41st INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior O-Level Paper, Fall 2019

SOLUTIONS OF PROBLEMS

1. (4 points) An illusionist lays the 52 cards of a standard deck in a row. In each step, the audience chooses an integer k not greater than the length of the row, and the illusionist removes either the kth card from the left or the k card from the right. The illusionist announces in advance that the Three of Clubs should be the last card which remains. For which initial positions of the Three of Clubs can the illusionist guarantee the success of the trick?

   Answer: for the first or last position. **Solution.** Let us call the first position from either end an outside position, and every other position an inside position. We can be forced to remove the Three of Clubs only if it is the central card at some point of the process, otherwise we always have the opportunity to throw away another card. If the Three of Clubs was in an outside position it will never become the middle one until the very last turn, which allows us to guarantee the success.

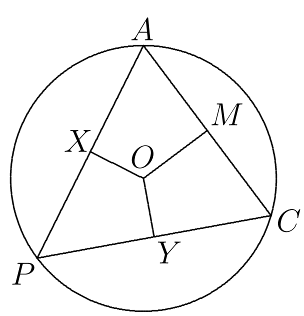
   Suppose now that the Three of Clubs was in an inside position. Let us show two possible strategies for the audience:

   Strategy 1. The audience keeps naming inside positions. This guarantees that the illusionist cannot remove the first and the last card. When there will be only 3 cards in the row, the Three of Clubs will be the middle one. Then the audience shall choose the number 2 and force the illusionist to discard the middle card.

   Strategy 2. The audience always names the number of the position of the Three of Clubs. The illusionist is forced to discard the card in the mirror symmetric position, which decreases the bigger of the distances from the Three of Clubs to the border of the row. Thus at some point those distances will be the same and the Three of Clubs will be in the middle.

2. (4 points) A and C are fixed points on a circle with centre O and P is a variable point on the same circle. X and Y are the respective midpoints of PA and PC, and H is the orthocenter of triangle OXY. Prove that H is a fixed point.

   Solution. Let M be the midpoint of AC. Then OM is perpendicular to AC. Since X and Y are the respective midpoints of PA and PC. XY is parallel to AC, and therefore perpendicular to OM. Similarly, OX is perpendicular to YM and OY is perpendicular to XM. It follows that M is the orthocentre of triangle OXY. Hence H coincides with M, which is a fixed point.



3. (4 points) Counters numbered 1 to 100 are arranged in order in a row. It costs 1 dollar to interchange two adjacent counters, but nothing to interchange two counters with exactly 3 other counters between them. What is the minimum cost for rearranging the 100 counters in reverse order?

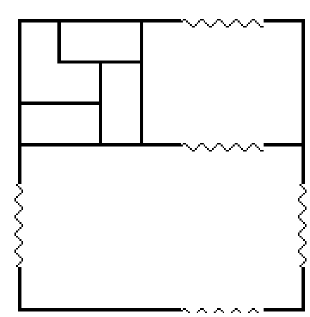
   Answer: 50 dollars. **Solution.** *Lower bound.* Note that the free (chargeless) operation does not change the parity of the numbers of places of the counters. As each counter needs to change the parity of the place where it stands, we need at least 100 such changes, which means at least 50 nonfree operations. *Algorithm.* Let us color the places in the row into four colors: $abcdabcdabcd \dots abcd$. As the free operation can interchange two counters in the consecutive cells of the same color, it is possible by using it several times to arrange the counters on the cells of the same color in any order we wish. Let us interchange the counters in all pairs bc and all pairs da for 49 dollars. Now let us move the counters 1 and 100 to the adjacent positions using the free operations and interchange them for the remaining 1 dollar. Now all the counters that were standing on the color a are standing on the color d, and similarly for other colors. Thus we may rearrange them in the reverse order using the free operations.

4. (5 points) Each of 1000 points on a circle is labelled with the square of an integer. The sum of any 41 adjacent labels is a multiple of 41^2. Is it necessarily true that each of the 1000 integers is a multiple of 41?

   Answer: yes. **Solution.** Let the integers be $a_1, a_2, \dots, a_{1000}$. Then $a_i^2 \equiv a_j^2 (mod 41^2)$ if $i \equiv j (mod 41)$. As 1000 and 41 are relatively prime it immediately follows that all the a_i^2 have the same residue modulo 41^2. The sum of any 41 adjacent labels will be congruent to $41a_i^2$ nodulo 41^2 and is divisible by 41^2, which means that a_i^2 is divisible by 41. As 41 is prime it follows that a_i is divisible by 41.

5. (5 points) Basil has sufficiently many copies of the I-tricube, which is a $1 \times 1 \times 3$ block, and of the V-tricube, which is a $1 \times 2 \times 2$ block with a unit cube missing at a corner. Basil builds with these pieces a solid rectangular box each dimension of which is at least 2. Prove that he does not have to use any copy of the I-tricube.

