42nd INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior A-Level Paper, Spring 2021

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

---

points problems

4 1. The number $2021 = 43 \cdot 47$ is composite. Prove that if we insert any number of digits "8" between 20 and 21 then the number remains composite.

Mikhail Evdikomov

5 2. In a room there are several children and a pile of 1000 sweets. The children come to the pile one after another in some order. Upon reaching the pile each of them divides the current number of sweets in the pile by the number of children in the room, rounds the result if it is not integer, takes the resulting number of sweets from the pile and leaves the room. All the boys round upwards and all the girls round downwards. The process continues until everyone leaves the room. Prove that the total number of sweets received by the boys does not depend on the order in which the children reach the pile.

Maxim Didin

6 3. There is an equilateral triangle ABC. Let E, F and K be points such that E lies on side AB, F lies on side AC, K lies on the extension of side AB and $AE = CF = BK$. Let P be the midpoint of segment EF. Prove that the angle KPC is right.

Vladimir Rastorguev

7 4. A traveller arrived to an island where 50 natives lived. All the natives stood in a circle and each announced firstly the age of his left neighbour, then the age of his right neighbour. Each native is either a knight who told both numbers correctly or a knave who increased one of the numbers by 1 and decreased the other by 1 (on his choice). Is it always possible after that to establish which of the natives are knights and which are knaves?

Alexandr Gribalko

5. In the center of each cell of a checkered rectangle M there is a pointlike light bulb. All the light bulbs are initially switched off. In one turn it is allowed to choose a straight line not intersecting any light bulbs such that on one side of it all the bulbs are switched off, and to switch all of them on. In each turn at least one bulb should be switched on. The task is to switch on all the light bulbs using the largest possible number of turns. What is the maximum number of turns if:

4 a) M is a square of size 21×21;

4 b) M is a rectangle of size 20×21?

Alexandr Shapovalov

10 6. 100 tourists arrive to a hotel at night. They know that in the hotel there are single rooms numbered as $1, 2, \ldots, n$, and among them k (the tourists do not know which) are under repair, the other rooms are free. The tourists, one after another, check the rooms in any order (maybe different for different tourists), and the first room not under repair is taken by the tourist. The tourists don't know whether a room is occupied until they check it. However it is forbidden to check an occupied room, and the tourists may coordinate their strategy beforehand to avoid this situation. For each k find the smallest n for which the tourists may select their rooms for sure.

Fyodor Ivlev

12 7. Let p and q be two coprime positive integers. A frog hops along the integer line so that on every hop it moves either p units to the right or q units to the left. Eventually, the frog returns to the initial point. Prove that for every positive integer d with $d < p + q$ there are two numbers visited by the frog which differ just by d.

Nikolay Belukhov

42nd INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Senior A-Level Paper, Spring 2021

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

points	problems
4	1. In a room there are several children and a pile of 1000 sweets. The children come to the pile one after another in some order. Upon reaching the pile each of them divides the current number of sweets in the pile by the number of children in the room, rounds the result if it is not integer, takes the resulting number of sweets from the pile and leaves the room. All the boys round upwards and all the girls round downwards. The process continues until everyone leaves the room. Prove that the total number of sweets received by the boys does not depend on the order in which the children reach the pile. *Maxim Didin*
5	2. Does there exist a positive integer n such that for any real x and y there exist real numbers $a_1, \ldots, a_n$ satisfying $$x = a_1 + \ldots + a_n \quad \text{and} \quad y = \frac{1}{a_1} + \ldots + \frac{1}{a_n}?$$ *Artemiy Sokolov*
5	3. Let M be the midpoint of the side BC of the triangle ABC. The circle ω passes through A, touches the line BC at M, intersects the side AB at the point D and the side AC at the point E. Let X and Y be the midpoints of BE and CD respectively. Prove that the circumcircle of the triangle MXY touches ω. *Alexey Doledenok*
8	4. There is a row of $100N$ sandwiches with ham. A boy and his cat play a game. In one *action* the boy eats the first sandwich from any end of the row. In one *action* the cat either eats the ham from one sandwich or does nothing. The boy performs 100 actions in each of his turns, and the cat makes only 1 action each turn; the boy starts first. The boy wins if the last sandwich he eats contains ham. Is it true that he can win for any positive integer N no matter how the cat plays? *Ivan Mitrofanov*
8	5. 100 tourists arrive to a hotel at night. They know that in the hotel there are single rooms numbered as $1, 2, \ldots, n$, and among them k (the tourists do not know which) are under repair, the other rooms are free. The tourists, one after another, check the rooms in any order (maybe different for different tourists), and the first room not under repair is taken by the tourist. The tourists don't know whether a room is occupied until they check it. However it is forbidden to check an occupied room, and the tourists may coordinate their strategy beforehand to avoid this situation. For each k find the smallest n for which the tourists may select their rooms for sure. *Fyodor Ivlev*
10	6. Find at least one real number A such that for any positive integer n the distance between $\lceil A^n \rceil$ and the nearest square of an integer is equal to 2. (By $\lceil x \rceil$ we denote the smallest integer not less than x.) *Dmitry Krekov*
	7. An integer $n > 2$ is given. Peter wants to draw n arcs of length α of great circles on a unit sphere so that they do not intersect each other. Prove that
6	a) for all $\alpha < \pi + \frac{2\pi}{n}$ it is possible;
7	b) for all $\alpha > \pi + \frac{2\pi}{n}$ it is impossible. *Ilya Bogdanov*