41st INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS, Fall 2019
SOLUTIONS OF PROBLEMS

Junior A-Level Paper

1. (2+2 points) Let us call the number of factors in the prime decomposition of an integer $n > 1$ the *complexity* of n. For example, complexity of numbers 4 and 6 is equal to 2. Find all n such that all integers between n and $2n$ have complexity
a) not greater than the complexity of n;
b) less than the complexity of n.

Answer: a) all n of the form $n = 2^k$; b) there is no such n.
Solution. a) It is obvious that 2^k is the smallest number of complexity k. Thus all numbers between 2^k and 2^{k+1} have complexity not greater than k. Let n be not a power of 2, then there exists some power of 2 between n and $2n$. Its complexity is larger then the complexity of n.
b) From the previous part we already know that we may check only the case $n = 2^k$. Consider $m = 3 \cdot 2^{k-1}$. This number is between n and $2n$ and has the same complexity as n.

Sidenote: one may deduce b) from the famous Bertrand's postulate that states that if p is prime then the next prime is less than $2p$. Indeed, let $n = pr$ where p is prime, then let q be the next prime after p. Consider $m = qr$, then $n < m < 2n$ and the complexities of m and n are equal.

2. (7 points) Two acute triangles ABC and $A_1B_1C_1$ are such that B_1 and C_1 lie on BC, and A_1 lies inside the triangle ABC. Let S and S_1 be the areas of those triangles respectively. Prove that

$$\frac{S}{AB + AC} > \frac{S_1}{A_1B_1 + A_1C_1}.$$

Solution. Let D and D_1 be symmetric to A and A_1 with respect to BC. Consider the bisectors of our triangles AK and A_1K_1. Then K and K_1 are the incenters of the quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ respectively and the inequality turns into $r > r_1$ where r and r_1 are the radii of the corresponding circles.

3. (7 points) There are 100 visually identical coins of three types: golden, silver and copper. There is at least one coin of each type. Each golden coin weighs 3 grams, each silver coin weighs 2 grams, and each copper coin weighs 1 gram. How to find the type of each coin performing no more than 101 measurements on a balance scale with no weights?

Solution 1 (A.Shapovalov). Let us consider the current situation to be *victorious* if we have determined the weights of k coins, there is one silver or two copper among them and we have performed no more than $k + 1$ measurements. Indeed, if we are in a victorious situation we may find the weight of any remaining coin by comparing it with 2 grams. Repeating this operation we easily find the weights of all the remaining coins and the number of operations is no more than 101. Thus what remains is to show how to obtain a victorious situation.

Let us fix one coin and compare it to others until we find a coin of some other weight. Let A be the lighter one and B be the heavier one. Let us compare B to other new coins until we find a coin of some other weight, let us name it C. The current situation is as follows: we have one or several coins of weight a, one or several coins of weight $b > a$ and one coin of weight $c \neq b$. Now let us compare C to A thus comparing c and a. There are two possible cases:
1) $c = a$, then let us compare B to $A+C$, thus comparing b to $2a$. If $b > 2a$ then $b = 3, a = 1$, if $b = 2a$ then $b = 2, a = 1$, if $b < 2a$ then $b = 3, a = 2$. In all those situations we have determined the weight of all coins we have considered and we have found either a coin of weight 2 or two coins of weight 1. The number of measurements we performed is greater than the number of considered coins by 1 thus the situation is victorious.

2) $c \neq a$, then a, b, c are three different weights. As we have compared them pairwise, we know their order, so we know the weights of all the coins we have considered. The situation is victorious.

Note: sometimes some of the measurements may be skipped, for example if C is heavier than B there is no need to compare C to A. Also if in the beginning we found at least 2 coins of weight a we may consider one of them to be the coin C.

Solution 2 (A.Ryabichev). We start by proving the following lemma by induction:
Lemma 1. Assume there are k coins such that there are coins of each type among them. Also there is a pair of coins A and a such that we already know that $A > a$. Then in $k-1$ measurements we may determine the weights of all k coins.
The base case of Lemma 1. If we have 3 coins we may compare the remaining coin with A and a, thus determining weights of 3 coins in 2 more measurements.
The induction step of Lemma 1. Let us compare any two coins other than A and a. If their weights are equal then we may throw one of them away, memorising which one it was equal to, and use the inductive hypothesis. Let us assume they are not equal, let us denote them $B > b$. Now let us compare $A + a$ and $B + b$. If they are equal then $A = B$ and $a = b$ thus we may throw away B and b and use the inductive hypothesis for $k - 2$ coins.

