42nd INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior A-Level Paper, Fall 2020

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points problem

4 1. Let us say that a circle intersects a quadrilateral *properly* if it intersects each of the quadrilateral's sides at two distinct interior points. Is it true that for each convex quadrilateral there exists a circle which intersects it *properly*?

Alexandr Perepechko

7 2. Let us say that a pair of disctinct positive integers is *nice* if their arithmetic mean and their geometric mean are both integer. Is it true that for each nice pair there is another nice pair with the same arithmetic mean?
(The pairs (a, b) and (b, a) are considered to be the same pair.)

Boris Frenkin

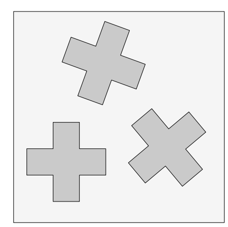
3. Alice and Bob are playing the following game. Each turn Alice suggests an integer number and Bob writes down either that number or the sum of that number with all previously written numbers. Is it always possible for Alice to ensure that at some moment among the written numbers there are

3 a) at least a hundred copies of number 5?

4 b) at least a hundred copies of number 10?

Andrey Arzhantsev

7 4. The X pentomino consists of 5 squares 1×1 where four squares are all adjacent to the fifth one. Is it possible to cut 9 such pentominoes from a 8×8 chessboard, not necessarily cutting along grid lines?
(The picture shows how to cut three such X pentominoes.)



Alexandr Gribalko

8 5. Do there exist 100 positive distinct integers such that a cube of one of them equals the sum of the cubes of all the others?

Mikhail Evdokimov

10 6. There are two round tables with n dwarves sitting at each table. Each dwarf has only two friends: his neighbours to the left and to the right. A good wizard wants to seat the dwarves at one round table so that each two neighbours are friends. His magic allows him to make any $2n$ pairs of dwarves into pairs of friends (the dwarves in a pair may be from the same or from different tables). However, he knows that an evil sorcerer will break n of those new friendships. For which n is the good wizard able to achieve his goal no matter what the evil sorcerer does?

Mikhail Svyatlovskiy

7. There is a convex quadrangle $ABCD$ such that no three of its sides can form a triangle. Prove that

6 a) one of its angles is not greater than $60°$;

6 b) one of its angles is at least $120°$.

Maxim Didin

42nd INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Senior A-Level Paper, Fall 2020

(The result is computed from the three problems with the highest scores.)

points	problem
4	1. There were n positive integers. For each pair of those integers Boris wrote their arithmetic mean onto a blackboard and their geometric mean onto a whiteboard. It so happened that for each pair at least one of those means was integer. Prove that on at least one of the boards all the numbers are integer. *Boris Frenkin*
5	2. Baron Munchausen presented a new theorem: if a polynomial $x^n - ax^{n-1} + bx^{n-2} + \dots$ has n positive integer roots then there exist a lines in the plane such that they have exactly b intersection points. Is the baron's theorem true? *Fedor Ivlev*
6	3. Two circles α and β with centers A and B respectively intersect at points C and D. The segment AB intersects α and β at points K and L respectively. The ray DK intersects the circle β for the second time at the point N, and the ray DL intersects the circle α for the second time at the point M. Prove that the intersection point of the diagonals of the quadrangle $KLMN$ coincides with the incenter of the triangle ABC. *Konstantin Knop*
7	4. There are two round tables with n dwarves sitting at each table. Each dwarf has only two friends: his neighbours to the left and to the right. A good wizard wants to seat the dwarves at one round table so that each two neighbours are friends. His magic allows him to make any $2n$ pairs of dwarves into pairs of friends (the dwarves in a pair may be from the same or from different tables). However, he knows that an evil sorcerer will break n of those new friendships. For which n is the good wizard able to achieve his goal no matter what the evil sorcerer does? *Mikhail Svyatlovskiy*
7	5. Does there exist a rectangle which can be cut into a hundred of rectangles such that all of them are similar to the original one but no two are congruent? *Mikhail Murashkin*
10	6. Alice and Bob play the following game. They write some fractions of the form $1/n$, where n is positive integer, onto the blackboard. The first move is made by Alice. Alice writes only one fraction in each her turn and Bob writes one fraction in his first turn, two fractions in his second turn, three fractions in his third turn and so on. Bob wants to make the sum of all the fractions on the board to be an integer number after some turn. Can Alice prevent this? *Andrey Arzhantsev*
12	7. A white bug sits in one corner square of a $1000 \times n$ chessboard, where n is an odd positive integer and $n > 2020$. In the two nearest corner squares there are two black chess bishops. On each move, the bug either steps into a square adjacent by side or moves as a chess knight. The bug wishes to reach the opposite corner square by never visiting a square occupied or attacked by a bishop, and visiting every other square exactly once. Show that the number of ways for the bug to attain its goal does not depend on n. *Nikolay Belukhov*