40th INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior A-Level Paper, Spring 2019

Grades 8 – 9 (ages 13-15)

(The result is computed from the three problems with the highest scores.)

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points	problems
5	1. The King gives the following task to his two wizards. The First Wizard should choose 7 distinct positive integers with total sum 100 and secretly submit them to the King. To the Second Wizard he should tell only the fourth largest number. The Second Wizard must figure out all the chosen numbers. Can the wizards succeed for sure? The wizards cannot discuss their strategy beforehand. *Mikhail Evdokimov*
7	2. 2019 point grasshoppers sit on a line. At each move one of the grasshoppers jumps over another one and lands at the point the same distance away from it. Jumping only to the right, the grasshoppers are able to position themselves so that some two of them are exactly 1 mm apart. Prove that the grasshoppers can achieve the same, jumping only to the left and starting from the initial position. *Sergey Dorichenko*
7	3. Two equal non-intersecting wooden disks, one gray and one black, are glued to a plane. A triangle with one gray side and one black side can be moved along the plane so that the disks remain outside the triangle, while the colored sides of the triangle are tangent to the disks of the same color (the tangency points are not the vertices). Prove that the line that contains the bisector of the angle between the gray and black sides always passes through some fixed point of the plane. *Egor Bakaev, Pavel Kozhevnikov, Vladimir Rastorguev*
8	4. Each segment whose endpoints are the vertices of a given regular 100-gon is colored red, if the number of vertices between its endpoints is even, and blue otherwise. (For example, all sides of the 100-gon are red.) A number is placed in every vertex so that the sum of their squares is equal to 1. On each segment the product of the numbers at its endpoints is written. The sum of the numbers on the blue segments is subtracted from the sum of the numbers on the red segments. What is the greatest possible result? *Ilya Bogdanov*
9	5. One needs to fill the cells of an $n \times n$ table ($n > 1$) with distinct integers from 1 to n^2 so that every two consecutive integers are placed in cells that share a side, while every two integers with the same remainder if divided by n are placed in distinct rows and distinct columns. For which n is this possible? *Alexandr Gribalko*
9	6. A point K is marked inside an isosceles triangle ABC so that $CK = AB = BC$ and $\angle KAC = 30°$. Find the angle AKB. *Egor Bakaev*
12	7. There are 100 piles of 400 stones each. At every move, Pete chooses two piles, removes one stone from each of them, and is awarded the number of points, equal to the non-negative difference between the numbers of stones in two new piles. Pete has to remove all stones. What is the greatest total score Pete can get, if his initial score is 0? *Maxim Didin*

40th INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Senior A-Level Paper, Spring 2019

Grades 10 – 11 (ages 15 and older)

(The result is computed from the three problems with the highest scores, the scores for the individual parts of a single problem are summed up.)

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points	problems	
5	1.	Some positive integer, divisible by 7, is shown on a computer screen. The cursor marks a gap between some two of its consecutive digits. Prove that there is a digit that can be inserted into the marked gap any number of times so that the resulting number is always divisible by 7. *Alexandr Galochkin*
6	2.	2019 point grasshoppers sit on a line. At each move one of the grasshoppers jumps over another one and lands at the point the same distance away from it. Jumping only to the right, the grasshoppers are able to position themselves so that some two of them are exactly 1 mm apart. Prove that the grasshoppers can achieve the same, jumping only to the left and starting from the initial position. *Sergey Dorichenko*
7	3.	Two not necessarily equal non-intersecting wooden disks, one gray and one black, are glued to a plane. An infinite angle with one gray side and one black side can be moved along the plane so that the disks remain outside the angle, while the colored sides of the angle are tangent to the disks of the same color (the tangency points are not the vertices). Prove that it is possible to draw a ray in the angle, starting from the vertex of the angle and such that no matter how the angle is positioned, the ray passes through some fixed point of the plane. [1] *Egor Bakaev, Ilya Bogdanov, Pavel Kozhevnikov, Vladimir Rastorguev*
8	4.	One needs to fill the cells of an $n \times n$ table ($n > 1$) with distinct integers from 1 to n^2 so that every two consecutive integers are placed in cells that share a side, while every two integers with the same remainder if divided by n are placed in distinct rows and distinct columns. For which n is this possible? *Alexandr Gribalko*
	5.	The orthogonal projection of a tetrahedron onto a plane containing one of its faces is a trapezoid of area 1, which has only one pair of parallel sides.
4	a)	Is it possible that the orthogonal projection of this tetrahedron onto a plane containing another its face is a square of area 1?
4	b)	The same question for a square of area 1/2019. *Mikhail Evdokimov*
8	6.	For each five distinct variables from the set $x_1, \ldots, x_{10}$ there is a single card on which their product is written. Peter and Basil play the following game. At each move, each player chooses a card, starting with Peter. When all cards have been taken, Basil assigns values to the variables as he wants, so that $0 \leqslant x_1 \leqslant \ldots, \leqslant x_{10}$. Can Basil ensure that the sum of the products on his cards is greater than the sum of the products on Peter's cards? *Ilya Bogdanov*
12	7.	On the grid plane all possible broken lines with the following properties are constructed: each of them starts at the point $(0, 0)$, has all its vertices at integer points, each linear segment goes either up or to the right along the grid lines. For each such broken line consider the corresponding *worm*, the subset of the plane consisting of all the cells that share at least one point with the broken line. Prove that the number of worms that can be divided into dominoes (rectangles 2×1 and 1×2) in exactly $n > 2$ different ways, is equal to the number of positive integers that are less than n and relatively prime to n. *Ilke Chanakchi, Ralf Schiffler*

[1] There was a mistake in the text of the problem 3, we publish here the correct version. The solutions were estimated according to the text published originally.