41st INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS
Junior A-Level Paper, Fall 2019
Grades 8 – 9 (ages 13-15)
(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points problems

1. Let us call the number of factors in the prime decomposition of an integer $n > 1$ the *complexity* of n. For example, complexity of numbers 4 and 6 is equal to 2. Find all n such that all integers between n and $2n$ have complexity
   - 2 a) not greater than the complexity of n;
   - 2 b) less than the complexity of n.

Boris Frenkin

2. (7) Two acute triangles ABC and $A_1B_1C_1$ are such that B_1 and C_1 lie on BC, and A_1 lies inside the triangle ABC. Let S and S_1 be the areas of those triangles respectively. Prove that
$$\frac{S}{AB+AC} > \frac{S_1}{A_1B_1+A_1C_1}.$$

Nairi Sedrakyan, Ilya Bogdanov

3. (7) There are 100 visually identical coins of three types: golden, silver and copper. There is at least one coin of each type. Each golden coin weighs 3 grams, each silver coin weighs 2 grams, and each copper coin weighs 1 gram. How to find the type of each coin performing no more than 101 measurements on a balance scale with no weights?

Vladislav Novikov

4. (7) Let OP and OQ be the perpendiculars from the circumcenter O of a triangle ABC to the internal and external bisectors of the angle B. Prove that the line PQ divides the segment connecting midpoints of CB and AB into two equal parts.

Artemiy Sokolov

5. (8) Let us say that the pair (m, n) of two positive different integers m and n is *nice* if mn and $(m+1)(n+1)$ are perfect squares. Prove that for each positive integer m there exists at least one $n > m$ such that the pair (m, n) is nice.

Yury Markelov

6. (8) Peter has several 100 ruble notes and no other money. He starts buying books; each book costs a positive integer number of rubles, and he gets change in 1 ruble coins. Whenever Peter is buying an expensive book for 100 rubles or higher he uses only 100 ruble notes in the minimum necessary number. Whenever he is buying a cheap one (for less than 100 rubles) he uses his coins if he has enough, otherwise using a 100 ruble note. When the 100 ruble notes have come to the end, Peter has expended exactly a half of his money. Is it possible that he has expended 5000 rubles or more?

Tatiana Kazitsina

7. (10) Peter has a wooden square stamp divided into a grid. He coated some 102 cells of this grid with black ink. After that, he pressed this stamp 100 times on a list of paper so that each time just those 102 cells left a black imprint on the paper. Is it possible that after his actions the imprint on the list is a square 101×101 such that all the cells except one corner cell are black?

Alexsandr Gribalko

41st INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Senior A-Level , Fall 2019

Grades 10 – 11 (ages 15 and older)

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points	problems
5	1. The polynomial $P(x,y)$ is such that for any integer $n \geq 0$ each of the polynomials $P(n,y)$ and $P(x,n)$ either is the constant zero or has the degree not greater than n. Is it possible that the polynomial $P(x,x)$ has an odd degree? *Boris Frenkin*
5	2. Let ABC be an acute triangle. Suppose the points A', B', C' lie on its sides BC, AC, AB respectively and the segments AA', BB', CC' intersect in a common point P inside the triangle. For each of those segments let us consider the circle such that the segment is its diameter, and the chord of this circle that contains the point P and is perpendicular to this diameter. All three these chords occurred to have the same length. Prove that P is the orthocenter of the triangle ABC. *Grigory Galperin*
6	3. There are 100 visually identical coins of three types: golden, silver and copper. There is at least one coin of each type. Each golden coin weighs 3 grams, each silver coin weighs 2 grams, and each copper coin weighs 1 gram. How to find the type of each coin performing no more than 101 measurements on a balance scale with no weights? *Vlasislav Novikov*
10	4. Consider a increasing sequence of positive numbers $$\ldots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \ldots,$$ infinite in both directions. For a positive integer k let b_k be the minimal integer such that the ratio of the sum of any k consecutive elements of the sequence to the largest of those k elements is not greater than b_k. Prove that the sequence $b_1, b_2, b_3, \ldots$ either coincides with the sequence $1, 2, 3, \ldots$ or is constant after some point. *Ivan Mitrofanov*
	5. The point M inside a convex quadrilateral $ABCD$ is equidistant from the lines AB and CD and is equidistant from the lines BC and AD. The area of $ABCD$ occurred to be equal to $MA \cdot MC + MB \cdot MD$. Prove that the quadrilateral $ABCD$ is
6	a) tangential (circumscribed);
6	b) cyclic (inscribed). *Nairi Sedrakyan*
	6. A cube consisting of $(2N)^3$ unit cubes is pierced by several needles parallel to the edges of the cube (each needle pierces exactly $2N$ unit cubes). Each unit cube is pierced by at least one needle. Let us call any subset of these needles "regular" if there are no two needles in this subset that pierce the same unit cube.
6	a) Prove that there exists a regular subset consisting of $2N^2$ needles such that all of them have either the same direction or two different directions.
6	b) What is the maximum size of a regular subset that does exist for sure? *Nikita Gladkov, Alexandr Zimin*
12	7. Some of the integers $1, 2, 3, \ldots, n$ have been colored red so that for each triplet of red numbers a, b, c (not necessarily distinct) if $a(b-c)$ is a multiple of n then $b = c$. Prove that there are no more than $\varphi(n)$ red numbers, where $\varphi(n)$ is the number of positive integers up to n that are relatively prime to n. *Alexandr Semenov*