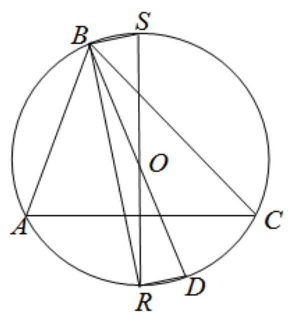
Now let us assume that $A + a > B + b$. Then necessarily $A = 3$ and $b = 1$. If we compare $A+b$ with $B+a$ we will be able to determine the weights of all four coins. Moreover, there is a coin of weight 2 among them thus we may compare all the remaining coins with it in $k-4$ measurements. This concludes the proof of Lemma 1.

Now let us prove the following statement by induction:
Statement 2. If there are k coins such that there are coins of each type then it is possible to find the weights of all coins in k measurements.
The base case of Statement 2. If there are 3 coins we may compare them pairwise in 3 measurements.
The induction step of Statement 2. Let us compare any two coins. If they are equal we may throw away one of them, memorising which one it was equal two and using the inductive hypothesis. Otherwise we obtain a pair $A > a$ and use Lemma 1.

Note: We performed 100 measurements which is even less than required. Also the only relation between weights that we used was that the weight of a copper coin plus the weight of a golden coin is equal to twice the weight of a silver coin. Thus it was not important for this solution that the weight of a silver coin is exactly two times bigger than the weight of a copper coin.

4. (7 points) Let OP and OQ be the perpendiculars from the circumcenter O of a triangle ABC to the internal and external bisectors of the angle B. Prove that the line PQ divides the segment connecting midpoints of CB and AB into two equal parts.

Solution. Let us consider the homothety with the center B and scale factor 2. Let us denote the circumcircle of ABC as Ω. The image of O is the point D opposite to B on Ω. The image of P is the point R, the intersection of Ω with the bisector of the angle ABC. The image of Q is the point S opposite to R on the circle Ω. Finally, the image of the "segment connecting ..." is the segment AC. Now all that remains is to note that RS bisects AC as R bisects the arc AC.



5. (8 points) Let us say that the pair (m, n) of two positive different integers m and n is *nice* if mn and $(m+1)(n+1)$ are perfect squares. Prove that for each positive integer m there exists at least one $n > m$ such that the pair (m, n) is nice.

Solution. For any m the pair $(m, m(4m+3)^2)$ is nice. Indeed, $(m+1)(m(4m+3)^2+1) = (m+1)(16m^3+24m^2+9m+1) = (m+1)^2(16m^2+8m+1) = ((m+1)(4m+1))^2$.

Possible approaches to finding the solution. It is natural to try finding n such that it has form ma^2 and $n+1$ is $(m+1)k^2$ for some integer k. Since n/m is a square, we have

$$n/m = (n+1-1)/m = ((m+1)k^2-1)/m = k^2 + (k^2-1)/m.$$

The easiest way to ensure that this is a perfect square is to set $(k^2-1)/m$ to be equal to $4k+4$. Then $(k-1)/m = 4$ which implies $k = 4m+1$.

6. (8 points) Peter has several 100 ruble notes and no other money. He starts buying books; each book costs a positive integer number of rubles, and he gets change in 1 ruble coins. Whenever Peter is buying an expensive book for 100 rubles or higher he uses only 100 ruble notes in the minimum necessary number. Whenever he is buying a cheap one (for less than 100 rubles) he uses his coins if he has enough, otherwise using a 100 ruble note. When the 100 ruble notes have come to the end, Peter has expended exactly a half of his money. Is it possible that he has expended 5000 rubles or more?

Answer: no, it is not possible.

Solution 1. Let us first consider the case where every cheap book was purchased with coins. Let the number of expensive books be n, then Peter spent at least $200n$ rubles in 100 ruble notes and got no more than $99n$ rubles in coins (and spent some of them for cheap books). Thus in the end he had less than a half of his money which contradicts the task. Thus this case is impossible.