   Solution. Basil can easily build a $1 \times 2 \times 3$ block with two copies of the V-tricube. He can build a $3 \times 3 \times 3$ block with nine copies of the V-tricube by stacking three of them on top of one another, and filling the remaining space with three copies of the $1 \times 2 \times 3$ block. Since Basil can build the box with copies of the I-tricube and of the V-tricube, at least one dimension of the box, say the altitude, is divisible by 3. We can cut the box into $m \times n$ horizontal slabs with altitude 3, where $m \geq 2$ and $n \geq 2$. Basil then builds one slab at a time, by filling the $m \times n$ base with copies of the 1×2 rectangle and of the 3×3 square. We consider two cases.
   Case 1. Either m or n is even.
   Each row has even length and can be filled with copies of the 1×2 rectangle. Hence the whole base can be filled with copies of the 1×2 rectangle.
   Case 2. Both m and n are odd.
   Then $m \geq 3$ and $n \geq 3$. Basil can start the bottom 3 rows with a copy of the 3×3 square. The remaining part of each of these 3 rows has even length and can be filled with copies of the 1×2 rectangle. Since the number of remaining rows is even, Basil can fill them with copies of the 1×2 rectangle.



Fall 2019 Senior O-Level Paper

1. (3 points) An illusionist lays the 52 cards of a standard deck in a row. In each step, the audience chooses an integer k not greater than the length of the row, and the illusionist removes either the kth card from the left or the k card from the right. The illusionist announces in advance that the Three of Clubs should be the last card which remains. For which initial positions of the Three of Clubs can the illusionist guarantee the success of the trick?

   Answer: for the first or last position. **Solution.** Let us call the first position from either end an outside position, and every other position an inside position. We can be forced to remove the Three of Clubs only if it is the central card at some point of the process, otherwise we always have the opportunity to throw away another card. If the Three of Clubs was in an outside position it will never become the middle one until the very last turn, which allows us to guarantee the success.

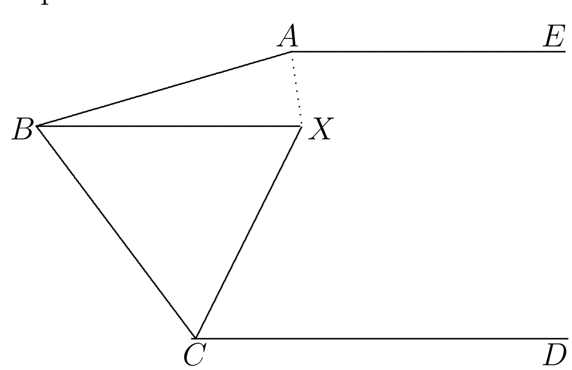
   Suppose now that the Three of Clubs was in an inside position. Let us show two possible strategies for the audience:

   Strategy 1. The audience keeps naming inside positions. This guarantees that the illusionist cannot remove the first and the last card. When there will be only 3 cards in the row, the Three of Clubs will be the middle one. Then the audience shall choose the number 2 and force the illusionist to discard the middle card.

   Strategy 2. The audience always names the number of the position of the Three of Clubs. The illusionist is forced to discard the card in the mirror symmetric position, which decreases the bigger of the distances from the Three of Clubs to the border of the row. Thus at some point those distances will be the same and the Three of Clubs will be in the middle.

2. (4 points) Let $ABCDE$ be a convex pentagon such that $AB = BC$, and AE is parallel to CD. Let K be the point of intersection of the bisectors of $\angle A$ and $\angle C$. Prove that BK is parallel to AE.

   Solution. Let the line through B parallel to AE intersect the bisector of $\angle C$ at the point X. Then BX is parallel to CD so that $\angle XCD = \angle CXB$. Hence $\angle BCX = \angle BXC$ so that $BC = BX$. Since $BA = BC$, we have $BA = BX$ so that $\angle BAX = \angle BXA$. Since BX is parallel to AE, $\angle BXA = \angle XAE$. It follows that BX is the bisector of $\angle A$, so that X and K coincide. Hence BK is parallel to AE.



3. (4 points) In each step, we may multiply a positive integer by 3 and then add 1 to the product. If the positive integer is even, we may divide it by 2. If the positive integer is odd, we may subtract 1 from it and then divide the difference by 2. Prove that starting with 1, we can obtain any positive integers in a finite number of steps.

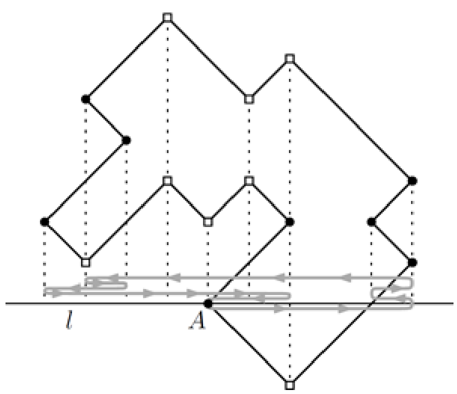
Solution: Let us assume all the integers from 1 to $3k-2$ have been obtained. Then the integer $3k-1$ can be obtained by the sequence $2k-1 \to 6k-2 \to 3k-1$, the integer $3k$ can be obtained by $2k \to 6k+1 \to 3k$ and finally $3k+1$ by $k \to 3k+1$. This allows us to use induction by k.

