *pole* of a hemisphere we mean the point on that hemisphere which is the outermost point with respect to its boundary.

Denote by B_i the arc of a great circle complementary to A_i (its length is equal to $\pi - \alpha_i$). Consider all (open) hemispheres containing B_i, suppose X_i and Y_i are endpoints of B_i (we assume that B_i contains them). A hemisphere contains B_i if and only if it contains X_i and Y_i, i.e. its pole lies on open hemispheres with poles X_i and Y_i. Then set $\mathcal{S}_i$ of the poles of such hemispheres is an intersection of these two hemispheres, i.e. it is a spherical "slice" of angle α_i and its area equals $2\alpha_i$.

Now we prove that sets $\mathcal{S}_i$ are pairwise disjoint. Since area of the sphere equals 4π, we will get the required inequality. Suppose that some point Z lies in $\mathcal{S}_1 \cap \mathcal{S}_2$, then the hemisphere with pole Z contains B_1 and B_2, and therefore the (closed) hemisphere $\mathcal{H}$ complementary to it intersects A_1 and A_2 by whole semicircles (and not their parts). But any two such semicircles on a hemisphere $\mathcal{H}$ do not intersect (since endpoints of any of them are diametrically opposite points on a hemisphere's boundary). This contradicts our assumption.

Note. For any positive numbers $\alpha_1, \ldots, \alpha_n$ with the sum less than 2π, we can apply the method from the solution of part a) to place disjoint arcs of lengths $\pi + \alpha_i$ on a sphere.