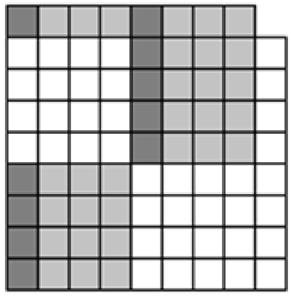
Now let us consider the latest cheap purchase that increased the number of coins, let us name it "purchase A". If the cost of this purchase was x than as Peter couldn't buy it with coins he had no more than $x-1$ rubles in coins before this purchase. Thus after the purchase he had no more than $(x-1)+(100-x) = 99$ rubles in coins. Now let the number of expensive books after A be equal to m, then in the end Peter had no more than $99+99m$ rubles in coins. In the beginning he had at least $200m+100$ rubles as he spent at least two 100 ruble notes for each expensive book and also at least one such note for the purchase A. Thus in the end he had at least $100m+50$ rubles. We obtain the following inequality: $99+99m \geq 100m+50$, which implies $m \leq 49$. Thus in the end there was no more than $99+99m \leq 99 \cdot 50 < 5000$ rubles.

Solution 2. Let us consider some cheap book that was bought for coins. Those coins were obtained as change in some previous purchases. Let us increase their costs by corresponding values instead of purchasing this cheap book. Repeating this procedure we can guarantee that there were no cheap books purchased for coins. As the amount of coins decreases, no new purchases for coin may arise.

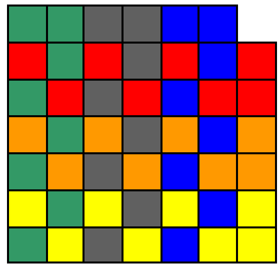
There existed a *small* purchase for 50 or less rubles as otherwise Peter would have spent more than half the money. There was only one small purchase. Indeed, if there was a second one then it could have been bought for coins that were obtained from the first one. However we have eliminated all the purchases for coins in our previous step.

Let us consider the difference between the change and the price for each purchase. For the small purchase it is not greater than $99-1 = 98$ rubles. For each other purchase it is negative and even because the sum of change and price is divisible by 100. Thus it lies between -98 and -2. This means that the number of purchases other than the small one is not greater than $98 : 2 = 49$ and for each of them Peter paid no more than 200 (as $99-201 < -98$). Thus there were no more than $1+2\cdot 49 = 99$ notes of 100 rubles and the half of this sum is not greater than $9900 : 2 = 4950 < 5000$.

7. (10 points) Peter has a wooden square stamp divided into a grid. He coated some 102 cells of this grid with black ink. After that, he pressed this stamp 100 times on a list of paper so that each time just those 102 cells left a black imprint on the paper. Is it possible that after his actions the imprint on the list is a square 101×101 such that all the cells except one corner cell are black?
Answer. Yes, it is possible.
Solution. Let us show a construction that covers any square $(2N+1) \times (2N+1)$ without one corner cell with $2N$ imprints of a stamp with $2N+2$ cells. On the picture there is exhibited a particular case of $N = 4$.



Let us assume that the missing corner is top-right. We divide our square into four $N \times N$ squares in the bottom-left part and two $1 \times 2N$ rectangles, one on top and one on the right. Let us consider the following set of cells: the left column of the bottom-left $N \times N$ square, the left column of the top-right square, the left cell of the top rectangle and the $(N+1)$-th from the left cell of the same top rectangle. This is a set of $2N+2$ cells. If we shift it to the right $N-1$ times, those shifted copies cover the entire bottom-left and top-right $N \times N$ squares and the entire top rectangle. Now all that remains is to note that the remaining area is the image of the already colored under rotation by 90° degrees.



Note. There are other possible solutions. Here we include a picture of another one, drawn for simplicity for the case $N = 3$.

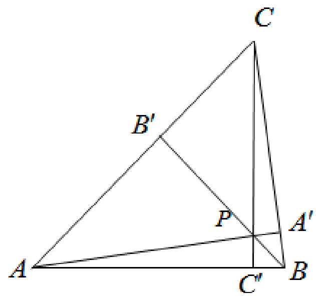
Senior A-Level Paper

1. (5 points) The polynomial $P(x, y)$ is such that for any integer $n \geq 0$ each of the polynomials $P(n, y)$ and $P(x, n)$ either is the constant zero or has the degree not greater than n. Is it possible that the polynomial $P(x, x)$ has an odd degree?