4. (5 points) In a polygon, not necessarily convex, any two adjacent sides are perpendicular to each other. Prove that for each vertex A, the number of other vertices B such that the bisector of $\angle A$ is perpendicular to the bisector of $\angle B$ is even.

Solution 1. Let us rotate the polygon so that the sides are horizontal and vertical. Let the number of the horizontal ones be k, then the number of the vertical ones is k too. All the vertices are of four possible types: ⌜, ⌝, ⌞, ⌟. Let us suppose without any loss of generality that vertex A has the type 2. Then what we want to prove is that the total number of vertices of types 1 and 4 is even.

Each vertex of type 1 or 3 is the left end of a horizontal side. Hence the total number of vertices of types 1 and 3 is k. Let the number of vertices of type 1 be x, then for type 3 it is $k-x$. By considering the bottom ends ot the vertical sides we may see that the total number of vertices of types 3 and 4 is k, thus the number of vertices of type 4 is x. Then the total number of vertices of types 1 and 4 is $2x$ which is even.

Solution 2. Let us rotate the polygon so that the bisector l of the angle at the vertex A is horizontal. Consider a point moving along the outline of the polygon with a constant speed starting and finishing in A. Then its projection onto l also moves with a constant speed and the direction of its motion changes each time it passes through a vertex whose bisector is parallel to l or through vertex A itself. Thus the number of such vertices is even. As the total number of all vertices is also even, we obtain the result of the problem.



Solution 3. Let us rotate the polygon so that the sides are horizontal and vertical. Let the number of the horizontal ones be k, then the number of the vertical ones is k too. The slope of each bisector is either 1 or -1.

Let us numerate the vertices counterclockwise with integers from 1 to $2k$ and let a_i be 1 if the angle in i-th vertex is $90°$ and -1 if $270°$. If we go around the outline of the polygon counterclockwise then each time we pass a vertex with $a_i = 1$ we rotate by $90°$ counterclockwise and each time we pass a vertex with $a_i = -1$ we rotate by $90°$ clockwise. As we rotate by $360°$ after the whole path we may find that the number of the angles of the second type is greater by 4 than the number of the angles of the first type. This means that this number is $\frac{2k-4}{2} = k-2$, which means $a_1 a_2 \ldots a_{2k} = (-1)^{k-2}$.

Note that the slopes of the bisectors of two consecutive vertices are the same if and only if the angles are different. Let us denote the slope of the bisector at i-th vertex as b_i. Assuming without loss of generality that $a_1 = b_1$ we see that for odd i we have $a_i = b_i$ and for even i we have $a_i = -b_i$. Thus $b_1 b_2 \ldots b_{2k} = (-1)^k a_1 a_2 \ldots a_{2k} = (-1)^{2k-2} = 1$. This means that the number of bisectors with the slope -1 is even. The same holds for the bisectors with the other slope.

5. (5 points) Counters numbered 1 to 100 are arranged in order in a row. It costs 1 dollar to interchange two adjacent counters, but nothing to interchange two counters with exactly 4 other counters between them. What is the minimum cost for rearranging the 100 counters in reverse order?

   Answer: 61 dollar. **Solution.** Note that the free (chargeless) operation does not change the residue of the counter modulo 5. Let us color the places in the row into 5 colors corresponding to their residue modulo 5, for simplicity let us name those color simply as 0, 1, 2, 3, 4. Note that any counters that stay on cells of the same color may be rearranged in any order by using the free operations, thus we may think of the nonfree operation as if it just traded a pair of counters between two adjacent colors. In this context we may reformulate the problem as follows: there are 5 piles of counters in a circle, one may interchange two counters between two adjacent piles for 1 dollar. What is the minimum cost for exchanging all the counters between 1 and 0 and between 2 and 4?

   Lower bound. If every counter from pile 0 travelled to pile 1, they all were used in at least 1 nonfree operation. The same holds for the counters from pile 1. Now as every counter from pile 2 travelled to pile 4, each of them was used in at least 2 nonfree operations, the same stays for counters from pile 4. As each operation works with two counters simultaneously, we may find that the minimum number of nonfree operations is $(20 + 20 + 40 + 40) : 2 = 60$. However if it were possible to perform exactly 60 operations then each counter from pile 2 passed through pile 3, thus at least one counter originally from pile 3 also was used in the operations and so there were more operations than 60.

   Algorithm. After we have reformulated the problem, the algorithm is easy. Let us exchange the counters between 0 and 1 directly for 20 dollars. Let us select a counter X in the pile 3. Then we trade the counter X for some counter a_1 from pile 2, now we trade a_1 for some counter b_1 in pile 4, now b_1 is in pile 3, so we may trade it for some counter a_2 in pile 2 and so on. In the end we will have b_2 in pile 3, so we will interchange it with X. This part of the algorithm costs 41 dollar.