41st INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior A-Level Paper, Spring 2020

Grades 8 – 9 (ages 13-15)

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points	problems
4	1. Does there exist a positive integer that is divisible by 2020 and has equal numbers of digits $0, 1, 2, \ldots, 9$? *Mikhail Evdokimov*
5	2. Three legendary knights are fighting against a multiheaded dragon. Whenever the first knight attacks, he cuts off half of the current number of heads plus one more. Whenever the second knight attacks, he cuts off one third of the current number of heads plus two more. Whenever the third knight attacks, he cuts off one fourth of the current number of heads plus three more. They repeatedly attack in an arbitrary order so that at each step an integer number of heads is being cut off. If all the knights cannot attack as the number of heads would become non-integer, the dragon eats them. Will the knights be able to cut off all the dragon's heads if it has 41! heads? (Note: $41! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 41$) *Alexey Zaslavsky*
	3. Is it possible to inscribe an N-gon in a circle so that all the lengths of its sides are different and all its angles (in degrees) are integer, where
4	a) $N = 19$;
3	b) $N = 20$? *Mikhail Malkin*
8	4. For which integers N it is possible to write real numbers into the cells of a square of size $N \times N$ so that among the sums of each pair of adjacent cells there are all integers from 1 to $2(N-1)N$ (each integer once)? *Maxim Didin*
9	5. Let $ABCD$ be an inscribed trapezoid. The base AB is 3 times longer than CD. Tangents to the circumscribed circle at the points A and C intersect at the point K. Prove that the angle KDA is a right angle. *Alexandr Yuran*
9	6. Alice has a deck of 36 cards, 4 suits of 9 cards each. She picks any 18 cards and gives the rest to Bob. Now each turn Alice picks any of her cards and lays it face-up onto the table, then Bob similarly picks any of his cards and lays it face-up onto the table. If this pair of cards has the same suit or the same value, Bob gains a point. What is the maximum number of points he can guarantee regardless of Alice's actions? *Mikhail Evdokimov*
12	7. Gleb picked positive integers N and a ($a < N$). He wrote the number a on a blackboard. Then each turn he did the following: he took the last number on the blackboard, divided the number N by this last number with remainder and wrote the remainder onto the board. When he wrote the number 0 onto the board, he stopped. Could he pick N and a such that the sum of the numbers on the blackboard would become greater than $100N$? *Ivan Mitrofanov*

41st INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS
Senior A-Level Paper, Spring 2020
Grades 10 – 11 (ages 15 and older)
(The result is computed from the three problems with the highest scores.)

points	problems
4	1. Consider two parabolas $y = x^2$ and $y = x^2 - 1$. Let U be the set of points between the parabolas (including the points on the parabolas themselves). Does U contain a line segment of length greater than 10^6? *Alexey Tolpygo*
5	2. Alice had picked positive integers a, b, c and then tried to find positive integers x, y, z such that $a = \text{l.c.m.}(x, y), b = \text{l.c.m.}(x, z), c = \text{l.c.m.}(y, z)$. It so happened that such x, y, z existed and were unique. Alice told this fact to Bob and also told him the numbers a and b. Prove that Bob can find c. (Note: l.c.m = least common multiple.) *Boris Frenkin*
8	3. Is it possible that two cross-sections of a tetrahedron by two different cutting planes are two squares, one with a side of length no greater than 1 and another with a side of length at least 100? *Mikhail Evdokimov*
9	4. Henry invited $2N$ guests to his birthday party. He has N white hats and N black hats. He wants to place hats on his guests and split his guests into one or several dancing circles so that in each circle there would be at least two people and the colors of hats of any two neighbours would be different. Prove that Henry can do this in exactly $(2N)!$ different ways. (All the hats with the same color are identical, all the guests are obviously distinct; $(2N)! = 1 \cdot 2 \cdot \ldots \cdot (2N)$.) *Gleb Pogudin*
9	5. Let $ABCD$ be an inscribed quadrilateral. Let the circles with diameters AB and CD intersect at two points X_1 and Y_1, the circles with diameters BC and AD intersect at two points X_2 and Y_2, the circles with diameters AC and BD intersect at two points X_3 and Y_3. Prove that the lines X_1Y_1, X_2Y_2 and X_3Y_3 are concurrent. *Maxim Didin*
10	6. There are $2n$ consecutive integers on a board. It is permitted to split them into pairs and simultaneously replace each pair by their difference (not necessarily positive) and their sum. Prove that it is impossible to obtain any $2n$ consecutive integers again. *Alexandr Gribalko*
12	7. Consider an infinite white plane divided into square cells. For which k it is possible to paint a positive finite number of cells black so that on each horisontal, vertical and diagonal line of cells there is either exactly k black cells or none at all? *A. Dinev, K. Garov, N Belukhov*