   Answer: no, it is not possible.
   Solution. Let the highest power of x that appears be m and let the highest power of y that appears be n. Without loss of generality let us assume that $n \geq m$. Let us represent $P(x, y)$ as $A(x)y^n + B(x)y^{n-1} + \ldots$ where $A(x), B(x), \ldots$ are polynomials in x. For all $0 \leq k < n$ consider $P(k, y) = A(k)y^n + B(k)y^{n-1} + \ldots$. It immediately follows from the properties of $P(x, y)$ that $P(k, y)$ has degree less than n, thus $A(k) = 0$. This means $A(0) = A(1) = A(2) = \ldots = A(n-1) = 0$. As $A(x)$ is not constant zero and has at least n roots it necessarily has degree at least n. On the other hand, the degree of $A(x)$ is not greater than m. Thus, $n = m$ and $A(x)y^n$ contains a term of the form $Cx^n y^n$. As this term does not appear in any other summands in the expression $A(x)y^n + B(x)y^{n-1} + \ldots$ it cannot cancel, thus $\deg P(x, x) = 2n$.

   Note: it is possible to show that a polynomial $P(x, y)$ satisfies the task if and only if it is of the form $c_0 + xy(c_1 + (x-1)(y-1)(c_2 + \ldots + (c_k + ((x-k)(y-k)c_{k+1}))\ldots)$, where k is a nonnegative integer and $c_0, \ldots c_{k+1}$ are some constants.

2. (5 points) Let ABC be an acute triangle. Suppose the points A', B', C' lie on its sides BC, AC, AB respectively and the segments AA', BB', CC' intersect in a common point P inside the triangle. For each of those segments let us consider the circle such that the segment is its diameter, and the chord of this circle that contains the point P and is perpendicular to this diameter. All three these chords occurred to have the same length. Prove that P is the orthocenter of the triangle ABC.



Solution. Let the length of those chords be equal to $2x$. By the intersecting chords theorem we have

$$x^2 = AP \cdot A'P = BP \cdot B'P = CP \cdot C'P.$$

By the converse theorem the points A, A', B, B' lie on a common circle, thus $\angle AA'B = \angle AB'B$. Similarly $\angle AA'C = \angle AC'C$, $\angle BB'C = \angle BC'C$. Thus

$$\angle AA'B = \angle AB'B = 180^\circ - \angle BB'C = 180^\circ - \angle BC'C = \angle AC'C = \angle AA'C,$$

which means that AA' is an altitude. Likewise, the other two segments are altitudes.

3. (6 points) There are 100 visually identical coins of three types: golden, silver and copper. There is at least one coin of each type. Each golden coin weighs 3 grams, each silver coin weighs 2 grams, and each copper coin weighs 1 gram. How to find the type of each coin performing no more than 101 measurements on a balance scale with no weights?

   Solution: see solution for problem Junior-3.

4. (10 points) Consider an increasing sequence of positive numbers

$$\ldots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \ldots,$$

infinite in both directions. For a positive integer k let b_k be the minimal integer such that the ratio of the sum of any k consecutive elements of the sequence to the largest of those k elements is not greater than b_k. Prove that the sequence $b_1, b_2, b_3, \ldots$ either coincides with the sequence $1, 2, 3, \ldots$ or is constant after some point.

Solution: It is obvious that $b_1 = 1$. Also for $k > 1$ the ratio from the task is less than k, thus $b_k \leq k$ for all positive integer k. If this sequence does not coincide with $1, 2, 3, \ldots$ then there is some k such that $b_k \leq k - 1$. Then $a_i + a_{i+1} + \ldots + a_{i+k-1} \leq (k-1)a_{i+k-1}$ for all integer i, thus $ka_i < (k-1)a_{i+k-1}$. Let us introduce $t = \frac{k-1}{k} < 1$, then $a_i < ta_{i+k-1} < ta_{i+k}$ for all integer i. It follows that

$$a_i < ta_{i+k} < t^2 a_{i+2k} < \ldots < t^q a_{i+qk} < \ldots$$

Using these inequalities let us establish an upper bound on b_n. Consider the ratio

$$\frac{a_{i+n} + a_{i+n-1} + \ldots + a_{i+1}}{a_{i+n}} = \frac{a_{i+n}}{a_{i+n}} + \frac{a_{i+n-1}}{a_{i+n}} + \ldots + \frac{a_{i+1}}{a_{i+n}}.$$

In this sum the first k summands are not greater than 1, the next k not greater than t, the next k not greater than t^2 etc. Thus

$$\frac{a_{i+n} + a_{i+n-1} + \ldots + a_{i+1}}{a_{i+n}} < k(1 + t + t^2 + \ldots + t^s) < k(1 + t + t^2 + \ldots) = \frac{k}{1-t} = k^2.$$

This implies that $b_n \leq k^2$ for all positive integer n. As the sequence (b_n) is a nondecreasing sequence of integers it necessarily stabilizes at some number not exceeding k^2.

5. (6+6 points) The point M inside a convex quadrilateral $ABCD$ is equidistant from the lines AB and CD and is equidistant from the lines BC and AD. The area of $ABCD$ occurred to be equal to $MA \cdot MC + MB \cdot MD$. Prove that the quadrilateral $ABCD$ is
   a) cyclic (inscribed);
   b) tangential (circumscribed).

Solution. a) Let us consider perpendiculars MP, MQ, MR, MT onto AB, BC, CD, DA. Then

$$S_{ABCD} = S_{AMB} + S_{BMC} + S_{CMD} + S_{DMA} \leq$$

$$\leq (S_{AMP} + S_{BMP}) + (S_{BMQ} + S_{CMQ}) + (S_{CMR} + S_{DMR}) + (S_{DMT} + S_{AMT}).$$

As the right triangles AMP and CMR have equal legs MP and MR, we may combine them into a triangle Δ that has two sides equal to MA and MC, thus

$$S_{AMP} + S_{CMR} = S_{\Delta} \leq \frac{1}{2} MA \cdot MC.$$

Similarly

$$S_{BMP} + S_{DMR} \leq \frac{1}{2} MB \cdot MD, S_{BMQ} + S_{DMS} \leq \frac{1}{2} MB \cdot MD, S_{CMQ} + S_{AMS} \leq \frac{1}{2} MA \cdot MC.$$

Altogether we obtain $S_{ABCD} \leq MA \cdot MC + MB \cdot MD$. We see now that all the inequalities shoud necessary be equalities. This implies that the points P, Q, R, T lie on the corresponding sides, not on their extensions, and that the triangle Δ is right, which means $\angle MAP + \angle MCR = 90°$. Similarly $\angle MAD + \angle MCQ = 90°$, thus

$$\angle BAD + \angle BCD = (\angle MAP + \angle MCR) + (\angle MAT + \angle MCQ) = 180°.$$

This proves that $ABCD$ is inscribed.

b) Considering the right triangle Δ again we see that $AP + RC = \sqrt{MA^2 + MC^2}$. Similarly $BP + RD = \sqrt{MB^2 + MD^2}$. Then $AB + CD = \sqrt{MA^2 + MC^2} + \sqrt{MB^2 + MD^2}$. If we compute $BC + DA$ the same way we shall get the same result, which proves that $ABCD$ is circumscribed.

Note: it is possible to prove that the area of any tangential cyclic quadrilateral $ABCD$ is equal to $MA \cdot MC + MB \cdot MD$ where M is its incenter.

6. (6+6 points) A cube consisting of $(2N)^3$ unit cubes is pierced by several needles parallel to the edges of the cube (each needle pierces exactly $2N$ unit cubes). Each unit cube is pierced by at least one needle. Let us call any subset of these needles "regular" if there are no two needles in this subset that pierce the same unit cube.
   a) Prove that there exists a regular subset consisting of $2N^2$ needles such that all of them have either the same direction or two different directions.
   b) What is the maximum size of a regular subset that does exist for sure?

Solution (A.Shapovalov). Let the cube's edges be parallel to the coordinate axes.

a) Let us split the cube into layers of height 1 parallel to the plane Oxy. In each such layer there is some number of needles parallel to Ox and some number of needles parallel to Oy. Let us consider the maximum of those two numbers. Similar maximums may be found for layers parallel to other coordinate planes. Altogether we get $6n$ maximums. Let k be the smallest of them and K be the layer where it occurs.

We may pick $2n - k$ rows and $2n - k$ columns in the layer K that do not contain needles. Then in their intersection we get $(2n - k)^2$ cubes that are not pierced by needles in this layer. Thus all those cubes are pierced by needles perpendicular to K. Let us paint those $(2n - k)^2$ needles into blue.

Now let us pick a side P of the cube that is orthogonal to K. Consider k layers that are parallel to P and do not contain blue needles. In each such layer we may pick at least k

needles with the same direction by definition of k. Let us color such needles red. The set of red and blue needles is regular. Furthermore the number of those needles is at least $2n^2$:

$$k^2 + (2n-k)^2 \geq \frac{1}{2}(k + (n-k))^2 = 2n^2.$$

b) **Answer:** $2n^2$ needles.

Consider two cubes $n \times n \times n$ in the opposite corners. There are $2n^3$ unit cubes in them together, let us color them green. Let us pierce each of them by three perpendicular needles. Then all unit cubes are pierced, so this set of needles satisfies the conditions in the task. Now let us prove that it is not possible to pick a regular subset of more than $2n^2$ needles.

Indeed, each needle pierces n green unit cubes. As in the regular subset each unit cube is pierced only by one needle, each needle requires its own n green cubes. Thus their number is not greater than $2n^3 : n = 2n^2$.

7. (12 points) Some of the integers $1, 2, 3, \ldots, n$ have been colored red so that for each triplet of red numbers a, b, c (not necessarily distinct) if $a(b-c)$ is a multiple of n then $b = c$. Prove that there are no more than $\varphi(n)$ red numbers, where $\varphi(n)$ is the number of positive integers up to n that are relatively prime to n.

   Lemma. Let D be some set of prime divisors of a positive integer n. Then the number of positive integers not greater than n and not divisible by any number from D is equal to $n \prod\limits_{p \in D} \left(1 - \frac{1}{p}\right)$.

   The proof of Lemma immediately follows from expanding the product and applying the inclusion-exclusion principle.

   Now let us assume that there are more than $\varphi(n)$ red numbers. Then some of those red numbers have a common prime divisor with n. Let q be the greatest of such prime divisors and let a be a red number that is divisible by q. We aim to find two distinct red numbers b and c that have the same residue modulo $\frac{n}{q}$, as then ab and ac would have the same residue modulo n. To accomplish that, let us show that $\varphi(n)$ is not less than the number of possible residues of red numbers modulo $\frac{n}{q}$.

   Let D be the set of all prime divisors of n and let D' be the set of all prime divisors of n greater than q. By assumption that q is the greatest prime divisor of n that appears in red numbers we see that all red numbers are not divisible by prime numbers from D'. Thus all their residues modulo $\frac{n}{q}$ are also not divisible by prime numbers from D'. By the Lemma the number of such possible residues is $\frac{n}{q} \prod\limits_{p \in D'} \left(1 - \frac{1}{p}\right)$. On the other hand, the number of red numbers is by our assumption greater than $\varphi(n)$ which by Lemma equals $n \prod\limits_{p \in D} \left(1 - \frac{1}{p}\right)$.

   To finish the proof it suffices to prove that $n \prod\limits_{p \in D} \left(1 - \frac{1}{p}\right)$ is not less than $\frac{n}{q} \prod\limits_{p \in D'} \left(1 - \frac{1}{p}\right)$.

   By cancelling n and all the terms with $p > q$ we obtain an equivalent inequality

$$\left(1 - \frac{1}{q}\right) \prod_{p \in D, p < q} \geq \frac{1}{q},$$

   which is equivalent to

$$q - 1 \geq \prod_{p \in D, p < q} \frac{p}{p-1}.$$

   The last inequality is true as

$$q - 1 = \frac{q-1}{q-2} \cdot \frac{q-2}{q-3} \cdot \ldots \cdot \frac{3}{2} \cdot \frac{2}{1}.$$